Algebraic properties and transformations of monographs

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Abstract

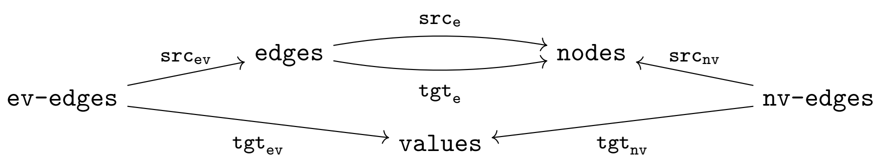
Monographs are graph-like structures with directed edges of unlimited length that are freely adjacent to each other. The standard nodes are represented as edges of length zero. They can be drawn in a way consistent with standard graphs and many others, like E-graphs or ∞-graphs. The category of monographs share many properties with the categories of graph structures (algebras of monadic many-sorted signatures, equivalent to presheaf toposes), except that there is no terminal monograph. It is universal in the sense that its slice categories (or categories of typed monographs) are equivalent to the categories of graph structures. Type monographs thus emerge as a natural way of specifying graph structures. A detailed analysis of single and double pushout transformations of monographs is provided, and a notion of attributed typed monographs generalizing typed attributed E-graphs is analyzed w.r.t. attribute-preserving transformations.

Keywords: Algebraic Graph Transformation, Graph Structures, Typed Graphs
Conflict of Interest: none.

1 Introduction

Many different notions of graphs are used in mathematics and computer science: simple graphs, directed graphs, multigraphs, hypergraphs, etc. One favourite notion in the context of logic and rewriting is that also known as *quivers*, i.e., structures of the form (N, E, s, t) where N, E are sets and s, t are functions from E (edges) to N (nodes), identifying the source and target tips of every edge (or arrow). One reason for this is that the category of quivers is isomorphic to the category of algebras of the many-sorted signature with two sorts `nodes` and `edges` and two operator names `src` and `tgt` of type `edges` → `nodes`. In conformity with this tradition, by *graph* we mean quiver throughout this paper.

In order to conveniently represent elaborate data structures it is often necessary to enrich the structure of graphs with attributes: nodes or edges may be labelled with elements from a fixed set, or with values taken in some algebra, or with sets of values as in [1], etc. An interesting example can be found in [2] with the notion of E-graphs, since the attributes are also considered as nodes. More precisely, an E-graph is an algebra whose signature can be represented by the following graph:



The names given to the sorts and operators help to understand the structure of E-graphs: the `edges` relate the `nodes` among themselves, the `nv-edges` relate the `nodes` to the `values`, and the `ev-edges` relate the `edges` to the `values`. Hence the sort `values` holds attributes that are also nodes. But then we see that in E-graphs the `ev-edges` are adjacent to `edges`. This is non-standard, but we may still accept such structures as some form of graph, if only because we understand how they can be drawn.

Hence the way of generalizing the notion of graphs seems to involve a generalization of the signature of graphs considered as algebras. This path has been followed by Michael Löwe in [3], where a *graph structure* is defined as a monadic many-sorted signature. Indeed in the examples above, and in many examples provided in [3], all operators have arity 1 and can therefore be considered as edges from their domain to their range sort. Is this the reason why they are called graph structures? But the example above shows that E-graphs are very different from the graph that represent their signature. Besides, it is not convenient that our understanding of such structures should be based on syntax, i.e., on the particular names given to sorts and operators in the signature.

Furthermore, it is difficult to see how the algebras of some very simple monadic signatures can be interpreted as graphs of any form. Take for instance the signature of graphs and reverse the target function to `tgt` : `nodes` → `edges`. Then there is a symmetry between the sorts `nodes` and `edges`, which means that in an algebra of this signature nodes and edges would be objects of the same nature. Is this still a graph? Can we draw it? Worse still, if the two sorts are collapsed into one, does it mean that a node/edge can be adjacent to itself?

We may address these problems by restricting graph structures to some class of monadic signatures whose algebras are guaranteed to behave in an orthodox way, say by exhibiting clearly separated edges and nodes. But this could be prone to arbitrariness, and it would still present another drawback: that the notion of graph structure does not easily give rise to a category. Indeed, it is difficult to define morphisms between algebras of different signatures, if only because they can have any number of carrier sets.

The approach adopted here is rather to reject any *structural* distinction between nodes and edges, hence to adopt a unified view of nodes as edges of length 0, and standard edges as edges of length 2 since they are adjacent to two nodes. This unified view logically allows edges to be adjacent to any edges and not just to nodes, thus generalizing the `ev-edges` of E-graphs, and even to edges that are adjacent to themselves. Finally, there is no reason to restrict the length of edges to 0 or 2, and we will find good reasons (in Section 6) for allowing edges of infinite, ordinal length. The necessary notions and notations are introduced in Section 2. The structure of *monograph* (together with morphisms) is defined in Section 3, yielding a bestiary of categories of monographs according to some of their characteristics. The properties of these categories w.r.t. the existence of limits and co-limits are analyzed in Section 4.

We then see in Section 5 how monographs can be accurately represented by drawings, provided of course that they have finitely many edges and that these have finite length. In particular, such drawings correspond to the standard way of drawing a graph for those monographs that can be identified with standard graphs, and similarly for E-graphs.

Section 6 is devoted to the comparison between monographs and graph structures, and the corresponding algebras (that we may call *graph structured algebras*). We show a property of universality of monographs, in the sense that all graph structured algebras can be represented (though usually not in a canonical way) as *typed monographs*, i.e., as morphisms of monographs.

The notion of graph structure has been introduced in [3] in order to obtain categories of partial homomorphisms in which techniques of algebraic graph rewriting could be carried out. The correspondence with monographs established in Section 6 calls for a similar development of partial morphisms of monographs in Section 7. The single and double pushout methods of rewriting monographs can then be defined, analyzed and compared in Section 8.

The notion of E-graph has been introduced in [2] in order to obtain well-behaved categories (w.r.t. graph rewriting) of *attributed graphs*, and hence to propose suitable representations of real-life data structures. This is achieved by enriching E-graphs with a data type algebra, and by identifying nodes of sort `value` with the elements of this algebra. We pursue a similar approach in Section 9 with the notion of *attributed typed monograph* by identifying elements of an algebra with edges, and obtain similarly well-behaved categories. Due to the universality of monographs we see that any Σ-algebra can be represented as an attributed typed monograph.

We conclude in Section 10. Note that parts of Sections 4 to 6 have been published in [4].

2 Basic Definitions and Notations

2.1 Sets

For any sets A, B, relation $R \subseteq A \times B$ and subset $X \subseteq A$, let $R[X] \stackrel{\text{def}}{=} \{y \in B \mid x \in X \wedge (x,y) \in R\}$. For any $x \in A$, by abuse of notation we write $R[x]$ for $R[\{x\}]$. If R is functional we write $R(x)$ for the unique element of $R[x]$, and if $S \subseteq C \times D$ is also functional and $R[A] \subseteq C$ let $S \circ R \stackrel{\text{def}}{=} \{(x, S(R(x))) \mid x \in A\}$.

A *function* $f : A \to B$ is a triple (A, R, B) where $R \subseteq A \times B$ is a functional relation. We write $f[X]$ and $f(x)$ for $R[X]$ and $R(x)$ respectively. For any $Y \supseteq f[X]$, let $f|_X^Y \stackrel{\text{def}}{=} (X, R \cap (X \times Y), Y)$ and $f|_X \stackrel{\text{def}}{=} f|_X^B$. If $A \subseteq B$ then $(A, \{(x,x) \mid x \in A\}, B)$ is an *inclusion function*. A function $g = (C, S, D)$ may be composed on the left with f if $B = C$, and then $g \circ f \stackrel{\text{def}}{=} (A, S \circ R, D)$. If $R[A] \subseteq C$ we may write $g \circ R$ or $S \circ f$ for $S \circ R$.

Sets and functions form the category **Set** with identities $\mathrm{Id}_A \stackrel{\text{def}}{=} (A, \{(x,x) \mid x \in A\}, A)$. In **Set** we use the standard product $(A \times B, \pi_1, \pi_2)$ and coproduct $(A + B, \mu_1, \mu_2)$ of pairs of sets (A, B). The elements $p \in A \times B$ are pairs of elements of A and B, i.e., $p = (\pi_1(p), \pi_2(p))$. For functions $f : C \to A$ and $g : C \to B$ we write $\langle f, g\rangle : C \to A \times B$ for the unique function such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$, i.e., $\langle f, g\rangle(z) \stackrel{\text{def}}{=} (f(z), g(z))$ for all $z \in C$. The elements of $A + B$ are pairs $\mu_1(x) \stackrel{\text{def}}{=} (x, 0)$ or $\mu_2(y) \stackrel{\text{def}}{=} (y, 1)$ for all $x \in A$ and $y \in B$, so that $A' \subseteq A$ and $B' \subseteq B$ entail $A' + B' = \mu_1[A'] \cup \mu_2[B']$.

An *ordinal* is a set α such that every element of α is a subset of α, and such that the restriction of the membership relation $\in$ to α is a strict well-ordering of α (a total order where every non-empty subset of α has a minimal element). Every member of an ordinal is an ordinal, and we write $\lambda < \alpha$ for $\lambda \in \alpha$. For any two ordinals α, β we have either $\alpha < \beta$, $\alpha = \beta$ or $\alpha > \beta$ (see e.g. [5]). Every ordinal α has a successor $\alpha \cup \{\alpha\}$, denoted $\alpha + 1$. Natural numbers n are identified with finite ordinals, so that $n = \{0, 1, \ldots, n-1\}$ and $\omega \stackrel{\text{def}}{=} \{0, 1, \ldots\}$ is the smallest infinite ordinal.

2.2 Sequences

For any set E and ordinal λ, an *E-sequence s of length λ* is an element of E^λ, i.e., a function $s : \lambda \to E$. Let ε be the only element of E^0 (thus leaving E implicit), and for any $e \in E$ let $e{\uparrow}\lambda$ be the only element of $\{e\}^\lambda$. For any $s \in E^\lambda$ and $\iota < \lambda$, the image of ι by s is written s_ι. If λ is finite and non-zero then s can be described as $s = s_0 \cdots s_{\lambda-1}$. For any ordinal α, let $E^{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\lambda<\alpha} E^\lambda$; this is a disjoint union. For any $s \in E^{<\alpha}$ let $|s|$ be the length of s, i.e., the unique $\lambda < \alpha$ such that $s \in E^\lambda$.

We will often use the following notational convention: for any $x \in E$ and $s \in E^\lambda$, we write $x \mid s$ (and say that *x occurs in s*) if there exists $\iota < \lambda$ such that $s_\iota = x$.

For any set F and function $f : E \to F$, let $f^{<\alpha} : E^{<\alpha} \to F^{<\alpha}$ be the function defined by $f^{<\alpha}(s) \stackrel{\text{def}}{=} f \circ s$ for all $s \in E^{<\alpha}$. We have $\mathrm{Id}_E^{<\alpha} = \mathrm{Id}_{E^{<\alpha}}$ and $(g \circ f)^{<\alpha} = g^{<\alpha} \circ f^{<\alpha}$ for all $g : F \to G$. Since $s \in E^\lambda$ entails $f \circ s \in F^\lambda$, then $|f^{<\alpha}(s)| = |s|$.

If s and s' are respectively E- and F-sequences of length λ, then they are both functions with domain λ hence there is a function $\langle s, s'\rangle$ of domain λ. Thus $\langle s, s'\rangle$ is an $(E \times F)$-sequence of length λ, and then $\pi_1^{<\alpha}(\langle s, s'\rangle) = \pi_1 \circ \langle s, s'\rangle = s$ and similarly $\pi_2^{<\alpha}(\langle s, s'\rangle) = s'$ for all $\alpha > \lambda$. If $f : E \to F$ and $g : E \to G$ then $\langle f, g\rangle : E \to F \times G$, hence for all $s \in E^{<\alpha}$ of length $\lambda < \alpha$ we have $\langle f, g\rangle^{<\alpha}(s) = \langle f, g\rangle \circ s = \langle f \circ s, g \circ s\rangle = \langle f^{<\alpha}(s), g^{<\alpha}(s)\rangle$ is an $(F \times G)$-sequence of length λ.

For $s \in E^{<\omega}$ and $(A_e)_{e\in E}$ an E-indexed family of sets, let $A_s \stackrel{\text{def}}{=} \prod_{\iota<|s|} A_{s_\iota}$. In particular we take $A_\varepsilon \stackrel{\text{def}}{=} 1$ as a terminal object in **Set**. For $(B_e)_{e\in E}$ an E-indexed family of sets and $(f_e : A_e \to B_e)_{e\in E}$ an E-indexed family of functions, let $f_s \stackrel{\text{def}}{=} \prod_{\iota<|s|} f_{s_\iota} : A_s \to B_s$.

2.3 Signatures and Algebras

A *signature* is a function[1] $\Sigma : \Omega \to S^{<\omega}$, such that $\Sigma(o) \neq \varepsilon$ for all $o \in \Omega$. The elements of Ω are called *operator names* and those of S *sorts*. The *arity* of an operator name $o \in \Omega$ is the finite ordinal $n \stackrel{\text{def}}{=} |\Sigma(o)| - 1$, its *range* is $\mathrm{Rng}(o) \stackrel{\text{def}}{=} \Sigma(o)_n$ (the last element of the S-sequence $\Sigma(o)$) and its *domain* is $\mathrm{Dom}(o) \stackrel{\text{def}}{=} \Sigma(o)|_n$ (the rest of the sequence). o is *monadic* if $n = 1$. The signature Σ is *finite* if Ω and S are finite, it is a *graph structure* if all its operator names are monadic.

A *Σ-algebra* $\mathcal{A}$ is a pair $((\mathcal{A}_s)_{s\in S}, (o^{\mathcal{A}})_{o\in\Omega})$ where $(\mathcal{A}_s)_{s\in S}$ is an S-indexed family of sets and $(o^{\mathcal{A}} : \mathcal{A}_{\mathrm{Dom}(o)} \to \mathcal{A}_{\mathrm{Rng}(o)})_{o\in\Omega}$ is an Ω-indexed family of functions. A *Σ-homomorphism* h from $\mathcal{A}$ to a Σ-algebra $\mathcal{B}$ is an S-indexed family of functions $(h_s : \mathcal{A}_s \to \mathcal{B}_s)_{s\in S}$ such that

$$o^{\mathcal{B}} \circ h_{\mathrm{Dom}(o)} = h_{\mathrm{Rng}(o)} \circ o^{\mathcal{A}}$$

for all $o \in \Omega$. Let $1_{\mathcal{A}} \stackrel{\text{def}}{=} (\mathrm{Id}_{\mathcal{A}_s})_{s\in S}$ and for any Σ-homomorphism $k : \mathcal{B} \to \mathcal{C}$, the Σ-homomorphism $k \circ h : \mathcal{A} \to \mathcal{C}$ is defined by $(k \circ h)_s \stackrel{\text{def}}{=} k_s \circ h_s$ for all $s \in S$. Let Σ-**Alg** be the category of Σ-algebras and Σ-homomorphisms.

2.4 Categories

We assume familiarity with the notions of functors, limits, colimits and their preservation and reflection by functors, see [7]. Isomorphism between objects in a category is denoted by $\simeq$ and equivalence between categories by $\approx$.

For any object T of a category $\boldsymbol{C}$, the *slice category* $\boldsymbol{C}\backslash T$ has as objects the morphisms of codomain T of $\boldsymbol{C}$, as morphisms from object $a : A \to T$ to object $b : B \to T$ the morphisms $f : A \to B$ of $\boldsymbol{C}$ such that $b \circ f = a$, and the composition of morphisms in $\boldsymbol{C}\backslash T$ is defined as the composition of the underlying morphisms in $\boldsymbol{C}$ (see [2] or [7, Definition 4.19]).

3 Monographs and their Morphisms

Definition 3.1 (monographs, edges, ordinal for A)**.** A set A is a *monograph* if there exists a set E (whose elements are called *edges* of A) and an ordinal α (said to be an ordinal *for* A) such that $(E, A, E^{<\alpha})$ is a function.

A monograph is therefore a functional relation, which means that its set of edges is uniquely determined. On the contrary, there are always infinitely many ordinals for a monograph. As running example we consider the monograph $A = \{(x, x\,y\,x), (y, y\,x\,y)\}$

[1]For the sake of simplicity we do not allow the overloading of operator names as in [6]. These names will turn out to be irrelevant anyway.

then its set of edges is $E = \{x, y\}$. Since $A(x)$ and $A(y)$ are elements of $E^3 \subseteq E^{<4}$, then $(E, A, E^{<4})$ is a function. Hence 4 is an ordinal for A, and so are all the ordinals greater than 4.

It is easy to see that for any set of monographs there exists a common ordinal for all its members.

Definition 3.2 (length $|x|$, edge x_ι, trace $\mathrm{tr}(A)$, O-monographs)**.** For any monograph A with set of edges E, the *length* of an edge $x \in E$ is the length $|A(x)|$, also written $|x|$ if there is no ambiguity. Similarly, for any $\iota < |x|$ we may write x_ι for $A(x)_\iota$. The *trace of* A is the set $\mathrm{tr}(A) \stackrel{\mathrm{def}}{=} \{|x| \mid x \in E\}$. For any set O of ordinals, A is an *O-monograph* if $\mathrm{tr}(A) \subseteq O$.

Since any ordinal is a set of ordinals, we see that an ordinal α is for a monograph iff this is an α-monograph. Hence all edges of a monograph have finite length iff it is an ω-monograph.

Definition 3.3 (adjacency, nodes N_A, standard monographs)**.** For any monograph A and edges x, y of A, x is *adjacent to* y if $y \mid A(x)$ (see Section 2.2). A *node* is an edge of length 0, and the set of nodes of A is written N_A. A is *standard* if $y \mid A(x)$ entails $y \in \mathrm{N}_A$, i.e., all edges are sequences of nodes.

The running example A has no nodes and is therefore not standard. Since $A(x) = x\,y\,x$ then x is adjacent to y and to itself. Similarly, $A(y) = y\,x\,y$ yields that y is adjacent to x and to itself. In this case the adjacency relation is symmetric, but this is not generally the case, e.g., a node is never adjacent to any edge, while edges may be adjacent to nodes.

Definition 3.4 (morphisms of monographs)**.** A *morphism* f from monograph A to monograph B with respective sets of edges E and F, denoted $f : A \to B$, is a function $f : E \to F$ such that $f^{<\alpha} \circ A = B \circ f$, where α is any ordinal for A. If $E \subseteq F$ and f is the inclusion function then it is an *inclusion morphism*; these morphisms will be depicted as hooked arrows $f : A \hookrightarrow B$.

Building on the running example, we consider the permutation $f = (x\ y)$ of E (in cycle notation), we see that $f^{<4} \circ A(x) = f^{<4}(x\,y\,x) = y\,x\,y = A(y) = A \circ f(x)$ and similarly that $f^{<4} \circ A(y) = f^{<4}(y\,x\,y) = x\,y\,x = A(x) = A \circ f(y)$, hence $f^{<4} \circ A = A \circ f$ and f is therefore a morphism from A to A. Since $f \circ f = \mathrm{Id}_E$ is obviously the identity morphism 1_A then f is an isomorphism.

Note that the terms of the equation $f^{<\alpha} \circ A = B \circ f$ are functional relations and not functions. One essential feature is that this equation holds for all ordinals α for A iff it holds for one. Thus if we are given a morphism then we know that the equation holds *for all* big enough α's, and if we want to prove that a function is a morphism then we need only prove that *there exists* a big enough α such that the equation holds.

This equation is of course equivalent to $f^{<\alpha} \circ A(x) = B \circ f(x)$ for all $x \in E$. The terms of this last equation are F-sequences that should therefore have the same length:

$$|x| = |A(x)| = |f^{<\alpha} \circ A(x)| = |B \circ f(x)| = |f(x)|,$$

i.e., the length of edges are preserved by morphisms. Hence $\mathrm{tr}(A) \subseteq \mathrm{tr}(B)$, and the equality holds if f is surjective. This means that if B is an O-monograph then so is A, and that every ordinal for B is an ordinal for A. This also means that the images of nodes can only be nodes:

$$f^{-1}[\mathrm{N}_B] = \{x \in E \mid |f(x)| = 0\} = \{x \in E \mid |x| = 0\} = \mathrm{N}_A.$$

The sequences $f^{<\alpha} \circ A(x)$ and $B \circ f(x)$ should also have the same elements

$$(f^{<\alpha} \circ A(x))_\iota = (f \circ (A(x)))_\iota = f(A(x)_\iota) = f(x_\iota)$$
$$\text{and } (B \circ f(x))_\iota = B(f(x))_\iota = f(x)_\iota$$

for all $\iota < |x|$. Thus $f : E \to F$ is a morphism iff

$$|f(x)| = |x| \text{ and } f(x_\iota) = f(x)_\iota \text{ for all } x \in E \text{ and all } \iota < |x|.$$

Assuming that $f : A \to B$ is a morphism and that B is standard, we have $f(x_\iota) = f(x)_\iota \in \mathrm{N}_B$ thus $x_\iota \in f^{-1}[\mathrm{N}_B] = \mathrm{N}_A$ for all $x \in E$ and $\iota < |x|$, hence A is also standard.

Given morphisms $f : A \to B$ and $g : B \to C$, we see that $g \circ f$ is a morphism from A to C by letting α be an ordinal for B, so that

$$(g \circ f)^{<\alpha} \circ A = g^{<\alpha} \circ f^{<\alpha} \circ A = g^{<\alpha} \circ B \circ f = C \circ g \circ f.$$

Definition 3.5 (categories of monographs, functor E)**.** Let **Monogr** be the category of monographs and their morphisms. Let **SMonogr** be its full subcategory of standard monographs. For any set O of ordinals, let O-**Monogr** (resp. O-**SMonogr**) be the full subcategory of O-monographs (resp. standard O-monographs). Let **FMonogr** be the full subcategory of finite ω-monographs (i.e. monographs with finitely many edges that all have finite length).

Let E be the forgetful functor from **Monogr** to **Set**, i.e., for every monograph A let $\mathsf{E}A$ be the set of edges of A, and for every morphism $f : A \to B$ let $\mathsf{E}f : \mathsf{E}A \to \mathsf{E}B$ be the underlying function, usually denoted f.

There is an obvious similitude between standard $\{0,2\}$-monographs and graphs. It is actually easy to define a functor $\mathsf{M} : \mathbf{Graph} \to \{0,2\}\text{-}\mathbf{SMonogr}$ by mapping any graph $G = (N, E, s, t)$ to the monograph $\mathsf{M}G$ whose set of edges is the coproduct $N + E$, and that maps every edge $e \in E$ to the sequence of nodes $s(e)\,t(e)$ (and of course every node $x \in N$ to ε). Similarly graph morphisms are transformed into morphisms of monographs through a coproduct of functions. It is easy to see that M is an equivalence of categories.

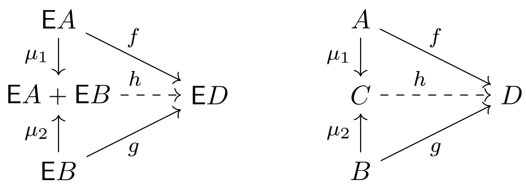
It is customary in Algebraic Graph Transformation to call *typed graphs* the objects of $\mathbf{Graph}\backslash G$, where G is a graph called *type graph*, see e.g. [2]. We will extend this terminology to monographs and refer to the objects of $\mathbf{Monogr}\backslash T$ as the *monographs typed by T* and T as a *type monograph*.

4 Limits and Colimits

The colimits of monographs follow the standard constructions of colimits in **Set** and **Graph**.

Lemma 4.1. *Every pair (A, B) of monographs has a coproduct $(A + B, \mu_1, \mu_2)$ such that $\mathrm{tr}(A + B) = \mathrm{tr}(A) \cup \mathrm{tr}(B)$ and if A and B are finite (resp. standard) then so is $A + B$.*

Proof. Let α be an ordinal for A and B, and $(\mathsf{E}A + \mathsf{E}B, \mu_1, \mu_2)$ be the coproduct of $(\mathsf{E}A, \mathsf{E}B)$ in **Set**. Since every element of $\mathsf{E}A + \mathsf{E}B$ is either a $\mu_1(x)$ or a $\mu_2(y)$ for some $x \in \mathsf{E}A$, $y \in \mathsf{E}B$, we can define a monograph C by taking $\mathsf{E}C \overset{\text{def}}{=} \mathsf{E}A + \mathsf{E}B$ with $C(\mu_1(x)) \overset{\text{def}}{=} \mu_1^{<\alpha} \circ A(x)$ and $C(\mu_2(y)) \overset{\text{def}}{=} \mu_2^{<\alpha} \circ B(y)$ for all $x \in \mathsf{E}A$, $y \in \mathsf{E}B$, so that $\mu_1 : A \to C$ and $\mu_2 : B \to C$ are morphisms. It is obvious that $\mathrm{tr}(C) = \mathrm{tr}(A) \cup \mathrm{tr}(B)$ and if A and B are finite (resp. standard) then so is C.



Let $f : A \to D$ and $g : B \to D$, there exists a unique function h from $\mathsf{E}A + \mathsf{E}B = \mathsf{E}C$ to $\mathsf{E}D$ such that $f = h \circ \mu_1$ and $g = h \circ \mu_2$, hence

$$h^{<\alpha} \circ C(\mu_1(x)) = (h \circ \mu_1)^{<\alpha} \circ A(x) = f^{<\alpha} \circ A(x) = D \circ f(x) = D \circ h(\mu_1(x))$$

for all $x \in \mathsf{E}A$, and similarly $h^{<\alpha} \circ C(\mu_2(y)) = D \circ h(\mu_2(y))$ for all $y \in \mathsf{E}B$, hence $h^{<\alpha} \circ C = D \circ h$, i.e., $h : C \to D$ is a morphism. □

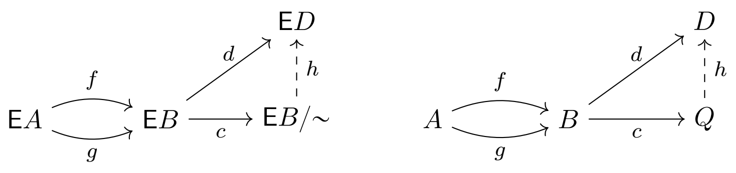
Lemma 4.2. *Every pair of parallel morphisms $f, g : A \to B$ has a coequalizer (Q, c) such that $\mathrm{tr}(Q) = \mathrm{tr}(B)$ and if B is finite (resp. standard) then so is Q.*

Proof. Let α be an ordinal for B and $\sim$ be the smallest equivalence relation on $\mathsf{E}B$ that contains $R = \{(f(x), g(x)) \mid x \in \mathsf{E}A\}$ and $c : \mathsf{E}B \to \mathsf{E}B/{\sim}$ be the canonical surjection, so that $c \circ f = c \circ g$. We thus have for all $x \in \mathsf{E}A$ that

$$c^{<\alpha} \circ B \circ f(x) = (c \circ f)^{<\alpha} \circ A(x) = (c \circ g)^{<\alpha} \circ A(x) = c^{<\alpha} \circ B \circ g(x).$$

For all $y, y' \in \mathsf{E}B$ such that $c(y) = c(y')$, i.e., $y \sim y'$, since $\sim$ is the symmetric and transitive closure of R then there exists a finite sequence $y_0, \ldots, y_n$ of elements of $\mathsf{E}B$ such that $y_0 = y$, $y_n = y'$ and $y_i \; R \; y_{i+1}$ or $y_{i+1} \; R \; y_i$ for all $0 \leqslant i < n$, hence $c^{<\alpha} \circ B(y_i) = c^{<\alpha} \circ B(y_{i+1})$, and therefore $c^{<\alpha} \circ B(y) = c^{<\alpha} \circ B(y')$.

We can now define a monograph Q by taking $\mathsf{E}Q = \mathsf{E}B/{\sim}$ with $Q(c(y)) \overset{\text{def}}{=} c^{<\alpha} \circ B(y)$, so that $c : B \to Q$ is a morphism. Since c is surjective then $\mathrm{tr}(Q) = \mathrm{tr}(B)$ and if B is finite (resp. standard) then so is Q.



Let $d : B \to D$ such that $d \circ f = d \circ g$. Since c is a coequalizer of f, g in **Set** (see [7, Examples 16.3 (2)]) then there exists a unique function h from $\mathsf{E}Q$ to $\mathsf{E}D$ such that $d = h \circ c$, and $h : Q \to D$ is a morphism since for all $y \in \mathsf{E}B$,

$$D \circ h(c(y)) = D \circ d(y) = d^{<\alpha} \circ B(y) = h^{<\alpha} \circ c^{<\alpha} \circ B(y) = h^{<\alpha} \circ Q(c(y)).$$

□

Corollary 4.3. *The epimorphisms in* **Monogr** *are the surjective morphisms.*

Proof. Assume $f : A \to B$ is an epimorphism. Let $(B + B, \mu_1, \mu_2)$ be a coproduct of (B, B) and (Q, c) be the coequalizer of $\mu_1 \circ f, \mu_2 \circ f : A \to B + B$ constructed in the proof of Lemma 4.2, then $c \circ \mu_1 \circ f = c \circ \mu_2 \circ f$, hence $c \circ \mu_1 = c \circ \mu_2$. For all $y \in \mathsf{E}B$ we thus have $\mu_1(y) \sim \mu_2(y)$, and since $\mu_1(y) \neq \mu_2(y)$ then $\mu_1(y)$ must be related by R to some element of $\mathsf{E}(B + B)$, hence there is an $x \in \mathsf{E}A$ such that $\mu_1(y) = \mu_1 \circ f(x)$, thus $y = f(x)$ since μ_1 is injective; this proves that f is surjective. The converse is obvious. □

A well-known consequence of Lemmas 4.1, 4.2 and that $\varnothing$ is the initial monograph is that all finite diagrams have colimits.

Theorem 4.4. *The categories of Definition 3.5 are finitely co-complete.*

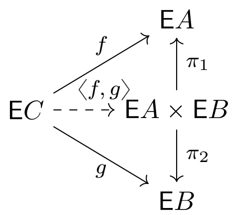
We next investigate the limits in categories of monographs. Products of monographs are more difficult to build than products of graphs. This is due to the fact that edges of identical length may be adjacent to edges of different lengths.

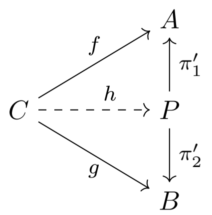
Lemma 4.5. *Every pair* (A, B) *of monographs has a product* $(A \times B, \pi'_1, \pi'_2)$ *such that* $A \times B$ *is finite whenever* A *and* B *are finite.*

Proof. Let α be an ordinal for A and B, let $(\mathsf{E}A \times \mathsf{E}B, \pi_1, \pi_2)$ be the product of $(\mathsf{E}A, \mathsf{E}B)$ in **Set**, we consider the set of subsets H of $\{(x, y) \in \mathsf{E}A \times \mathsf{E}B \mid |x| = |y|\}$ such that $(x, y) \in H$ entails $(x_\iota, y_\iota) \in H$ for all $\iota < |x|$. This set contains $\varnothing$ and is closed under union, hence it has a greatest element $\mathsf{E}P$, and we let $P(x, y) \stackrel{\text{def}}{=} \langle A(x), B(y)\rangle$ for all $(x, y) \in \mathsf{E}P$; this is obviously an $\mathsf{E}P$-sequence, hence P is a monograph. Let $\pi'_1 \stackrel{\text{def}}{=} \pi_1|_{\mathsf{E}P}$ and $\pi'_2 \stackrel{\text{def}}{=} \pi_2|_{\mathsf{E}P}$, we have

$$\pi_1'^{<\alpha} \circ P(x, y) = A(x) = A \circ \pi'_1(x, y)$$

for all $(x, y) \in \mathsf{E}P$, hence $\pi'_1 : P \to A$ and similarly $\pi'_2 : P \to B$ are morphisms.





Let $f : C \to A$ and $g : C \to B$, then $\langle f, g\rangle : \mathsf{E}C \to \mathsf{E}A \times \mathsf{E}B$ and for all $z \in \mathsf{E}C$ we have $|f(z)| = |z| = |g(z)|$ hence $\langle f, g\rangle[\mathsf{E}C] \subseteq \{(x, y) \in \mathsf{E}A \times \mathsf{E}B \mid |x| = |y|\}$. Assume that $(x, y) \in \langle f, g\rangle[\mathsf{E}C]$, then there exists a $z \in \mathsf{E}C$ such that $x = f(z)$ and $y = g(z)$, hence $|x| = |y|$, $f(z_\iota) = f(z)_\iota = x_i$ and $g(z_\iota) = g(z)_\iota = y_\iota$ for all $\iota < |x|$, hence $(x_\iota, y_\iota) \in \langle f, g\rangle[\mathsf{E}C]$. Thus $\langle f, g\rangle[\mathsf{E}C] \subseteq \mathsf{E}P$ and we let $h \stackrel{\text{def}}{=} \langle f, g\rangle|^{\mathsf{E}P}_{\mathsf{E}C}$, then h is the unique function such that $\pi'_1 \circ h = f$ and $\pi'_2 \circ h = g$, and $h : C \to P$ is a morphism since for all $z \in \mathsf{E}C$,

$$P \circ h(z) = P(f(z), g(z)) = \langle A \circ f(z), B \circ g(z)\rangle = \langle f^{<\alpha} \circ C(z), g^{<\alpha} \circ C(z)\rangle = h^{<\alpha} \circ C(z).$$

□

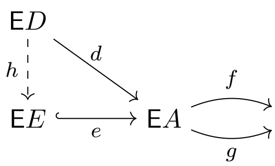
We therefore see that $\mathsf{E}(A \times B)$ is only a subset of $\mathsf{E}A \times \mathsf{E}B$.

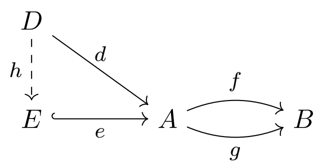
Lemma 4.6. *Every pair of parallel morphisms* $f, g : A \to B$ *has an equalizer* (E, e) *such that* E *is finite whenever* A *is finite.*

Proof. Let α be an ordinal for A, $\mathsf{E}E \stackrel{\text{def}}{=} \{x \in \mathsf{E}A \mid f(x) = g(x)\}$, $e : \mathsf{E}E \hookrightarrow \mathsf{E}A$ be the inclusion function and $E(x) \stackrel{\text{def}}{=} A(x)$ for all $x \in \mathsf{E}E$. Since

$$f^{<\alpha} \circ A(x) = B \circ f(x) = B \circ g(x) = g^{<\alpha} \circ A(x)$$

then $E(x)$ is an $\mathsf{E}E$-sequence, hence E is a monograph. Besides $e^{<\alpha} \circ E(x) = A(x) = A \circ e(x)$, hence $e : E \to A$ is a morphism such that $f \circ e = g \circ e$.





For any $d : D \to A$ such that $f \circ d = g \circ d$, we have $d(y) \in \mathsf{E}E$ for all $y \in \mathsf{E}D$, hence $h \stackrel{\text{def}}{=} d|_{\mathsf{E}D}^{\mathsf{E}E}$ is the unique function such that $d = e \circ h$. We have

$$e^{<\alpha} \circ h^{<\alpha} \circ D = d^{<\alpha} \circ D = A \circ d = A \circ e \circ h = e^{<\alpha} \circ E \circ h$$

and $e^{<\alpha} : (\mathsf{E}E)^{<\alpha} \hookrightarrow (\mathsf{E}A)^{<\alpha}$ is the inclusion function, hence $h^{<\alpha} \circ D = E \circ h$ and $h : D \to E$ is a morphism. □

Corollary 4.7. *The monomorphisms in* **Monogr** *are the injective morphisms.*

Proof. Assume $f : A \to B$ is a monomorphism. Let $(A \times A, \pi_1, \pi_2)$ be a product of (A, A) and (E, e) be the equalizer of $f \circ \pi_1, f \circ \pi_2 : A \times A \to B$ constructed in the proof of Lemma 4.6, then $f \circ \pi_1 \circ e = f \circ \pi_2 \circ e$, hence $\pi_1 \circ e = \pi_2 \circ e$. For all $x, y \in \mathsf{E}A$, if $f(x) = f(y)$ then $f \circ \pi_1(x, y) = f \circ \pi_2(x, y)$ hence $(x, y) \in \mathsf{E}E$ and therefore $x = \pi_1 \circ e(x, y) = \pi_2 \circ e(x, y) = y$, hence f is injective. The converse is obvious. □

A well-known consequence of Lemmas 4.5 and 4.6 is that all non-empty finite diagrams in **Monogr** have limits. Since a limit of O-monographs (resp. standard monographs) is an O-monograph (resp. standard), this holds for all categories of Definition 3.5. In particular they all have pullbacks.

We shall now investigate the limits of the empty diagram in these categories, i.e., their possible terminal objects.

Definition 4.8. For any set of ordinals O, let

$$\mathrm{T}_O = \begin{cases} \{(\lambda, 0{\uparrow}\lambda) \mid \lambda \in O\} & \text{if } 0 \in O \\ \varnothing & \text{otherwise.} \end{cases}$$

If $0 \in O$ then 0 is a node of T_O and obviously $\mathsf{E}\mathrm{T}_O = \mathrm{tr}(\mathrm{T}_O) = O$. Hence in all cases T_O is a standard O-monograph.

Lemma 4.9. T_O *is terminal in O-***SMonogr***.*

Proof. If $0 \notin O$ then $\varnothing = \mathrm{T}_O$ is the only standard O-monograph, hence it is terminal. Otherwise let A be any standard O-monograph, α an ordinal for A and $\ell : \mathsf{E}A \to O$ be the function that maps every edge $x \in \mathsf{E}A$ to its length $|x|$. Since A is standard then $(\ell^{<\alpha} \circ A(x))_\iota = |A(x)_\iota| = 0$ for all $\iota < |x|$, hence $\ell^{<\alpha} \circ A(x) = 0{\uparrow}|x| = \mathrm{T}_O \circ \ell(x)$, so that $\ell : A \to \mathrm{T}_O$ is a morphism. Since morphisms preserve the length of edges and there is exactly one edge of each length in T_O, then ℓ is unique. □

We now use the fact that every ordinal is a set of ordinals.

Lemma 4.10. *For any monograph T and morphism $f : \mathrm{T}_\alpha \to T$, any ordinal for T is equal to or greater than α.*

Proof. Let β be an ordinal for T, then by the existence of f we have $\alpha = \mathrm{tr}(\mathrm{T}_\alpha) \subseteq \mathrm{tr}(T) \subseteq \beta$, hence $\alpha \leqslant \beta$. □

Lemma 4.11. **Monogr***,* **SMonogr** *and* **FMonogr** *have no terminal object.*

Proof. Suppose that T is a terminal monograph, then there is an ordinal β for T and there is a morphism from $\mathrm{T}_{\beta+1}$ to T; by Lemma 4.10 this implies that $\beta + 1 \leqslant \beta$, a contradiction. This still holds if T is standard since $\mathrm{T}_{\beta+1}$ is standard. And it also holds if T is a finite ω-monograph, since then β can be chosen finite, and then $\mathrm{T}_{\beta+1}$ is also a finite ω-monograph. □

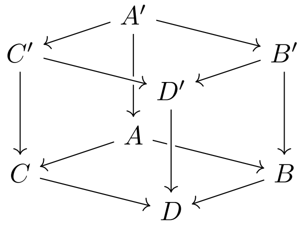
Since terminal objects are limits of empty diagrams obviously these categories are not finitely complete.

Theorem 4.12. *O-***SMonogr** *is finitely complete for every set of ordinals O. The categories* **Monogr***,* **SMonogr** *and* **FMonogr** *are not finitely complete.*

Proof. By Lemmas 4.5, 4.6, 4.9 and 4.11. □

The category **Graph** is also known to be adhesive, a property of pushouts and pullbacks that has important consequences on algebraic transformations (see [8]) and that we shall therefore investigate.

Definition 4.13 (van Kampen squares, adhesive categories)**.** A pushout square (A, B, C, D) is a *van Kampen square* if for any commutative cube



where the back faces (A', A, B', B) and (A', A, C', C) are pullbacks, it is the case that the top face (A', B', C', D') is a pushout iff the front faces (B', B, D', D) and (C', C, D', D) are both pullbacks.

A category *has pushouts along monomorphisms* if all sources (A, f, g) have pushouts whenever f or g is a monomorphism.

A category is *adhesive* if it has pullbacks, pushouts along monomorphisms and all such pushouts are van Kampen squares.

As in the proof that **Graph** is adhesive, we will use the fact that the category **Set** is adhesive.

Lemma 4.14. E *reflects isomorphisms.*

Proof. Let $f : A \to B$ such that f is bijective, then it has an inverse $f^{-1} : \mathsf{E}B \to \mathsf{E}A$. For all $y \in \mathsf{E}B$ and all $\iota < |y|$, let $x = f^{-1}(y)$, we have

$$f^{-1}(y_\iota) = f^{-1}(f(x)_\iota) = f^{-1}(f(x_\iota)) = x_\iota = f^{-1}(y)_\iota$$

hence $f^{-1} : B \to A$ is a morphism, and f is therefore an isomorphism. □

A side consequence is that **Monogr** is balanced, i.e., if f is both a monomorphism and an epimorphism, then by Corollaries 4.3 and 4.7 f is bijective, hence is an isomorphism. More important is that we can use [7, Theorem 24.7], i.e., that a faithful and isomorphism reflecting functor from a category that has some limits or colimits and preserves them, also reflects them.

Lemma 4.15. E *preserves and reflects finite colimits.*

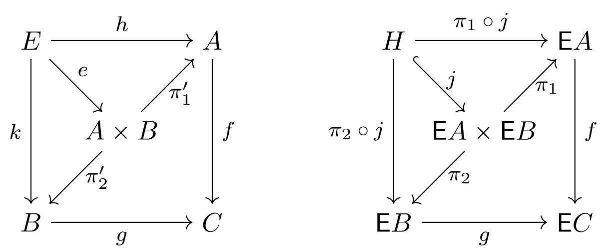
Proof. It is easy to see from the proofs of Lemmas 4.1 and 4.2 that E preserves both coproducts and coequalizers, so that E preserves all finite co-limits and hence also reflects them. □

This is particularly true for pushouts. The situation for pullbacks is more complicated since E does not preserve products.

Lemma 4.16. E *preserves and reflects pullbacks.*

Proof. We first prove that E preserves pullbacks. Let $f : A \to C$, $g : B \to C$ and α be an ordinal for A and B, we assume w.l.o.g. a canonical pullback (E, h, k) of (f, g, C), i.e., let $(A \times B, \pi'_1, \pi'_2)$ be the product of (A, B) and (E, e) be the equalizer of $(f \circ \pi'_1, g \circ \pi'_2)$ with $h = \pi'_1 \circ e$ and $k = \pi'_2 \circ e$. Let $(\mathsf{E}A \times \mathsf{E}B, \pi_1, \pi_2)$ be the product of $(\mathsf{E}A, \mathsf{E}B)$ in **Set**, we have by the proof of Lemma 4.5 that $\mathsf{E}(A \times B) \subseteq \mathsf{E}A \times \mathsf{E}B$, $\pi'_1 = \pi_1|_{\mathsf{E}(A\times B)}$ and $\pi'_2 = \pi_2|_{\mathsf{E}(A\times B)}$.

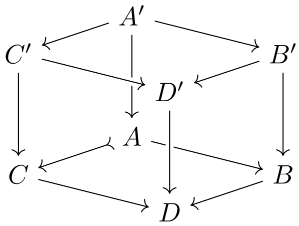
Let $H \stackrel{\text{def}}{=} \{(x, y) \in \mathsf{E}A \times \mathsf{E}B \mid f(x) = g(y)\}$ and $j : H \hookrightarrow \mathsf{E}A \times \mathsf{E}B$ be the inclusion function. By canonical construction $(H, \pi_1 \circ j, \pi_2 \circ j)$ is a pullback of $(f, g, \mathsf{E}C)$ in **Set**; we next prove that it is the image by E of the pullback (E, h, k) of (f, g, C) in **Monogr**.

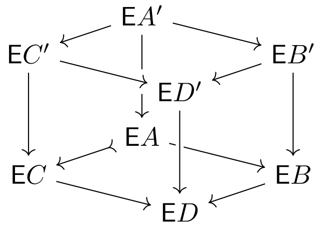


By the construction of E in Lemma 4.6 we have $\mathsf{E}E = \{(x, y) \in \mathsf{E}(A \times B) \mid f(x) = g(y)\} \subseteq H$ and $e : \mathsf{E}E \hookrightarrow \mathsf{E}(A \times B)$ is the inclusion function. For all $(x, y) \in H$ we have $|x| = |f(x)| = |g(y)| = |y|$, and for all $\iota < |x|$ we have $f(x_\iota) = f(x)_\iota = g(y)_\iota = g(y_\iota)$ so that $(x_\iota, y_\iota) \in H$ and therefore $H \subseteq \mathsf{E}(A \times B)$ by the construction of $A \times B$ in Lemma 4.5. We thus have $H = \mathsf{E}E$ hence $\pi_1 \circ j = \pi'_1 \circ e = h$ and $\pi_2 \circ j = \pi'_2 \circ e = k$, so that E preserves pullbacks and hence as above E also reflects them. □

Theorem 4.17. *The categories of Definition 3.5 are adhesive.*

Proof. The existence of pullbacks and pushouts is already established. Let (A, B, C, D) be a pushout square of monographs along a monomorphism and the commutative cube depicted below left,



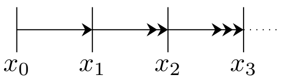


whose back faces are pullbacks. Its image by functor E is the commutative cube in **Set** depicted on the right. Its bottom face $(\mathsf{E}A, \mathsf{E}B, \mathsf{E}C, \mathsf{E}D)$ is a pushout along a monomorphism by Lemma 4.15 and Corollary 4.7, and its back faces are pullbacks by Lemma 4.16. If the top face (A', B', C', D') is a pushout then again so is $(\mathsf{E}A', \mathsf{E}B', \mathsf{E}C', \mathsf{E}D')$, and since **Set** is an adhesive category (see [8]) then the front faces of the right cube are pullbacks, hence so are the front faces of the left cube by Lemma 4.16. Conversely, if these front faces are pullbacks then by the same argument the top face of both cubes are pushouts. Hence (A, B, C, D) is a van Kampen square. □

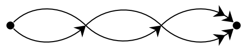
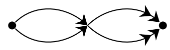
It is known that every topos is adhesive [9] but that some adhesive categories are not toposes [8, Example 3.8]. **Monogr**, **SMonogr** and **FMonogr** are new examples of adhesive categories that are not toposes (by Theorem 4.12).

5 Drawing Monographs

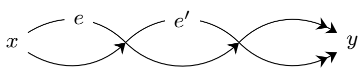
Obviously we may endeavour to draw a monograph A only if $\mathsf{E}A$ is finite and if its edges have finite lengths, i.e., if A is a finite ω-monograph. If we require that any monograph $\mathsf{M}G$ should be drawn as the graph G, then a node should be represented by a bullet • and an edge of length 2 by an arrow ⌒► joining its two adjacent nodes. But generally the adjacent edges may not be nodes and there might be more than 2 of them, hence we adopt the following convention: an edge e of length at least 2 is represented as a sequence of connected arrows with an increasing number of tips



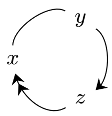
(where $A(e) = x_0x_1x_2x_3\cdots$) and such that any arrow should enter x_i at the same angle as the next arrow leaves x_i. For the sake of clarity we represent symmetric adjacencies by a pair of crossings rather than a single one, e.g., if $A(e) = xe'y$ and $A(e') = xey$, where x and y are nodes, the drawing may be

 but not 

It is sometimes necessary to name the edges in a drawing. We may then adopt the convention sometimes used for drawing diagrams in a category: the bullets are replaced by the names of the corresponding nodes, and arrows are interrupted to write their name at a place free from crossing, as in



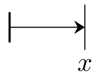
Note that no confusion is possible between the names of nodes and those of other edges, e.g., in



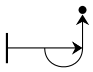
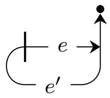
it is clear that x and z are nodes since arrow tips point to them, and that y is the name of an edge of length 3.

As is the case of graphs, monographs may not be planar and drawing them may require crossing edges that are not adjacent; in this case no arrow tip is present at the crossing and no confusion is possible with the adjacency crossings. However, it may seem preferable in such cases to erase one arrow in the proximity of the other, as in ╳.

There remains to represent the edges of length 1. Since $A(e) = x$ is standardly written $A : e \mapsto x$, the edge e will be drawn as

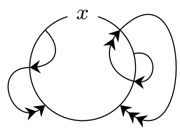


In order to avoid confusion there should be only one arrow out of the thick dash, e.g., if $A(e) = e'$ and $A(e') = ex$ where x is a node, the drawing may be

 but not 

since this last drawing may be interpreted as the monograph $A(e') = x$ and $A(e) = e'e'$, that is not isomorphic to the intended monograph.

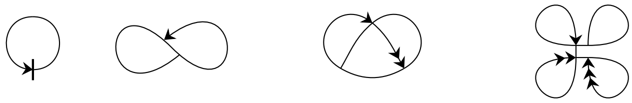
Other conventions may be more appropriate depending on the context or on specific monographs. Consider for instance a monograph with one node x and two edges $x{\uparrow}3$ and $x{\uparrow}4$. The concentration of many arrow tips on a single bullet would potentially confuse readers unless it is sufficiently large. One possibility is to replace the bullet by a circle and treat it as a standard edge without tips. This monograph could then be drawn as



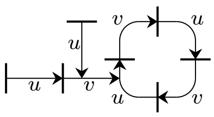
These conventions are designed so that it is only possible to read a drawing of any finite ω-monograph A as the monograph A itself if all edges are named in the drawing, or as some monograph isomorphic to A otherwise. This would not be possible if a monograph A was a function rather than a functional relation, since then its codomain $(\mathsf{E}A)^{<\alpha}$ would not be pictured. It would of course be possible to add the ordinal α to the drawing, but then would it still qualify as a drawing?

Note that the drawing of a graph or of a standard $\{0, 2\}$-monograph can be read either as a graph G or as a monograph A, and then $\mathsf{M}G \simeq A$.

One particularity of monographs is that edges can be adjacent to themselves, as in



We may also draw a typed monograph $a : A \to T$, where every edge $e \in \mathsf{E}A$ has a type $a(e)$ that can be written at the proximity of e. For instance, a monograph typed by $T = \{(u,\ v),\ (v,\ u)\}$ is drawn with labels u and v as in



Of course, knowing that a is a morphism sometimes allows to deduce the type of an edge, possibly from the types of adjacent edges. In the present case, indicating a single type would have been enough to deduce all the others.

In particular applications it may be convenient to adopt completely different ways of drawing (typed) monographs.

Example 5.1. In [10] term graphs are defined from structures (V, E, lab, att) where V is a set of *nodes*, E a set of *hyperedges*, $att : E \to V^{<\omega}$ defines the adjacencies and $lab : E \to \Omega$ such that $|att(e)|$ is 1 plus the arity of $lab(e)$ for all $e \in E$ (for the sake of simplicity, we consider only ground terms of a signature $\Sigma : \Omega \to S^{<\omega}$ such that $\Omega \cap S = \varnothing$). The first element of the sequence $att(e)$ is considered as the *result node* of e and the others as its *argument nodes*, so that e determines *paths* from its result node to all its argument nodes. *Term graphs* are those structures such that paths do not cycle, every node is reachable from a root node and is the result node of a unique hyperedge. This definition is given for unsorted signatures but can easily be generalized, as we do now.

We consider the type monograph T_Σ defined by $\mathsf{E}\mathrm{T}_\Sigma \stackrel{\text{def}}{=} S \cup \Omega$, and

$$\mathrm{T}_\Sigma(s) \stackrel{\text{def}}{=} \varepsilon \text{ for all } s \in S,$$
$$\mathrm{T}_\Sigma(o) \stackrel{\text{def}}{=} \Sigma(o) \text{ for all } o \in \Omega.$$

Note that T_Σ is a standard ω-monograph, and indeed that any standard ω-monograph has this form for a suitable Σ.

Any typed monograph $a : A \to \mathrm{T}_\Sigma$ corresponds to a structure (V, E, lab, att) where $V = \mathrm{N}_A$, $E = \mathsf{E}A \backslash \mathrm{N}_A$, $lab(e) = a(e)$ and $att(e) = A(e)$ for all $e \in E$. The only difference (due to our definition of signatures) is that the result node of e is now the last node of the sequence $A(e)$.

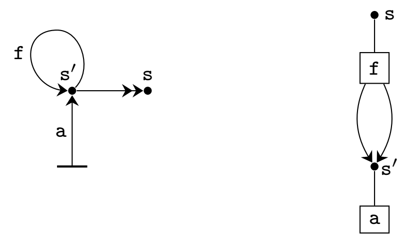
We now consider the signature Σ with two sorts $\mathtt{s}$, $\mathtt{s}'$, a binary function symbol $\mathtt{f}$ with $\Sigma(\mathtt{f}) = \mathtt{s}'\,\mathtt{s}'\,\mathtt{s}$ and a constant symbol $\mathtt{c}$ with $\Sigma(\mathtt{c}) = \mathtt{s}'$. We represent the term graph $\mathtt{f}(\mathtt{c}, \mathtt{c})$, where the two occurrences of $\mathtt{c}$ are shared, as a typed monograph $a : A \to \mathrm{T}_\Sigma$. We need two edges e, e' and their result nodes x, x', the first for $\mathtt{f}$ and the second for $\mathtt{c}$. Thus A is defined by

$$\mathsf{E}A = \{x, x', e, e'\},\ A(x) = A(x') = \varepsilon,\ A(e) = x'\,x'\,x \text{ and } A(e') = x'.$$

The typing morphism $a : A \to \mathrm{T}_\Sigma$ is given by

$$a(x) = \mathtt{s},\ a(x') = \mathtt{s}',\ a(e) = \mathtt{f} \text{ and } a(e') = \mathtt{c}.$$

We give below the standard drawing of the monograph A typed by a and the (clearly preferable) standard depiction of the corresponding term graph.



6 Graph Structures and Typed Monographs

The procedure of reading the drawing of a graph as a Γ_{g}-algebra $\mathcal{G}$, where Γ_{g} is the signature of graphs given in Section 1, is rather simple: every bullet is interpreted as an element of $\mathcal{G}_{\mathtt{nodes}}$, every arrow as an element of $\mathcal{G}_{\mathtt{edges}}$ and the images of this element by the functions $\mathtt{src}^{\mathcal{G}}$ and $\mathtt{tgt}^{\mathcal{G}}$ are defined according to geometric proximity in the drawing. A procedure for reading E-graphs would be similar, except that bullets may be interpreted either as $\mathtt{nodes}$ or $\mathtt{values}$, and this typing information should therefore be indicated in the drawing.

Since the drawing of a graph is nothing else than the drawing of a standard $\{0, 2\}$-monograph, we may skip the drawing step and directly transform a standard $\{0, 2\}$-monograph A as a Γ_{g}-algebra $\mathcal{G}$. Then

$$\mathcal{G}_{\mathtt{nodes}} = \mathrm{N}_A,\ \mathcal{G}_{\mathtt{edges}} = \{x \in \mathsf{E}A \mid |x| = 2\},\ \mathtt{src}^{\mathcal{G}}(x) = x_0 \text{ and } \mathtt{tgt}^{\mathcal{G}}(x) = x_1$$

for all $x \in \mathcal{G}_{\mathtt{edges}}$. Thus every node of A is typed by $\mathtt{nodes}$ and all other edges are typed by $\mathtt{edges}$. This typing is obviously a morphism from A to the monograph $\{(\mathtt{nodes}, \varepsilon), (\mathtt{edges}, \mathtt{nodes}\,\mathtt{nodes})\}$ that is isomorphic to the terminal object of $\{0, 2\}$-**SMonogr** (see Lemma 4.9).

More generally, for any given graph structure Γ (signatures of the form $\Gamma : \Omega \to S^2$, see Section 2.3) we may ask which monographs, equipped with a suitable morphism to a type monograph T, can be interpreted in this way as Γ-algebras. As above, the edges of T should be the sorts of Γ. But this is not sufficient since there is no canonical way of linking adjacencies in T (such as $\mathtt{edges}_0 = \mathtt{nodes}$ and $\mathtt{edges}_1 = \mathtt{nodes}$) with the operator names of Γ (such as $\mathtt{src}$ and $\mathtt{tgt}$). We will therefore use a notion of morphism between signatures in order to rename operators, and we also rename sorts in order to account for functoriality in T.

Definition 6.1 (categories **Sig**, **GrStruct**, $\mathbf{Sig}_{\mathrm{srt}}$)**.** A *morphism* r from $\Sigma : \Omega \to S^{<\omega}$ to $\Sigma' : \Omega' \to S'^{<\omega}$ is a pair $(r_{\mathrm{opn}}, r_{\mathrm{srt}})$ of functions $r_{\mathrm{opn}} : \Omega \to \Omega'$ and $r_{\mathrm{srt}} : S \to S'$ such that

$$r_{\mathrm{srt}}^{<\omega} \circ \Sigma = \Sigma' \circ r_{\mathrm{opn}}.$$

For any morphism $r' : \Sigma' \to \Sigma''$ let $r' \circ r \stackrel{\mathrm{def}}{=} (r'_{\mathrm{opn}} \circ r_{\mathrm{opn}}, r'_{\mathrm{srt}} \circ r_{\mathrm{srt}}) : \Sigma \to \Sigma''$, $1_\Sigma \stackrel{\mathrm{def}}{=} (\mathrm{Id}_\Omega, \mathrm{Id}_S)$, and **Sig** be the category of signatures and their morphisms. Let **GrStruct** be the full subcategory of graph structures.

Let $\mathbf{Sig}_{\mathrm{srt}}$ be the subcategory of **Sig** restricted to morphisms of the form (r_{opn}, j) where j is an inclusion function. We write $\doteq$ for the isomorphism relation between objects in $\mathbf{Sig}_{\mathrm{srt}}$.

Note that graph structures Γ can be interpreted as graphs with sorts as vertices and operator names as arrows (so that **GrStruct** is obviously equivalent to **Graph**).

The question is therefore to elucidate the link between T and Γ. As explained above, the edges of T correspond to the sorts of Γ (vertices in a graph). We also see that every adjacency in T corresponds to an operator name in Γ (arrows in a graph), e.g., an edge e of length 2 adjacent to e_0 and e_1 (i.e. such that $T(e) = e_0\, e_1$) corresponds to two operator names (two arrows), say $\mathtt{src}_e$ and $\mathtt{tgt}_e$, of domain sort e and range sort e_0 and e_1 respectively. Since edges may have length greater than 2, we create canonical operator names of the form $[e{\cdot}\iota]$ for the ι^{th} adjacency of the edge e for every $\iota < |e|$ (hence we favor $[e{\cdot}0]$ and $[e{\cdot}1]$ over $\mathtt{src}_e$ and $\mathtt{tgt}_e$).

Definition 6.2 (functor $\mathsf{S} : \mathbf{Monogr} \to \mathbf{GrStruct}$)**.** To every monograph T we associate the set of operator names $\Omega_T \stackrel{\text{def}}{=} \{[e{\cdot}\iota] \mid e \in \mathsf{E}T \text{ and } \iota < |e|\}$ and the graph structure $\mathsf{S}T : \Omega_T \to (\mathsf{E}T)^{<\omega}$ defined by $\mathsf{S}T([e{\cdot}\iota]) \stackrel{\text{def}}{=} e\, e_\iota$ for all $[e{\cdot}\iota] \in \Omega_T$, i.e., we let $\mathrm{Dom}([e{\cdot}\iota]) \stackrel{\text{def}}{=} e$ and $\mathrm{Rng}([e{\cdot}\iota]) \stackrel{\text{def}}{=} e_\iota$.

To every morphism $f : T \to T'$ in **Monogr** we associate the morphism $\mathsf{S}f : \mathsf{S}T \to \mathsf{S}T'$ defined by: $(\mathsf{S}f)_{\text{opn}}$ is the function that maps every operator name $[e{\cdot}\iota] \in \Omega_T$ to the operator name $[f(e){\cdot}\iota] \in \Omega_{T'}$, and $(\mathsf{S}f)_{\text{srt}}$ is the function $f : \mathsf{E}T \to \mathsf{E}T'$.

We see that $\mathsf{S}f$ is indeed a morphism of graph structures:

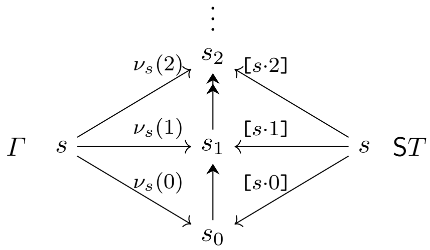
$$(\mathsf{S}f)_{\text{srt}}^{<\omega} \circ \mathsf{S}T([e{\cdot}\iota]) = f(e)\, f(e_\iota) = f(e)\, f(e)_\iota = \mathsf{S}T'([f(e){\cdot}\iota]) = \mathsf{S}T' \circ (\mathsf{S}f)_{\text{opn}}([e{\cdot}\iota])$$

for all $[e{\cdot}\iota] \in \Omega_T$, and it is obvious that S is a faithful functor. With the equivalence $\mathbf{GrStruct} \approx \mathbf{Graph}$ the functor S can be seen as a transformation of monographs as graphs.

The next lemma is central as it shows that no graph structure is omitted by the functor S if we allow sort-preserving isomorphisms of graph structures. We assume the Axiom of Choice through its equivalent formulation known as the Numeration Theorem [5].

Lemma 6.3. *For every graph structure Γ there exists a monograph T such that $\mathsf{S}T \doteq \Gamma$.*

Proof. Let $\Gamma : \Omega \to S^{<\omega}$ and for every sort $s \in S$ let Ω_s be the set of operator names $o \in \Omega$ whose domain sort is s, i.e., $\Omega_s \stackrel{\text{def}}{=} \mathrm{Dom}^{-1}[s]$. By the Numeration Theorem there exists an ordinal λ_s equipollent to Ω_s, i.e., such that there exists a bijection $\nu_s : \lambda_s \to \Omega_s$. Let T be the monograph such that $\mathsf{E}T \stackrel{\text{def}}{=} S$ and $T(s)_\iota \stackrel{\text{def}}{=} \mathrm{Rng}(\nu_s(\iota))$ for all $\iota < \lambda_s$, so that $T(s)$ is an S-sequence of length λ_s.



$T(s)$

We now consider the function $r_{\text{opn}} : \Omega_T \to \Omega$ defined by $r_{\text{opn}}([s{\cdot}\iota]) \stackrel{\text{def}}{=} \nu_s(\iota)$. This function is surjective since for all $o \in \Omega$, by taking $s = \mathrm{Dom}(o)$ and $\iota = \nu_s^{-1}(o)$ we get $\iota < \lambda_s = |s|$ hence $[s{\cdot}\iota] \in \Omega_T$ and obviously $r_{\text{opn}}([s{\cdot}\iota]) = o$. It is also injective since $r_{\text{opn}}([s{\cdot}\iota]) = r_{\text{opn}}([s'{\cdot}\iota'])$ entails $s = \mathrm{Dom}(\nu_s(\iota)) = \mathrm{Dom}(\nu_{s'}(\iota')) = s'$ hence $\iota = \iota'$ and therefore $[s{\cdot}\iota] = [s'{\cdot}\iota']$. Finally, we see that

$$\mathrm{Id}_S^{<\omega} \circ \mathsf{S}T([s{\cdot}\iota]) = s\, s_\iota = \mathrm{Dom}(\nu_s(\iota))\, \mathrm{Rng}(\nu_s(\iota)) = \Gamma(\nu_s(\iota)) = \Gamma \circ r_{\text{opn}}([s{\cdot}\iota])$$

for all $[s{\cdot}\iota] \in \Omega_T$, hence $(r_{\text{opn}}, \mathrm{Id}_S) : \mathsf{S}T \to \Gamma$ is an isomorphism, so that $\mathsf{S}T \doteq \Gamma$. □

The reason why monographs require edges of ordinal length now becomes apparent: the length of an edge s is the cardinality of Ω_s, i.e., the number of operator names whose domain sort is s, and no restriction on this cardinality is ascribed to graph structures. The bijections ν_s provide linear orderings of the sets Ω_s. Since $T(s)$ depends on ν_s the monograph T such that $\mathsf{S}T \doteq \Gamma$ may not be unique, even though S is injective on objects, as we now show.

Theorem 6.4. S *is an isomorphism-dense embedding of* **Monogr** *into* **GrStruct**.

Proof. It is trivial by Lemma 6.3 that S is isomorphism-dense since $\mathsf{S}T \doteq \Gamma$ entails $\mathsf{S}T \simeq \Gamma$. Assume that $\mathsf{S}T = \mathsf{S}T'$ then $\mathsf{E}T = \mathsf{E}T'$ and $\Omega_T = \Omega_{T'}$, hence $|T(e)| = |T'(e)|$ for all $e \in \mathsf{E}T$, and $T(e)_\iota = (\mathsf{S}T([e{\cdot}\iota]))_1 = (\mathsf{S}T'([e{\cdot}\iota]))_1 = T'(e)_\iota$ for all $\iota < |e|$, thus $T = T'$. □

It is clear that S cannot be an equivalence of categories (since **GrStruct** has a terminal object), so some of the structure of **Monogr** must be lost in **GrStruct**. We illustrate this on graphs.

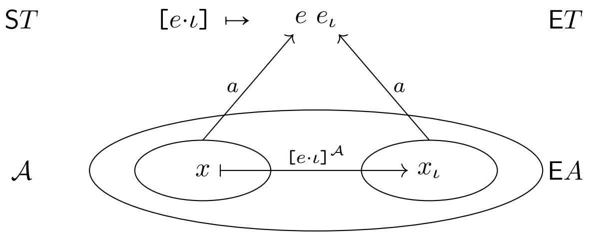


Figure 1: The $\mathsf{S}T$-algebra $\mathcal{A} = \mathsf{A}_T a$ where $a : A \to T$

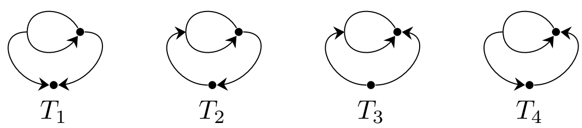
Example 6.5. We consider the graph structure Γ_{g} of graphs. We have $\Omega_{\texttt{nodes}} = \varnothing$ and $\Omega_{\texttt{edges}} = \{\texttt{src}, \texttt{tgt}\}$, hence $\lambda_{\texttt{edges}} = 2$. Let $\nu_{\texttt{edges}} : 2 \to \Omega_{\texttt{edges}}$ be the bijection defined by $\nu_{\texttt{edges}} : 0 \mapsto \texttt{src}, 1 \mapsto \texttt{tgt}$, the corresponding monograph is $\mathrm{T_g} \stackrel{\mathrm{def}}{=} \{(\texttt{nodes}, \varepsilon), (\texttt{edges}, \texttt{nodes}\,\texttt{nodes})\}$, and we easily check that $\mathsf{S}\mathrm{T_g} \doteq \Gamma_{\mathrm{g}}$. However, the only automorphism of $\mathrm{T_g}$ is $1_{\mathrm{T_g}}$, while Γ_{g} has a non-trivial automorphism $m = ((\texttt{src}\ \texttt{tgt}), \mathrm{Id}_{\{\texttt{nodes},\texttt{edges}\}})$ (in cycle notation), hence S is not surjective on morphisms.

This automorphism reflects the fact that a graph structure does not define an order between its operator names. Directing edges as arrows from $\texttt{src}$ to $\texttt{tgt}$ or the other way round is a matter of convention that is reflected in the choice of $\nu_{\texttt{edges}}$ in Example 6.5. This contrasts with monographs where edges are inherently directed by ordinals, and also with the structure of graphs where the source function comes first. In the translation from **Monogr** to **GrStruct** the direction of edges are necessarily lost.

Example 6.6. The signature Γ_{e} of E-graphs from [2] has five sorts $\texttt{edges}$, $\texttt{nv-edges}$, $\texttt{ev-edges}$, $\texttt{nodes}$, $\texttt{values}$ and six operator names $\texttt{src}_{\texttt{e}}$, $\texttt{tgt}_{\texttt{e}}$, $\texttt{src}_{\texttt{nv}}$, $\texttt{tgt}_{\texttt{nv}}$, $\texttt{src}_{\texttt{ev}}$, $\texttt{tgt}_{\texttt{ev}}$ whose domain and range sorts are defined as in Section 1. We have $\Omega_{\texttt{nodes}} = \Omega_{\texttt{values}} = \varnothing$, $\Omega_{\texttt{edges}} = \{\texttt{src}_{\texttt{e}}, \texttt{tgt}_{\texttt{e}}\}$, $\Omega_{\texttt{nv-edges}} = \{\texttt{src}_{\texttt{nv}}, \texttt{tgt}_{\texttt{nv}}\}$ and $\Omega_{\texttt{ev-edges}} = \{\texttt{src}_{\texttt{ev}}, \texttt{tgt}_{\texttt{ev}}\}$. There are four possible monographs T such that $\mathsf{S}T \doteq \Gamma_{\mathrm{e}}$ given by

$$\begin{array}{ll} T(\texttt{nodes}) = T(\texttt{values}) = \varepsilon, & T(\texttt{nv-edges}) = \texttt{nodes}\,\texttt{values} \text{ or } \texttt{values}\,\texttt{nodes}, \\ T(\texttt{edges}) = \texttt{nodes}\,\texttt{nodes}, & T(\texttt{ev-edges}) = \texttt{edges}\,\texttt{values} \text{ or } \texttt{values}\,\texttt{edges}. \end{array}$$

These four monographs are depicted below.



The type indicated by the syntax (and consistent with the drawings of E-graphs in [2]) is of course T_1.

The restrictions of S to the categories of Definition 3.5 are isomorphism-dense embeddings into full subcategories of **GrStruct** that are easy to define. The O-monographs correspond to graph structures $\Gamma : \Omega \to S^{<\omega}$ such that $|\Omega_s| \in O$ for all $s \in S$, and the standard monographs to $\Omega_{\mathrm{Rng}(o)} = \varnothing$ for all $o \in \Omega$. The finite monographs correspond to finite S, hence **FMonogr** corresponds to finite signatures.

We can now describe precisely how a monograph A typed by T through $a : A \to T$ can be read as an $\mathsf{S}T$-algebra $\mathcal{A}$. As mentioned above, every edge x of A is typed by $a(x) \in \mathsf{E}T$ and should therefore be interpreted as an element of $\mathcal{A}_{a(x)}$, hence $\mathcal{A}_{a(x)}$ is the set of all edges $x \in \mathsf{E}A$ that are typed by $a(x)$. Then, for every $\iota < |x| = |a(x)|$, the ι^{th} adjacent edge x_ι of x is the image of x by the ι^{th} operator name for this type of edge, that is $[a(x){\cdot}\iota]$. Note that the sort of this image is $a(x_\iota) = a(x)_\iota$ that is precisely the range sort of the operator name $[a(x){\cdot}\iota]$ in $\mathsf{S}T$ (see Definition 6.2), so that $\mathcal{A}$ is indeed an $\mathsf{S}T$-algebra. This leads to the following definition.

Definition 6.7 (functor A_T : **Monogr**$\backslash T \to \mathsf{S}T$-**Alg**)**.** Given a monograph T, we define the function A_T that maps every object $a : A \to T$ of **Monogr**$\backslash T$ to the $\mathsf{S}T$-algebra $\mathcal{A} = \mathsf{A}_T a$ defined by

- $\mathcal{A}_e \stackrel{\mathrm{def}}{=} a^{-1}[e]$ for all $e \in \mathsf{E}T$, and
- $[e{\cdot}\iota]^{\mathcal{A}}(x) \stackrel{\mathrm{def}}{=} x_\iota$ for all $[e{\cdot}\iota] \in \Omega_T$ and $x \in \mathcal{A}_e$.

Besides, A_T also maps every morphism $f : a \to b$, where $b : B \to T$, to the $\mathsf{S}T$-homomorphism $\mathsf{A}_T f$ from $\mathcal{A}$ to $\mathcal{B} = \mathsf{A}_T b$ defined by

$$(\mathsf{A}_T f)_e \stackrel{\mathrm{def}}{=} f|_{\mathcal{A}_e}^{\mathcal{B}_e} \text{ for all } e \in \mathsf{E}T.$$

The $\mathsf{S}T$-algebra $\mathcal{A}$ can be pictured as in Figure 1. The carrier sets $\mathcal{A}_e$ form a partition of $\mathsf{E}A$. Since $f : a \to b$ (not pictured) is a function $f : \mathsf{E}A \to \mathsf{E}B$ such that $b \circ f = a$, then $b \circ f[\mathcal{A}_e] = a[a^{-1}[e]] \subseteq \{e\}$ hence $f[\mathcal{A}_e] \subseteq b^{-1}[e] = \mathcal{B}_e$, so that $f|_{\mathcal{A}_e}^{\mathcal{B}_e}$ is well-defined.

We also see that $h = \mathsf{A}_T f$ is an $\mathsf{S}T$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ since for every operator name $[e{\cdot}\iota] \in \Omega_T$ we have $\mathrm{Dom}([e{\cdot}\iota]) = e$, $\mathrm{Rng}([e{\cdot}\iota]) = e_\iota$ and

$$[e{\cdot}\iota]^{\mathcal{B}} \circ h_e(x) = [e{\cdot}\iota]^{\mathcal{B}}(f(x)) = f(x)_\iota = f(x_\iota) = f([e{\cdot}\iota]^{\mathcal{A}}(x)) = h_{e_\iota} \circ [e{\cdot}\iota]^{\mathcal{A}}(x)$$

for all $x \in \mathcal{A}_e$. It is obvious from Definition 6.7 that A_T preserves identities and composition of morphisms, hence that it is indeed a functor.

Theorem 6.8. *For every monograph T, $\mathsf{A}_T : \mathbf{Monogr}\backslash T \approx \mathsf{S}T\text{-}\mathbf{Alg}$.*

Proof. Let $a : A \to T$ and $b : B \to T$ be objects of $\mathbf{Monogr}\backslash T$ and $\mathcal{A} \stackrel{\text{def}}{=} \mathsf{A}_T a$, $\mathcal{B} \stackrel{\text{def}}{=} \mathsf{A}_T b$. It is trivial that A_T is faithful.

A_T *is full.* For any $\mathsf{S}T$-homomorphism $h : \mathcal{A} \to \mathcal{B}$, let $f : \mathsf{E}A \to \mathsf{E}B$ be the function defined by $f(x) \stackrel{\text{def}}{=} h_{a(x)}(x)$ for all $x \in \mathsf{E}A$. Let $e = a(x)$ so that $x \in \mathcal{A}_e$, since $h_e(x) \in \mathcal{B}_e = b^{-1}[e]$ then $b \circ f(x) = b(h_e(x)) = e$, hence $b \circ f = a$ and $|f(x)| = |b(f(x))| = |a(x)| = |x|$. For all $\iota < |x|$ we have $a(x_\iota) = a(x)_\iota = e_\iota$ and since h is an $\mathsf{S}T$-homomorphism then

$$f(x_\iota) = h_{e_\iota}([e{\cdot}\iota]^{\mathcal{A}}(x)) = [e{\cdot}\iota]^{\mathcal{B}}(h_e(x)) = f(x)_\iota$$

hence $f : a \to b$ is a morphism. Since $(\mathsf{A}_T f)_e(x) = f|_{\mathcal{A}_e}^{\mathcal{B}_e}(x) = h_e(x)$ for all $e \in \mathsf{E}T$ and all $x \in \mathcal{A}_e$, then $\mathsf{A}_T f = h$.

A_T *is isomorphism-dense.* For any $\mathsf{S}T$-algebra $\mathcal{C}$, let

$$\mathsf{E}C \stackrel{\text{def}}{=} \bigcup_{e \in \mathsf{E}T} \mathcal{C}_e \times \{e\} \quad\text{and}\quad (C(x,e))_\iota \stackrel{\text{def}}{=} ([e{\cdot}\iota]^{\mathcal{C}}(x), e_\iota)$$

for all $(x,e) \in \mathsf{E}C$ and $\iota < |e|$. Since $\mathrm{Rng}([e{\cdot}\iota]) = e_\iota$ then $[e{\cdot}\iota]^{\mathcal{C}}(x) \in \mathcal{C}_{e_\iota}$ hence $(C(x,e))_\iota \in \mathsf{E}C$, so that C is a monograph such that $|(x,e)| = |e|$. Let $c : \mathsf{E}C \to \mathsf{E}T$ be defined by $c(x,e) \stackrel{\text{def}}{=} e$, we have

$$c((x,e)_\iota) = c([e{\cdot}\iota]^{\mathcal{C}}(x), e_\iota) = e_\iota = (c(x,e))_\iota,$$

hence $c : C \to T$ is a morphism. For all $e \in \mathsf{E}T$ we have $(\mathsf{A}_T c)_e = c^{-1}[e] = \mathcal{C}_e \times \{e\}$, and we let $h_e : \mathcal{C}_e \to (\mathsf{A}_T c)_e$ be defined by $h_e(x) \stackrel{\text{def}}{=} (x,e)$ for all $x \in \mathcal{C}_e$. The functions h_e are bijective and $h \stackrel{\text{def}}{=} (h_e)_{e \in \mathsf{E}T}$ is an $\mathsf{S}T$-homomorphism since

$$[e{\cdot}\iota]^{\mathsf{A}_T c} \circ h_e(x) = [e{\cdot}\iota]^{\mathsf{A}_T c}(x,e) = (x,e)_\iota = ([e{\cdot}\iota]^{\mathcal{C}}(x), e_\iota) = h_{e_\iota} \circ [e{\cdot}\iota]^{\mathcal{C}}(x),$$

for all $[e{\cdot}\iota] \in \Omega_T$ and $x \in \mathcal{C}_e$, hence $\mathcal{C} \simeq \mathsf{A}_T c$. □

It is easy to see that for any two signatures Σ and Σ', if $\Sigma \simeq \Sigma'$ then $\Sigma\text{-}\mathbf{Alg} \simeq \Sigma'\text{-}\mathbf{Alg}$. We conclude that all graph structured algebras can be represented as typed monographs.

Corollary 6.9. *For every graph structure Γ there exists a monograph T such that $\Gamma\text{-}\mathbf{Alg} \approx \mathbf{Monogr}\backslash T$.*

Proof. By Lemma 6.3 there exists T such that $\Gamma \simeq \mathsf{S}T$, hence $\mathbf{Monogr}\backslash T \approx \mathsf{S}T\text{-}\mathbf{Alg} \simeq \Gamma\text{-}\mathbf{Alg}$. □

It is worth mentioning that the categories $\Gamma\text{-}\mathbf{Alg}$ are equivalent to *presheaf toposes*, i.e., functor categories of the form $\mathbf{Set}^{\boldsymbol{C}}$ for small categories $\boldsymbol{C}$. Indeed, it is easy to see that $\mathbf{Set}^{\boldsymbol{C}} \approx \Gamma\text{-}\mathbf{Alg}$ if Γ is the underlying graph of $\boldsymbol{C}$, and similarly if $\boldsymbol{C}$ is the category freely generated by the graph Γ. Hence the categories of typed monographs are the presheaf toposes.

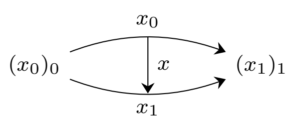
Example 6.10. Following [11], an ∞-*graph* $\mathcal{G}$ is given by a diagram of sets

$$\mathcal{G}_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \mathcal{G}_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} \cdots \underset{t_{n-1}}{\overset{s_{n-1}}{\leftleftarrows}} \mathcal{G}_n \underset{t_n}{\overset{s_n}{\leftleftarrows}} \mathcal{G}_{n+1} \underset{t_{n+1}}{\overset{s_{n+1}}{\leftleftarrows}} \cdots$$

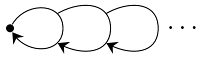
such that, for every $n \in \omega$, the following equations hold:

$$s_n \circ s_{n+1} = s_n \circ t_{n+1}, \qquad t_n \circ s_{n+1} = t_n \circ t_{n+1}.$$

This means that every element x of $\mathcal{G}_{n+2}$ is an edge whose source x_0 and target x_1 are edges of $\mathcal{G}_n$ that are parallel, i.e., that have same source $(x_0)_0 = (x_1)_0$ and same target $(x_0)_1 = (x_1)_1$. Graphically:



This is known as the *globular condition.* We consider the type monograph T_∞ defined by $\mathsf{E}\mathrm{T}_\infty = \omega$,$\mathrm{T}_\infty(0) = \varepsilon$ and $\mathrm{T}_\infty(n+1) = n\,n$ for all $n \in \omega$. This is an infinite non-standard $\{0, 2\}$-monograph that can be pictured as



We express the globular condition on typed monographs $g : G \to \mathrm{T}_\infty$ as:

$$\text{for all } x \in \mathsf{E}G,\ \text{if } g(x) \geqslant 2 \text{ then } G(x_0) = G(x_1).$$

We rapidly check that this is equivalent to the globular condition on the ST_∞-algebra $\mathcal{G} = \mathsf{A}_{\mathrm{T}_\infty} g$. The set of sorts of ST_∞ is ω and its operator names are $\mathtt{[}n+1\mathtt{\cdot 0]}$ and $\mathtt{[}n+1\mathtt{\cdot 1]}$ with domain sort $n+1$ and range sort n, for all $n \in \omega$. We let $s_n \stackrel{\text{def}}{=} \mathtt{[}n+1\mathtt{\cdot 0]}^{\mathcal{G}}$ and $t_n \stackrel{\text{def}}{=} \mathtt{[}n+1\mathtt{\cdot 1]}^{\mathcal{G}}$, that are functions from $\mathcal{G}_{n+1}$ to $\mathcal{G}_n$ as in the diagram of ∞-graphs.

By Definition 6.7 we have for all $x \in \mathcal{G}_{n+2} = g^{-1}[n+2]$ and all $i, j \in 2$ that

$$\mathtt{[}n+1\mathtt{\cdot}j\mathtt{]}^{\mathcal{G}} \circ \mathtt{[}n+2\mathtt{\cdot}i\mathtt{]}^{\mathcal{G}}(x) = \mathtt{[}n+1\mathtt{\cdot}j\mathtt{]}^{\mathcal{G}}(x_i) = (x_i)_j$$

hence

$$\begin{aligned} G(x_0) = G(x_1) &\text{ iff } (x_0)_0 = (x_1)_0 \text{ and } (x_0)_1 = (x_1)_1 \\ &\text{ iff } \mathtt{[}n+1\mathtt{\cdot 0]}^{\mathcal{G}} \circ \mathtt{[}n+2\mathtt{\cdot 0]}^{\mathcal{G}}(x) = \mathtt{[}n+1\mathtt{\cdot 0]}^{\mathcal{G}} \circ \mathtt{[}n+2\mathtt{\cdot 1]}^{\mathcal{G}}(x) \\ &\quad \text{ and } \mathtt{[}n+1\mathtt{\cdot 1]}^{\mathcal{G}} \circ \mathtt{[}n+2\mathtt{\cdot 0]}^{\mathcal{G}}(x) = \mathtt{[}n+1\mathtt{\cdot 1]}^{\mathcal{G}} \circ \mathtt{[}n+2\mathtt{\cdot 1]}^{\mathcal{G}}(x) \\ &\text{ iff } s_n \circ s_{n+1}(x) = s_n \circ t_{n+1}(x) \text{ and } t_n \circ s_{n+1}(x) = t_n \circ t_{n+1}(x). \end{aligned}$$

Example 6.11. The signature Γ_h of *hypergraphs* (see [3, Example 3.4]) is defined by the set of sorts $\mathrm{S}_\mathrm{h} \stackrel{\text{def}}{=} \{\mathtt{V}\} \cup \{\mathtt{H}_{n,m} \mid n, m \in \omega\}$ and for all $n, m \in \omega$ by n operator names $\mathtt{src}_i^{n,m}$ and m operator names $\mathtt{tgt}_j^{n,m}$ with domain sort $\mathtt{H}_{n,m}$ and range sort $\mathtt{V}$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Hence there are $n+m$ operator names of domain $\mathtt{H}_{n,m}$, and $(n+m)!$ bijections from the ordinal $n+m$ to this set of operator names. But since they all have the same range sort $\mathtt{V}$, the type monograph T_h does not depend on these bijections (one for every pair (n, m)). It is defined by $\mathsf{E}\mathrm{T}_\mathrm{h} \stackrel{\text{def}}{=} \mathrm{S}_\mathrm{h}$ and

$$\begin{aligned} \mathrm{T}_\mathrm{h}(\mathtt{V}) &= \varepsilon \\ \mathrm{T}_\mathrm{h}(\mathtt{H}_{n,m}) &= \mathtt{V}{\uparrow}(n+m) \text{ for all } n, m \in \omega. \end{aligned}$$

This is a standard ω-monograph. It is easy to see that any standard ω-monograph can by typed by T_h, though not in a unique way. Every edge of length $l > 0$ can be typed by any sort $\mathtt{H}_{n,m}$ such that $n + m = l$, and every node can be typed by $\mathtt{V}$ (or by $\mathtt{H}_{0,0}$ if it is not adjacent to any edge). To any such typing corresponds an ST_h-algebra by the equivalence $\mathsf{A}_{\mathrm{T}_\mathrm{h}}$, and thus a hypergraph (a Γ_h-algebra) since $\Gamma_\mathrm{h} \simeq \mathsf{ST}_\mathrm{h}$.

But to know which hypergraph $\mathcal{H}$ corresponds exactly to a typed monograph we need to be more specific, since there are infinitely many isomorphisms between Γ_h and ST_h. The natural isomorphism stems from the obvious orderings $\mathtt{src}_1^{n,m} < \cdots < \mathtt{src}_n^{n,m} < \mathtt{tgt}_1^{n,m} < \cdots < \mathtt{tgt}_m^{n,m}$ for all $n, m \in \omega$. In this isomorphism the canonical operator name $\mathtt{[H}_{n,m}\mathtt{\cdot}i\mathtt{]}$ for all $i < n+m$ corresponds to $\mathtt{src}_{i+1}^{n,m}$ if $i < n$, and to $\mathtt{tgt}_{i+1-n}^{n,m}$ if $i \geqslant n$. Thus an edge x, say of length 3 typed by $\mathtt{H}_{2,1}$, must be interpreted as an hyperedge $x \in \mathcal{H}_{\mathtt{H}_{2,1}}$ with $(\mathtt{src}_1^{2,1})^{\mathcal{H}}(x) = x_0$, $(\mathtt{src}_2^{2,1})^{\mathcal{H}}(x) = x_1$, $(\mathtt{tgt}_1^{2,1})^{\mathcal{H}}(x) = x_2$ and $x_0, x_1, x_2 \in \mathcal{H}_\mathtt{V}$.

The results of this section apply in particular to typed graphs. It is easy to see that $\mathsf{S} \circ \mathsf{M}$ is an isomorphism-dense embedding of **Graph** into the full subcategory of graph structures $\Gamma : \Omega \to S^{<\omega}$ such that for every operator name $o \in \Omega$ we have $|\Omega_{\mathrm{Dom}(o)}| = 2$ and $\Omega_{\mathrm{Rng}(o)} = \varnothing$. Hence for every such Γ there exists a graph G such that $\mathbf{Graph}\backslash G \approx \mathbf{Monogr}\backslash \mathsf{M}G \approx \Gamma\textbf{-Alg}$. The type graph G is determined only up to the orientation of its edges.

7 Submonographs and Partial Morphisms

Graph structures have been characterized in [3] as the signatures that allow the transformation of the corresponding algebras by the single pushout method. This method is based on the construction of pushouts in categories of partial homomorphisms, defined as standard homomorphisms from subalgebras of their domain algebra, just as partial functions are standard functions from subsets of their domain (in the categorical theoretic sense of the word *domain*). The results of Section 6 suggest that a similar approach can be followed with monographs. We first need a notion of submonograph, their (inverse) image by morphisms and restrictions of morphisms to submonographs.

Definition 7.1 (submonographs, (inverse) images, restrictions)**.** A monograph A is a *submonograph* of a monograph M if $A \subseteq M$. For any monograph N and morphism $f : M \to N$, let $f(A) \stackrel{\text{def}}{=} \{(f(x), N \circ f(x)) \mid x \in \mathsf{E}A\}$. For any submonograph $C \subseteq N$, let $f^{-1}(C) \stackrel{\text{def}}{=} \{(x, M(x)) \mid x \in f^{-1}[\mathsf{E}C]\}$. If $f(A) \subseteq C$, let $f|_A^C : A \to C$ be the morphism whose underlying function is $f|_{\mathsf{E}A}^{\mathsf{E}C}$.

In the sequel we will use the following obvious facts without explicit reference. $f(A)$ and $f^{-1}(C)$ are submonographs of N and M respectively. If A and B are submonographs of M then so are $A \cup B$ and $A \cap B$. We have $f(A \cup B) = f(A) \cup f(B)$ thus $A \subseteq B$ entails $f(A) \subseteq f(B)$. If C and D are submonographs of N we have similarly $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ and $C \subseteq D$ entails $f^{-1}(C) \subseteq f^{-1}(D)$. We also have $A \subseteq f^{-1}(f(A))$ and $f(f^{-1}(C)) = C \cap f(M)$. For any $g : N \to P$ and submonograph E of P, $(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$. If $(A + B, \mu_1, \mu_2)$ is the coproduct of (A, B) and C is a submonograph of $A + B$ then $C = \mu_1^{-1}(C) + \mu_2^{-1}(C)$.

We may now define the notion of partial morphisms of monographs, with a special notation in order to distinguish them from standard morphisms, and their composition.

Definition 7.2 (partial morphisms $\lceil f \rceil$)**.** A *partial morphism* $\lceil f \rceil : A \to B$ is a morphism $f : A' \to B$ where A' is a submonograph of A. f is called the *underlying morphism* of $\lceil f \rceil$. If the domain of f is not otherwise specified, we write $\lceil f \rceil : A \hookleftarrow A' \to B$. If the domain A' of f is specified but not the domain of $\lceil f \rceil$ then they are assumed to be identical, i.e., $\lceil f \rceil : A' \hookleftarrow A' \to B$. For any $\lceil g \rceil : B \hookleftarrow B' \to C$ we define the *composition of partial morphisms* as

$$\lceil g \rceil \circ \lceil f \rceil \overset{\text{def}}{=} \left\lceil g \circ f|_{f^{-1}(B')}^{B'} \right\rceil : A \hookleftarrow f^{-1}(B') \to C.$$

This composition is similar to the composition of partial maps in [12] in the sense that $(f^{-1}(B'), f|_{f^{-1}(B')}^{B'} : f^{-1}(B') \to B', j' : f^{-1}(B') \hookrightarrow A')$ is a pullback of $(j : B' \hookrightarrow B, f : A' \to B, B)$ (or *inverse image* of j along f), except that partial maps are taken modulo isomorphic variations of subobjects, while we consider only the (unique) inverse image with an inclusion morphism. This choice corresponds to [3] and to the standard notion of partial function in Computer Science.

It is obvious that the identities for this composition are the partial morphisms $\lceil 1_A \rceil : A \to A$.

Lemma 7.3. *Composition of partial morphisms is associative.*

Proof. Let $\lceil f \rceil$ and $\lceil g \rceil$ as in Definition 7.2, $\lceil h \rceil : C \hookleftarrow C' \to D$ and $k \overset{\text{def}}{=} g \circ f|_{f^{-1}(B')}^{B'} : f^{-1}(B') \to C$, so that

$$\lceil h \rceil \circ (\lceil g \rceil \circ \lceil f \rceil) = \lceil h \rceil \circ \lceil k \rceil = \left\lceil h \circ k|_{k^{-1}(C')}^{C'} \right\rceil : A \hookleftarrow k^{-1}(C') \to D.$$

We also have

$$\begin{aligned}(\lceil h \rceil \circ \lceil g \rceil) \circ \lceil f \rceil &= \left\lceil h \circ g|_{g^{-1}(C')}^{C'} \right\rceil \circ \lceil f \rceil \\ &= \left\lceil h \circ g|_{g^{-1}(C')}^{C'} \circ f|_{f^{-1}(g^{-1}(C'))}^{g^{-1}(C')} \right\rceil : A \hookleftarrow f^{-1}(g^{-1}(C')) \to D.\end{aligned}$$

The functions k and $g \circ f$ have the same images and codomain but their domain is respectively $f^{-1}(B')$ and A'. Since $g : B' \to C$ and $f : A' \to B$ then obviously $g^{-1}(C') \subseteq B'$ and $(g \circ f)^{-1}(C') = f^{-1}(g^{-1}(C')) \subseteq f^{-1}(B') \subseteq A'$, hence $k^{-1}(C') = (g \circ f)^{-1}(C')$. This proves that the functions $h \circ k|_{k^{-1}(C')}^{C'}$ and $h \circ g|_{g^{-1}(C')}^{C'} \circ f|_{f^{-1}(g^{-1}(C'))}^{g^{-1}(C')}$ have the same domain, and since they also have the same codomain and images, they are equal. □

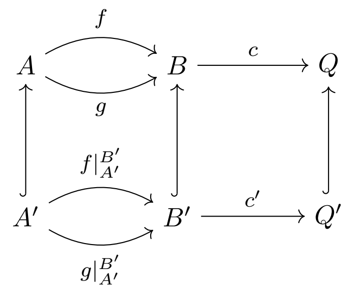
Definition 7.4 (categories of partial morphisms of monographs)**.** Let $\mathbf{Monogr^P}$ be the category of monographs and partial morphisms composed as in Definition 7.2. Let $\mathbf{SMonogr^P}$ be its full subcategory of standard monographs. For any set O of ordinals, let O-$\mathbf{Monogr^P}$ (resp. O-$\mathbf{SMonogr^P}$) be its full subcategory of O-monographs (resp. standard O-monographs). Let $\mathbf{FMonogr^P}$ be its full subcategory of finite ω-monographs.

We now see how these inverse images allow to formulate a sufficient condition ensuring that restrictions of coequalizers are again coequalizers.

Lemma 7.5 (coequalizer restriction)**.** *Let A' and B' be submonographs of A and B respectively and $f, g : A \to B$ be parallel morphisms such that*

$$f^{-1}(B') = A' = g^{-1}(B')$$

(i.e., the two left squares below are pullbacks), if (Q, c) is a coequalizer of (f, g) then (Q', c') is a coequalizer of $(f|_{A'}^{B'}, g|_{A'}^{B'})$, where $Q' = c(B')$, $c' = c|_{B'}^{Q'}$ and $c^{-1}(Q') = B'$ (the right square below is also a pullback).



Proof. We assume w.l.o.g. that (Q, c) is the coequalizer of (f, g) constructed in Lemma 4.2 with $\sim$ being the equivalence relation generated by $R = \{(f(x), g(x)) \mid x \in \mathsf{E}A\}$, and we let (Q', c') be the coequalizer of $(f|^{B'}_{A'}, g|^{B'}_{A'})$ constructed similarly with the equivalence relation $\approx$ generated by $R' = \{(f|^{B'}_{A'}(x), g|^{B'}_{A'}(x)) \mid x \in \mathsf{E}A'\}$. By the properties of f and g we have that

$$f(x) \in \mathsf{E}B' \text{ iff } x \in f^{-1}[\mathsf{E}B'] \text{ iff } x \in \mathsf{E}A' \text{ iff } x \in g^{-1}[\mathsf{E}B'] \text{ iff } g(x) \in \mathsf{E}B'$$

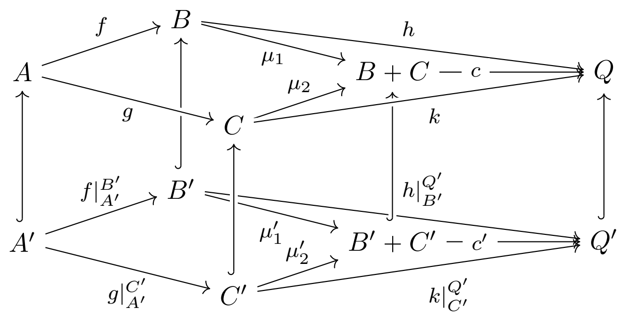
for all $x \in \mathsf{E}A$, hence for all $y, y' \in \mathsf{E}B$ we have that $y\ R'\ y'$ iff $y\ R\ y'$ and at least one of y, y' is in $\mathsf{E}B'$. By an easy induction we see that $y \approx y'$ iff $y \sim y'$ and $y' \in \mathsf{E}B'$, hence the $\approx$-classes are the $\sim$-classes of the elements of $\mathsf{E}B'$, i.e., $\mathsf{E}Q' = c[\mathsf{E}B']$. It follows trivially that $Q' = c(B')$, $c' = c|^{Q'}_{B'}$ and $c^{-1}(Q') = B'$. □

It is then easy to obtain a similar result on pushouts.

Lemma 7.6 (pushout restriction)**.** *Let A', B', C' be submonographs of A, B, C respectively and $f : A \to B$, $g : A \to C$ be morphisms such that*

$$f^{-1}(B') = A' = g^{-1}(C')$$

(the two left faces below are pullbacks), if (h, k, Q) is a pushout of (A, f, g) then $(h|^{Q'}_{B'}, k|^{Q'}_{C'}, Q')$ is a pushout of $(A', f|^{B'}_{A'}, g|^{C'}_{A'})$ where $Q' = h(B') \cup k(C')$, $h^{-1}(Q') = B'$ and $k^{-1}(Q') = C'$ (the two right faces are pullbacks).



Proof. We assume w.l.o.g. that (h, k, Q) is obtained by the canonical construction of pushouts, i.e., that $h = c \circ \mu_1$ and $k = c \circ \mu_2$ where (Q, c) is a coequalizer of $(\mu_1 \circ f, \mu_2 \circ g)$ and $(B + C, \mu_1, \mu_2)$ is the coproduct of (B, C). Let $(B' + C', \mu'_1, \mu'_2)$ be the coproduct of (B', C'), then obviously $B' + C' \subseteq B + C$, $\mu'_1 = \mu_1|^{B'+C'}_{B'}$ and $\mu'_2 = \mu_2|^{B'+C'}_{C'}$. Since

$$(\mu_1 \circ f)^{-1}(B' + C') = f^{-1}(B') = A' = g^{-1}(C') = (\mu_2 \circ g)^{-1}(B' + C')$$

then by Lemma 7.5 (Q', c') is a coequalizer of

$$((\mu_1 \circ f)|^{B'+C'}_{A'}, (\mu_2 \circ g)|^{B'+C'}_{A'}) = (\mu'_1 \circ f|^{B'}_{A'}, \mu'_2 \circ g|^{C'}_{A'})$$

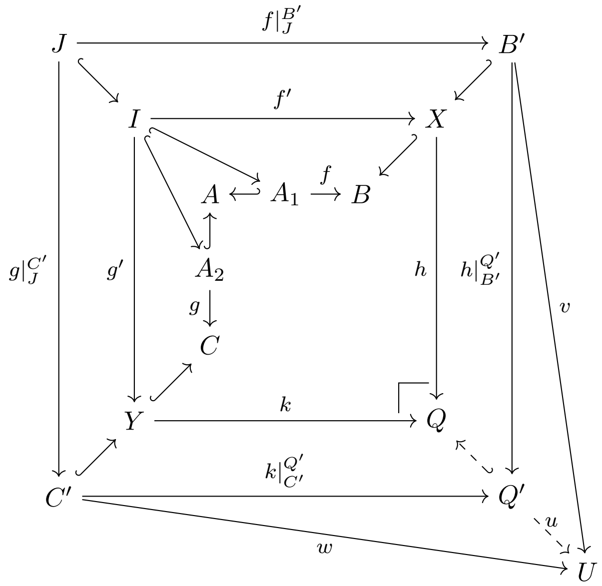
where $Q' = c(B' + C')$, $c' = c|^{Q'}_{B'+C'}$ and $c^{-1}(Q') = B' + C'$. We thus have $h^{-1}(Q') = (c \circ \mu_1)^{-1}(Q') = \mu_1^{-1}(B' + C') = B'$ and similarly $k^{-1}(Q') = C'$. We also have $h|^{Q'}_{B'} = (c \circ \mu_1)|^{Q'}_{B'} = c' \circ \mu'_1$ and $k|^{Q'}_{C'} = (c \circ \mu_2)|^{Q'}_{C'} = c' \circ \mu'_2$, hence $(h|^{Q'}_{B'}, k|^{Q'}_{C'}, Q')$ is the canonical pushout of $(A', f|^{B'}_{A'}, g|^{B'}_{A'})$, and therefore $Q' = h|^{Q'}_{B'}(B') \cup k|^{Q'}_{C'}(C') = h(B') \cup k(C')$. □

Note that the top face in the diagram above may not be a van Kampen square, because this property is only established for our special pullbacks with inclusion morphisms (and only in one direction). Besides, we could not use adhesivity since neither f nor g are assumed to be injective.

We can now show that categories of partial morphisms of monographs have pushouts. The following construction is inspired by [3, Construction 2.6, Theorem 2.7] though the proof uses pushout restriction.

Theorem 7.7. *The categories of Definition 7.4 have pushouts.*

Proof. Let $\lceil f \rceil : A \hookleftarrow A_1 \to B$ and $\lceil g \rceil : A \hookleftarrow A_2 \to C$. The set of submonographs $J \subseteq A_1 \cap A_2$ such that $f^{-1}(f(J)) = J$ and $g^{-1}(g(J)) = J$ contains $\varnothing$ and is closed under union, hence has a greatest element denoted I. There is also a greatest submonograph $X \subseteq B$ such that $f^{-1}(X) \subseteq I$, that must therefore be greater than $f(I)$, i.e., we have $f(I) \subseteq X$ hence $f^{-1}(f(I)) \subseteq f^{-1}(X)$ and this yields $f^{-1}(X) = I$. Similarly, there is a greatest submonograph $Y \subseteq C$ such that $g^{-1}(Y) \subseteq I$, so that $g(I) \subseteq Y$ and $g^{-1}(Y) = I$.



Let $f' = f|_I^X$, $g' = g|_I^Y$ and (h, k, Q) be a pushout of (I, f', g') in **Monogr**, we claim that $(\lceil h \rceil, \lceil k \rceil, Q)$ is a pushout of $(A, \lceil f \rceil, \lceil g \rceil)$ in $\mathbf{Monogr^P}$, where obviously $\lceil h \rceil : B \hookleftarrow X \to Q$ and $\lceil k \rceil : C \hookleftarrow Y \to Q$. We first see that

$$\lceil h \rceil \circ \lceil f \rceil = \left\lceil h \circ f|_{f^{-1}(X)}^X \right\rceil = \lceil h \circ f' \rceil = \lceil k \circ g' \rceil = \left\lceil k \circ g|_{g^{-1}(Y)}^Y \right\rceil = \lceil k \rceil \circ \lceil g \rceil .$$

We now consider any pair of partial morphisms $\lceil v \rceil : B \hookleftarrow B' \to U$ and $\lceil w \rceil : C \hookleftarrow C' \to U$ such that $\lceil v \rceil \circ \lceil f \rceil = \lceil w \rceil \circ \lceil g \rceil$, hence $v \circ f|_J^{B'} = w \circ g|_J^{C'}$ where $J \stackrel{\text{def}}{=} f^{-1}(B') = g^{-1}(C')$. Since $f(J) = f(f^{-1}(B')) \subseteq B'$ then $J \subseteq f^{-1}(f(J)) \subseteq f^{-1}(B') = J$, hence $f^{-1}(f(J)) = J$ and similarly $g^{-1}(g(J)) = J$, so that $J \subseteq I$. This can be written $f^{-1}(B') \subseteq I$ and thus entails $B' \subseteq X$ and similarly $C' \subseteq Y$, hence $f'^{-1}(B') = J = g'^{-1}(C')$.

We can therefore apply Lemma 7.6 and get that $(h|_{B'}^{Q'}, k|_{C'}^{Q'}, Q')$ is a pushout of $(J, f'|_J^{B'}, g'|_J^{C'})$ where $Q' = h(B') \cup k(C')$, $h^{-1}(Q') = B'$ and $k^{-1}(Q') = C'$. Since $v \circ f'|_J^{B'} = v \circ f|_J^{B'} = w \circ g|_J^{C'} = w \circ g'|_J^{C'}$ there exists a unique $u : Q' \to U$ such that $u \circ h|_{B'}^{Q'} = v$ and $w = u \circ k|_{C'}^{Q'}$. We thus have a partial morphism $\lceil u \rceil : Q \hookleftarrow Q' \to U$ such that

$$\lceil u \rceil \circ \lceil h \rceil = \left\lceil u \circ h|_{h^{-1}(Q')}^{Q'} \right\rceil = \left\lceil u \circ h|_{B'}^{Q'} \right\rceil = \lceil v \rceil$$

and similarly $\lceil u \rceil \circ \lceil k \rceil = \lceil w \rceil$.

Suppose there is a $\lceil u' \rceil : Q \hookleftarrow D \to U$ such that $\lceil u' \rceil \circ \lceil h \rceil = \lceil v \rceil$ and $\lceil u' \rceil \circ \lceil k \rceil = \lceil w \rceil$, then $u' \circ h|_{h^{-1}(D)}^D = v$ hence $h^{-1}(D) = B'$ and similarly $k^{-1}(D) = C'$. Since $D \subseteq Q = h(X) \cup k(Y)$ then

$$D = (D \cap h(X)) \cup (D \cap k(Y)) = h(h^{-1}(D)) \cup k(k^{-1}(D)) = h(B') \cup k(C') = Q'$$

and we get $\lceil u' \rceil = \lceil u \rceil$ by the unicity of u.

If B and C are finite (resp. standard, resp. O-monographs) then so are X and Y, hence so is Q by Theorem 4.4. ☐

One important feature of this construction is illustrated below.

Example 7.8. Suppose there are edges x of $A_1 \cap A_2$ and $y \in \mathsf{E}A_2 \backslash \mathsf{E}A_1$ such that $g(x) = g(y)$. If x is an edge of $I = g^{-1}(g(I))$ then so is y, which is impossible since $I \subseteq A_1 \cap A_2$. Hence x is not an edge of $I = f^{-1}(X)$ and therefore $f(x) \notin \mathsf{E}X$. Since y is not an edge of $I = g^{-1}(Y)$ then similarly $g(x) = g(y) \notin \mathsf{E}Y$. This means that even though x has images by both f and g, none of these has an image (by h or k) in Q, i.e., they are "deleted" from the pushout.

The result of the present section can be replicated by replacing every monograph, say A, by a typed monograph with a fixed type T, say $a : A \to T$. But then expressions like $A \subseteq B$ are replaced by $a \subseteq b$, which ought to be interpreted as $A \subseteq B$ *and* $a = b|_A$, so that $\mathsf{A}_T a$ is then a subalgebra of $\mathsf{A}_T b$. In this way the results of [3] on categories of partial homomorphisms could be deduced from Corollary 6.9. They cannot be obtained directly from Theorem 7.7.

8 Algebraic Transformations of Monographs

Rule-based transformations of graphs are conceived as substitutions of subgraphs (image of a left hand side of a rule) by subgraphs (image of its right hand side). Substitutions are themselves designed as an operation of deletion (of nodes or edges) followed by an operation of addition. This last operation is conveniently represented as a pushout, especially when edges are added between existing nodes (otherwise a coproduct would be sufficient).

The operation of deletion is however more difficult to represent in category theory, since there is no categorical notion of a complement. This is a central and active issue in the field of Algebraic Graph Transformation, and many definitions have been proposed, see [13, 14, 15, 16]. The most common and natural one, known as the double pushout method [17, 18, 19], assumes the operation of deletion as the inverse of the operation of addition.

More precisely, in the following pushout diagram

$$\begin{array}{ccc} L & \xleftarrow{\;l\;} & K \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle k} \\ M & \xleftarrow[\;f\;]{} & D \end{array}$$

we understand M as the result of adding edges to D as specified by l and k. Images of edges of K are present in both D and L, and therefore also in M, without duplications (since $f \circ k = m \circ l$). The edges that are added to D are therefore the images by m of the edges of L that do not occur in $l(K)$. We may then inverse this operation and understand D as the result of removing these edges from M. The monograph M and the morphisms m, l then appear as the input of the operation, and the monograph D and morphisms k, f as its output. The problem of course is that the pushout operation is not generally bijective, hence it cannot always be inverted. We first analyze the conditions of existence of D.

Definition 8.1 (pushout complement, gluing condition)**.** A *pushout complement* of morphisms $l : K \to L$ and $m : L \to M$ is a monograph D and a pair of morphisms $k : K \to D$ and $f : D \to M$ such that (m, f, M) is a pushout of (K, l, k).

The morphisms $l : K \to L$ and $m : L \to M$ satisfy the *gluing condition* (GC(l, m) for short) if, for $L' = \mathsf{E}L \backslash l[\mathsf{E}K]$,

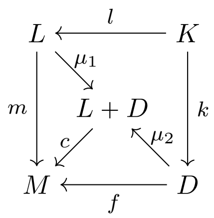
(1) for all $x, x' \in \mathsf{E}L$, $m(x) = m(x')$ and $x \in L'$ entail $x = x'$, and

(2) for all $e, e' \in \mathsf{E}M$, $e \mid M(e')$ and $e \in m[L']$ entail $e' \in m[L']$.

The edges of M that should be removed from M to obtain D are the elements of $m[L']$. We may say that an edge $m(x)$ of M is *marked for removal* if $x \in L'$ and *marked for preservation* if $x \in l[\mathsf{E}K]$. Condition (1) of the gluing condition states that the restriction of m to $m^{-1}[m[L']]$ should be injective, or in other words that an edge can be deleted if it is marked for removal once, and not marked for preservation. Condition (2) states that an edge can be deleted only if all the edges that are adjacent to it are also deleted (otherwise these edges would be adjacent to a non-existent edge). It is obvious that this gluing condition reduces to the standard one known on graphs, when applied to standard $\{0, 2\}$-monographs. We now prove that it characterizes the existence of pushout complements (note that l is not assumed to be injective).

Lemma 8.2. *The morphisms $l : K \to L$ and $m : L \to M$ have a pushout complement iff they satisfy the gluing condition.*

Proof. Necessary condition. We assume w.l.o.g. that the pushout (m, f, M) of (K, l, k) is obtained by canonical construction, i.e., let $(L + D, \mu_1, \mu_2)$ be the coproduct of (L, D), (M, c) be the coequalizer of $(\mu_1 \circ l, \mu_2 \circ k)$, $m = c \circ \mu_1$ and $f = c \circ \mu_2$. Thus $\mathsf{E}M$ is the quotient of $\mathsf{E}L + \mathsf{E}D$ by the equivalence relation $\sim$ generated by $R = \{(\mu_1 \circ l(z), \mu_2 \circ k(z)) \mid z \in \mathsf{E}K\}$. Let $L' = \mathsf{E}L \backslash l[\mathsf{E}K]$, we first prove (1) and then (2).



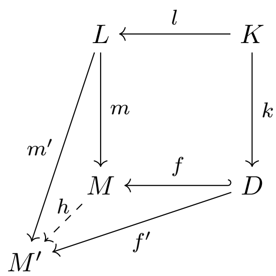
For all $x, x' \in \mathsf{E}L$, if $x \in L'$ then $x \notin l[\mathsf{E}K]$, hence $\mu_1(x)$ is not related by R to any element and is therefore alone in its $\sim$-class. Hence[2] if $m(x) = m(x')$ then $\mu_1(x) \sim \mu_1(x')$ and therefore $x = x'$.

For all $e, e' \in \mathsf{E}M$ such that $e \mid M(e')$ and $e \in m[L']$, let $x \in L'$ such that $e = m(x)$. Suppose that $e' = f(y')$ for some $y' \in \mathsf{E}D$ then $M(e') = f^{<\alpha} \circ D(y')$ hence there is a $y \mid D(y')$ such that $e = f(y)$, hence $m(x) \in f[\mathsf{E}D]$ which is impossible by note 2. Since $M = f(D) \cup m(L)$ there must be a $x' \in \mathsf{E}L$ such that $e' = m(x')$. Suppose now that $x' = l(z)$ for some $z \in \mathsf{E}K$ then $e' = m(l(z)) = f(k(z)) \in f[\mathsf{E}D]$, and we have seen this is impossible. Hence $x' \notin l[\mathsf{E}K]$ and therefore $e' \in m[L']$.

Sufficient condition. We assume (1) and (2), let α be an ordinal for M, $\mathsf{E}D \stackrel{\text{def}}{=} \mathsf{E}M \backslash m[L']$ and $D(e) \stackrel{\text{def}}{=} M(e)$ for all $e \in \mathsf{E}D$; by (2) this is an $\mathsf{E}D$-sequence, hence D is a submonograph of M and the inclusion function $f : D \hookrightarrow M$ is a morphism. By (1) we have $m[L'] \cap m \circ l[\mathsf{E}K] = \varnothing$, hence $m \circ l[\mathsf{E}K] \subseteq \mathsf{E}D$ and we let $k \stackrel{\text{def}}{=} (m \circ l)|^{\mathsf{E}D}_{\mathsf{E}K}$ so that $f \circ k = m \circ l$. We have

$$k^{<\alpha} \circ K = m^{<\alpha} \circ l^{<\alpha} \circ K = m^{<\alpha} \circ L \circ l = M \circ m \circ l = D \circ k$$

hence $k : K \to D$ is a morphism.



To prove that (m, f, M) is a pushout of (K, l, k), let $m' : L \to M'$ and $f' : D \to M'$ be morphisms such that $m' \circ l = f' \circ k$. Since $\mathsf{E}M = \mathsf{E}D \uplus m[L']$ we define $h : \mathsf{E}M \to \mathsf{E}M'$ as

$$h(e) \stackrel{\text{def}}{=} \begin{cases} f'(e) & \text{if } e \in \mathsf{E}D \\ m'(x) & \text{if } x \in L' \text{ and } e = m(x) \end{cases}$$

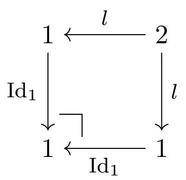
since x is unique by (1). For all $x \in \mathsf{E}L$, if $x \in L'$ then $h \circ m(x) = m'(x)$, otherwise there is a $z \in \mathsf{E}K$ such that $x = l(z)$ and then

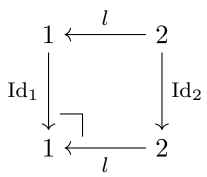
$$h \circ m(x) = h \circ m \circ l(z) = h \circ f \circ k(z) = f' \circ k(z) = m' \circ l(z) = m'(x),$$

hence $h \circ m = m'$. It is obvious that $h \circ f = f'$ and that these two equations uniquely determine h. Proving that $h : M \to M'$ is a morphism is straightforward. □

Note that D is finite whenever M is finite. This proves that this gluing condition is also valid in **FMonogr**, and it is obviously also the case in **SMonogr**, O-**Monogr** and O-**SMonogr** for every set O of ordinals. It therefore characterizes the existence of D, but by no means its unicity.

Example 8.3. In the category **Set** (equivalent to 1-**Monogr**), let l be the unique function from ordinal 2 to ordinal 1, the reader can easily check that





[2]Another consequence is that $\mu_1(x)$ is not related by $\sim$ to any element of $\mu_2[\mathsf{E}D]$, hence that $m(x) \notin f[\mathsf{E}D]$.

are both pushouts. Only the left one is constructed in the proof of Lemma 8.2 (sufficient condition). The right one can be built as a final pullback complement, see [13]. Note that final pullback complements also exist when the gluing condition is not met (in this case they are obviously not pushout complements).

We will therefore need some restrictions in order to ensure some form of determinism, i.e., that the result of double pushout transformations be determined (up to isomorphism) by the matching m. For this it is useful to observe in the proof of Lemma 8.2 (necessary condition) that $f[\mathsf{E}D]$ is invariant (whatever D).

Corollary 8.4. *If D, $k : K \to D$, $f : D \to M$ is a pushout complement of $l : K \to L$, $m : L \to M$ then $f[\mathsf{E}D] = \mathsf{E}M \backslash m[L']$, where $L' = \mathsf{E}L \backslash l[\mathsf{E}K]$.*

Proof. Since $m[\mathsf{E}L] \backslash (m \circ l)[\mathsf{E}K] \subseteq m[L']$ then

$$m[\mathsf{E}L] \backslash m[L'] \subseteq (m \circ l)[\mathsf{E}K] = (f \circ k)[\mathsf{E}K] \subseteq f[\mathsf{E}D].$$

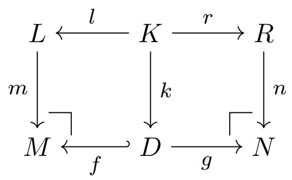
By property of pushouts we have $\mathsf{E}M = f[\mathsf{E}D] \cup m[\mathsf{E}L]$, and by note 2 we have $m[L'] \cap f[\mathsf{E}D] = \varnothing$, hence

$$\mathsf{E}M \backslash m[L'] = (f[\mathsf{E}D] \backslash m[L']) \cup (m[\mathsf{E}L] \backslash m[L']) = f[\mathsf{E}D].$$

□

One way of ensuring the unicity of D (up to isomorphism) is to assume that l is injective: this is a well-known consequence of Theorem 4.17 (see [8]). However, an analysis of the construction of D in the proof of Lemma 8.2 (sufficient condition) shows that we can always build D as a submonograph of M, hence we may as well assume that f is an inclusion morphism and avoid restrictions on l (though both will appear as equivalent, see Corollary 8.9 below). We therefore adopt the following restricted notion of double pushout transformation.

Definition 8.5 (span rules (l, r), matching m, relation $\overset{(l,r)}{\Longrightarrow}_m$)**.** A *span rule* is a pair (l, r) of morphisms $l : K \to L$, $r : K \to R$ with the same domain K. A *matching* of (l, r) in an object M is a morphism $m : L \to M$. For any object N we write $M \overset{(l,r)}{\Longrightarrow}_m N$ if there exists a double-pushout diagram



where f is an inclusion morphism.

We easily see that the relation $\overset{(l,r)}{\Longrightarrow}_m$ is deterministic up to isomorphism.

Corollary 8.6. *$M \overset{(l,r)}{\Longrightarrow}_m N$ and $M \overset{(l,r)}{\Longrightarrow}_m N'$ entail $N \simeq N'$.*

Proof. We have two pushout complements $k : K \to D$, $f : D \hookrightarrow M$ and $k' : K \to D'$, $f' : D' \hookrightarrow M$ of m, l, hence by Corollary 8.4

$$\mathsf{E}D = f[\mathsf{E}D] = \mathsf{E}M \backslash m[L'] = f'[\mathsf{E}D'] = \mathsf{E}D'$$

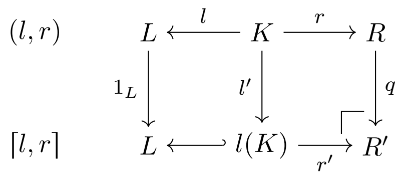
hence $D = D'$, $f = f'$, $k = (f \circ k)|_K^D = (m \circ l)|_K^{D'} = (f' \circ k')|_K^{D'} = k'$, and therefore $N \simeq N'$ by general property of pushouts. □

It is obvious by Theorem 4.4 and by the construction of D in Lemma 8.2 that, in the categories of Definition 3.5, there exists a N such that $M \overset{(l,r)}{\Longrightarrow}_m N$ if and only if l and m satisfy the gluing condition. This means in particular that an edge e of M may be deleted only if it is explicitly marked for removal, i.e., if there is an edge $x \in L'$ such that $m(x) = e$. All edges that are not marked for removal are guaranteed to be preserved. This conservative semantics for transformation rules is extremely safe but imposes a discipline of programming that may be tedious.

As noted in Example 7.8, pushout of partial morphisms have a potential of removing edges. Since such pushouts always exist, they can be used to define transformations that are not restricted by the gluing condition. This is the idea of the single pushout method, that was initiated in [20] and fully developed in [21, 3].

Definition 8.7 (partial rules $\lceil r \rceil$, $\lceil l, r \rceil$, span rule $\mathrm{sp}(\lceil r \rceil)$, relation $\overset{\lceil r \rceil}{\Longrightarrow}_m$). A *partial rule* is a partial morphism $\lceil r \rceil : L \hookleftarrow K \to R$, to which is associated the span rule $\mathrm{sp}(\lceil r \rceil) \overset{\text{def}}{=} (l, r)$ where $l : K \hookrightarrow L$ is the inclusion morphism. A *matching* of $\lceil r \rceil$ in a monograph M is a morphism $m : L \to M$. For any monograph N we write $M \overset{\lceil r \rceil}{\Longrightarrow}_m N$ if there exist partial morphisms $\lceil g \rceil$ and $\lceil n \rceil$ such that $(\lceil n \rceil, \lceil g \rceil, N)$ is a pushout of $(L, \lceil r \rceil, \lceil m \rceil)$.

To any span rule (l, r) where $l : K \to L$, $r : K \to R$ we associate a partial rule $\lceil l, r \rceil \overset{\text{def}}{=} \lceil r' \rceil : L \hookleftarrow l(K) \to R'$ such that (q, r', R') is a pushout of (K, r, l') where $l' \overset{\text{def}}{=} l|_K^{l(K)}$.

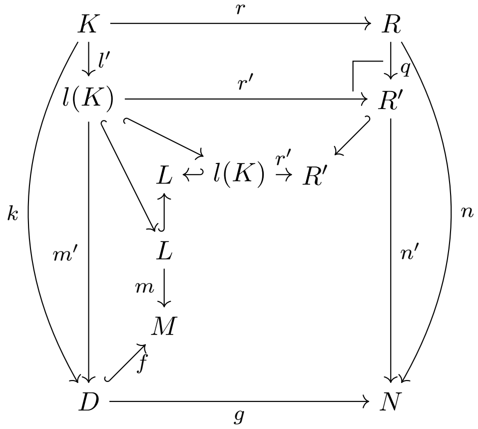


The relation $\overset{\lceil r \rceil}{\Longrightarrow}_m$ is also deterministic up to isomorphism since N is obtained as a pushout. Obviously a morphism m is a matching of (l, r) in M iff it is a matching of $\lceil l, r \rceil$ in M. The partial rule $\lceil l, r \rceil$ is designed to perform the same transformation as the span rule (l, r). We prove that this is indeed the case when the gluing condition holds.

Theorem 8.8. *For any span rule (l, r), monographs M, N and matching m of (l, r) in M, we have*

$$M \overset{(l,r)}{\Longrightarrow}_m N \quad \textit{iff} \quad M \overset{\lceil l,r \rceil}{\Longrightarrow}_m N \textit{ and } \mathrm{GC}(l, m).$$

Proof. Let R', l', q and r' be as in Definition 8.7. We first compute the pushout of $\lceil l, r \rceil$ and $\lceil m \rceil$ according to the construction in Lemma 7.7, by assuming the gluing condition $\mathrm{GC}(l, m)$ and that $D \subseteq M$, $k : K \to D$, $f : D \hookrightarrow M$ is a pushout complement of l, m.



Let I be the greatest submonograph of $l(K) \cap L$ such that $r'^{-1}(r'(I)) = I$ and $m^{-1}(m(I)) = I$. By $\mathrm{GC}(l, m)$ (1) we have for all $x \in \mathsf{E}L$ that $m(x) \in m[l[\mathsf{E}K]]$ entails $x \notin L' = \mathsf{E}L \backslash l[\mathsf{E}K]$, i.e., $x \in l[\mathsf{E}K]$, hence $m^{-1}(m(l(K))) \subseteq l(K)$ and since the reverse inclusion is always true we get $I = l(K)$. Hence the greatest monograph $X \subseteq R'$ such that $r'^{-1}(X) \subseteq I$ is R'.

Let Y be the greatest submonograph of M such that $m^{-1}(Y) \subseteq l(K)$, this entails $m^{-1}[\mathsf{E}Y] \cap L' = \varnothing$, hence $\mathsf{E}Y \cap m[L'] = \varnothing$ and by Corollary 8.4 $Y \subseteq f(D) = D$. Conversely, for all $x \in m^{-1}[\mathsf{E}D] = m^{-1}[\mathsf{E}M \backslash m[L']]$ we have $m(x) \notin m[L']$, hence by $\mathrm{GC}(l, m)$ (1) $x \notin L'$ and thus $x \in l[\mathsf{E}K]$, so that $m^{-1}(D) \subseteq l(K)$. Hence $D \subseteq Y$ and we get $Y = D$.

The pushout of $\lceil l, r \rceil$ and $\lceil m \rceil$ is therefore obtained from the pushout of r' and $m' \overset{\text{def}}{=} m|_{l(K)}^D$. Besides, we have $m' \circ l' = (m \circ l)|_K^D = (f \circ k)|_K^D = k$.

Sufficient condition. We assume $M \overset{(l,r)}{\Longrightarrow}_m N$ and the diagram in Definition 8.5. By Lemma 8.2 we have $\mathrm{GC}(l, m)$. By the above we get $(g \circ m') \circ l' = g \circ k = n \circ r$, and since (q, r', R') is a pushout of (K, r, l') then there exists a unique $n' : R' \to N$ such that $n' \circ r' = g \circ m'$ and $n' \circ q = n$. Since (n, g, N) is a pushout of (K, r, k) then by pushout decomposition (n', g, N) is a pushout of $(l(K), r', m')$, hence $M \overset{\lceil l,r \rceil}{\Longrightarrow}_m N$.

Necessary condition. By $\mathrm{GC}(l, m)$ and Lemma 8.2 we can build a pushout complement $D \subseteq M$, $k : K \to D$, $f : D \hookrightarrow M$ of l, m. By $M \overset{\lceil l,r \rceil}{\Longrightarrow}_m N$ and the above there is a pushout (n', g, N) of $(l(K), r', m')$, hence by pushout composition $(N, n' \circ q, g)$ is a pushout of (K, r, k), hence $M \overset{(l,r)}{\Longrightarrow}_m N$. □

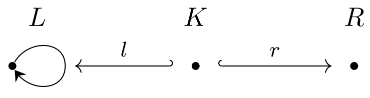
It is easy to see that $\lceil \_, \_ \rceil$ is a left inverse to sp, i.e., $\lceil \mathrm{sp}(\lceil r \rceil) \rceil = \lceil r \rceil$ for any partial rule $\lceil r \rceil$. Besides, $\mathrm{GC}(l, m)$ is equivalent to $\mathrm{GC}(i, m)$, where $i : l(K) \hookrightarrow L$. We conclude that, with respect to double pushout transformations, any span rule can be mimicked by a rule with an inclusion morphism on the left.

Corollary 8.9. $M \overset{(l,r)}{\Longrightarrow}_m N$ *iff* $M \overset{\mathrm{sp}(\lceil l,r \rceil)}{\Longrightarrow}_m N$.

In other words, it is equivalent (on monographs) to restrict f or l to inclusion morphisms (or simply to monomorphisms).

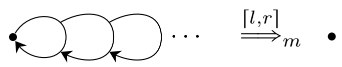
When the gluing condition holds, single and double pushout transformations are therefore equivalent. Single pushout transformations are more expressive since they also apply when the gluing condition does not hold, as illustrated in the following example.

Example 8.10. We consider the following "loop removing" rule:



and try to apply it to monograph T_∞ from Example 6.10. There is a unique morphism $m : L \to \mathrm{T}_\infty$ but it does not satisfy the gluing condition. Indeed, we see that condition (2) is breached since $1 \mid \mathrm{T}_\infty(2)$ and $1 \in m[L']$ and yet $2 \notin m[L']$. Hence the only way to apply the rule to T_∞ is through a single pushout transformation.

For this we first compute the rule $\lceil l, r \rceil$. Since l is the inclusion morphism of $l(K) = K$ into L, then $r' = r$ (and $R' = R = K$) and hence $\lceil l, r \rceil = \lceil r \rceil : L \hookleftarrow K \to R$. The monograph D is the greatest one such that $D \subseteq \mathrm{T}_\infty$ and $m^{-1}(D) \subseteq l(K)$, hence obviously $D = \{(0, \varepsilon)\}$. Since $l(K)$ and R are both isomorphic to D then so is the result of the transformation, i.e.,



Hence removing the edge 1 from T_∞ silently removes the edges n for all $n > 1$.

We therefore see that single pushouts implement a semantics where edges can be silently removed, but minimally so for a monograph to be obtained. This may remove edges in a cascade, a feature that does not appear on graphs. Note that item (1) of the gluing condition may also be breached when an edge is marked more than once for removal, in which case it is deleted, but also when an edge is marked both for removal and for preservation. Example 7.8 shows that in such cases the edge is also removed. All edges marked for removal are guaranteed to be deleted, and the other edges are preserved only if this does not conflict with deletions. This semantics of transformation rules is thus dual to the previous one, and should be more appealing to the daring (or lazy) programmer.

9 Attributed Typed Monographs

The notion of E-graph has been designed in [2] in order to obtain an adhesive category of graphs with attributed nodes and edges. This follows from a line of studies on Typed Attributed Graph Transformations, see [22, 23, 24]. The attributes are taken in a data type algebra and may be of different sorts (booleans, integers, strings, etc.). In the case of E-graphs only the nodes of sort `values` represent such attributes. But they are also typed by E-graphs, and in the type E-graphs each node of sort `values` represent a sort of the data type algebra. This should recall the constructions of Section 6 that we now use in order to generalize the notion of typed attributed graphs given in [2]. The idea is similarly to impose that the edges typed by a sort of a data type algebra are the elements of the corresponding carrier set.

Definition 9.1 (categories $\mathbf{ATM}(T, \Sigma)$)**.** For any monograph T and signature $\Sigma : \Omega \to S^{<\omega}$, an *attributed typed monograph* (ATM for short) over T, Σ is a pair $(a, \mathcal{A})$ of an object $a : A \to T$ in $\mathbf{Monogr}\backslash T$ and a Σ-algebra $\mathcal{A}$ such that $\mathcal{A}_s = (\mathsf{A}_T a)_s$ for all $s \in S \cap \mathsf{E}T$.

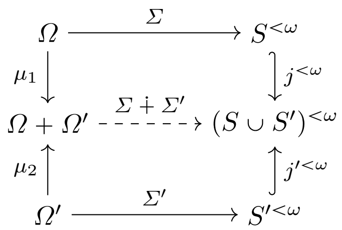
A *morphism* m from $(a, \mathcal{A})$ to an ATM $(b, \mathcal{B})$ over T, Σ is a pair $(\vec{m}, \dot{m})$ of a morphism $\vec{m} : a \to b$ in $\mathbf{Monogr}\backslash T$ and a Σ-homomorphism $\dot{m} : \mathcal{A} \to \mathcal{B}$ such that $\dot{m}_s = (\mathsf{A}_T \vec{m})_s$ for all $s \in S \cap \mathsf{E}T$.

Let $1_{(a,\mathcal{A})} \overset{\mathrm{def}}{=} (1_a, 1_{\mathcal{A}})$ and for any morphism $m' : (b, \mathcal{B}) \to (c, \mathcal{C})$ let $m' \circ m \overset{\mathrm{def}}{=} (\vec{m}' \circ \vec{m}, \dot{m}' \circ \dot{m})$ that is a morphism from $(a, \mathcal{A})$ to $(c, \mathcal{C})$. Let $\mathbf{ATM}(T, \Sigma)$ be the category of ATMs over T, Σ and their morphisms.

The edges that are considered as attributes are not the nodes of a specific sort as in E-graphs; they are characterized by the fact that they are typed by an edge of T that happens to be also a sort of the data type signature Σ, i.e., an element of S. This is consistent with the typed attributed E-graphs of [2].

We therefore see that the signatures $\mathsf{S}T$ and Σ share sorts but we shall consider them as otherwise distinct, in particular w.r.t. operator names. To account for this property we need the following construction.

Definition 9.2 (signature $\Sigma \dotplus \Sigma'$)**.** Given two signatures $\Sigma : \Omega \to S^{<\omega}$ and $\Sigma' : \Omega' \to S'^{<\omega}$, let $(\Omega + \Omega', \mu_1, \mu_2)$ be the coproduct of (Ω, Ω') in **Set** and j, j' be the inclusion functions of S, S' respectively into $S \cup S'$, let $\Sigma \dotplus \Sigma' : \Omega + \Omega' \to (S \cup S')^{<\omega}$ be the unique function such that $(\Sigma \dotplus \Sigma') \circ \mu_1 = j^{<\omega} \circ \Sigma$ and $(\Sigma \dotplus \Sigma') \circ \mu_2 = j'^{<\omega} \circ \Sigma'$.



We leave it to the reader to check that this construction defines a coproduct in the category $\mathbf{Sig}_{\text{srt}}$ and therefore that $\Sigma_1 \doteq \Sigma_2$ and $\Sigma'_1 \doteq \Sigma'_2$ entail $\Sigma_1 \dotplus \Sigma'_1 \doteq \Sigma_2 \dotplus \Sigma'_2$. For the sake of simplicity we will assume in the sequel that $\mathsf{S}T$ and Σ have no operator name in common, thus assimilate $\Omega_T + \Omega$ to $\Omega_T \cup \Omega$ and omit the inclusion functions, so that $\mathsf{S}T = (\mathsf{S}T \dotplus \Sigma)|_{\Omega_T}^{(\mathsf{E}T)^{<\omega}}$ and $\Sigma = (\mathsf{S}T \dotplus \Sigma)|_{\Omega}^{S^{<\omega}}$.

Definition 9.3 (functor $\mathsf{D} : \mathbf{ATM}(T, \Sigma) \to (\mathsf{S}T \dotplus \Sigma)$-$\mathbf{Alg}$)**.** For every signature $\Sigma : \Omega \to S^{<\omega}$ and monograph T such that $\Omega_T \cap \Omega = \varnothing$, let $\Sigma' \stackrel{\text{def}}{=} \mathsf{S}T \dotplus \Sigma$ and $\mathsf{D} : \mathbf{ATM}(T, \Sigma) \to \Sigma'$-$\mathbf{Alg}$ be the functor defined as follows: for every object $(a, \mathcal{A})$ of $\mathbf{ATM}(T, \Sigma)$ let $\mathsf{D}(a, \mathcal{A})$ be the Σ'-algebra $\mathcal{A}'$ defined by

- $\mathcal{A}'_s \stackrel{\text{def}}{=} \mathcal{A}_s$ for all $s \in S$ and $\mathcal{A}'_e \stackrel{\text{def}}{=} (\mathsf{A}_T a)_e$ for all $e \in \mathsf{E}T$,
- $o^{\mathcal{A}'} \stackrel{\text{def}}{=} o^{\mathcal{A}}$ for all $o \in \Omega$ and $[e{\cdot}\iota]^{\mathcal{A}'} \stackrel{\text{def}}{=} [e{\cdot}\iota]^{\mathsf{A}_T a}$ for all $[e{\cdot}\iota] \in \Omega_T$.

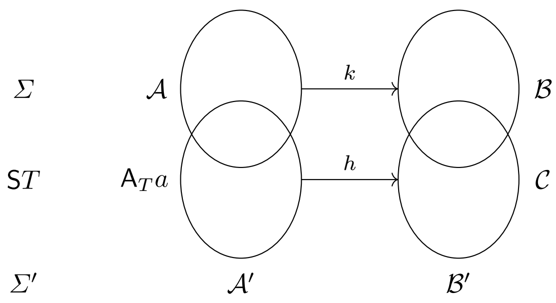
For every morphism $m : (a, \mathcal{A}) \to (b, \mathcal{B})$, let $(\mathsf{D}m)_s \stackrel{\text{def}}{=} \dot{m}_s$ for all $s \in S$ and $(\mathsf{D}m)_e \stackrel{\text{def}}{=} (\mathsf{A}_T \vec{m})_e$ for all $e \in \mathsf{E}T$.

It is straightforward to check that $\mathsf{D}m$ is a Σ'-homomorphism from $\mathsf{D}(a, \mathcal{A})$ to $\mathsf{D}(b, \mathcal{B})$, and hence that D is a functor.

Theorem 9.4. *D is an equivalence from $\mathbf{ATM}(T, \Sigma)$ to $(\mathsf{S}T \dotplus \Sigma)$-$\mathbf{Alg}$.*

Proof. It is easy to see that D is full and faithful by the same property of A_T.

We prove that D is isomorphism-dense. For any Σ'-algebra $\mathcal{B}'$, let $\mathcal{B}$ (resp. $\mathcal{C}$) be its restriction to Σ (resp. $\mathsf{S}T$). Since A_T is isomorphism-dense by Theorem 6.8, there exist an object $a : A \to T$ in $\mathbf{Monogr}\backslash T$ and an $\mathsf{S}T$-isomorphism $h : \mathsf{A}_T a \to \mathcal{C}$. We define simultaneously a set $\mathcal{A}_s$ and a function $k_s : \mathcal{A}_s \to \mathcal{B}_s$ for all $s \in S$ by taking $\mathcal{A}_s \stackrel{\text{def}}{=} \mathcal{B}_s$ and $k_s \stackrel{\text{def}}{=} 1_{\mathcal{A}_s}$ if $s \in S \backslash \mathsf{E}T$, and $\mathcal{A}_s \stackrel{\text{def}}{=} (\mathsf{A}_T a)_s$ and $k_s \stackrel{\text{def}}{=} h_s$ if $s \in S \cap \mathsf{E}T$ (in this case we have $\mathcal{C}_s = \mathcal{B}'_s = \mathcal{B}_s$). We then define for every $o \in \Omega$ the function $o^{\mathcal{A}} \stackrel{\text{def}}{=} k^{-1}_{\mathrm{Rng}(o)} \circ o^{\mathcal{B}} \circ k_{\mathrm{Dom}(o)} : \mathcal{A}_{\mathrm{Dom}(o)} \to \mathcal{A}_{\mathrm{Rng}(o)}$, and the Σ-algebra $\mathcal{A} \stackrel{\text{def}}{=} \big((\mathcal{A}_s)_{s \in S}, (o^{\mathcal{A}})_{o \in \Omega}\big)$. By construction $(a, \mathcal{A})$ is obviously an ATM over T, Σ and $k \stackrel{\text{def}}{=} (k_s)_{s \in S}$ is a Σ-isomorphism $k : \mathcal{A} \to \mathcal{B}$.



Let $\mathcal{A}' \stackrel{\text{def}}{=} \mathsf{D}(a, \mathcal{A})$, $h'_s \stackrel{\text{def}}{=} k_s : \mathcal{A}'_s \to \mathcal{B}'_s$ for all $s \in S$ and $h'_e \stackrel{\text{def}}{=} h_e : \mathcal{A}'_e \to \mathcal{B}'_e$ for all $e \in \mathsf{E}T$, since $h_s = k_s$ for all $s \in S \cap \mathsf{E}T$ then $h' \stackrel{\text{def}}{=} (h'_s)_{s \in S \cup \mathsf{E}T}$ is well-defined. It is then easy to see that $h' : \mathcal{A}' \to \mathcal{B}'$ is a Σ'-isomorphism, so that $\mathsf{D}(a, \mathcal{A}) \simeq \mathcal{B}'$. □

Theorem 9.4 generalizes[3] [2, Theorem 11.3] that establishes an isomorphism between the category of attributed E-graphs typed by an attributed E-graph ATG and the category of algebras of a signature denoted AGSIG(ATG). In particular Theorem 11.3 of

[3] Our proof is also much shorter than the 6 pages taken by the corresponding result on attributed typed E-graphs. This is due partly to our use of A_T (Definition 6.7) and of Theorem 6.8, but also to the simplicity of monographs compared to the 5 sorts and 6 operator names of E-graphs.

[2] requires the hypothesis that AGSIG(ATG) should be *well-structured*, which means that if there is an operator name of $\mathsf{S}T$ whose domain sort is s then s is not a sort of the data type signature Σ. Obviously this is equivalent to requiring that only nodes of T can be considered as sorts of Σ and is linked to the fact that only `values` nodes of E-graphs are supposed to hold attributes. Since we are not restricted to E-graphs there is no need to require that attributes should only be nodes. This has an interesting consequence:

Corollary 9.5. *For every signatures Σ, Σ' and graph structure Γ such that $\Sigma' = \Gamma \dotplus \Sigma$ there exists a monograph T such that Σ'-$\mathbf{Alg} \approx \mathbf{ATM}(T, \Sigma)$.*

Proof. By Lemma 6.3 there exists a monograph T such that $\mathsf{S}T \doteq \Gamma$, hence $\mathsf{S}T \dotplus \Sigma \doteq \Gamma \dotplus \Sigma = \Sigma'$ and therefore Σ'-$\mathbf{Alg} \simeq (\mathsf{S}T \dotplus \Sigma)$-$\mathbf{Alg} \approx \mathbf{ATM}(T, \Sigma)$. □

Obviously, any signature Σ' can be decomposed as $\Gamma \dotplus \Sigma$ by putting some of its monadic operators (and the sorts involved in these) in Γ and all other operators in Σ. And then any Σ'-algebra can be represented as an ATM over T, Σ, where $\mathsf{S}T \doteq \Gamma$. This opens the way to applying graph transformations to these algebras, but this requires some care since it is not generally possible to remove or add elements to a Σ'-algebra and obtain a Σ'-algebra as a result.

The approach adopted in [2, Definition 11.5] is to restrict the morphisms used in span rules to a class of monomorphisms that are extensions of Σ-isomorphisms to $(\Gamma \dotplus \Sigma)$-homomorphisms. It is then possible to show [2, Theorem 11.11] that categories of typed attributed E-graphs are adhesive HLR categories (a notion that generalizes Definition 4.13, see [25]) w.r.t. this class of monomorphisms.

A similar result holds on categories of ATMs. For the sake of simplicity, and since rule-based graph transformations are unlikely to modify attributes such as booleans, integers or strings (and if they do they should probably not be considered as graph transformations), we will only consider morphisms that leave the data type algebra unchanged, element by element. This leaves the possibility to transform the edges whose sort is in Γ but not in Σ.

Definition 9.6 (categories $\mathbf{ATM}(T, \mathcal{A})$, functor U, f stabilizes $\mathcal{A}$)**.** For any Σ-algebra $\mathcal{A}$ let $\mathbf{ATM}(T, \mathcal{A})$ be the subcategory of $\mathbf{ATM}(T, \Sigma)$ restricted to objects $(a, \mathcal{A})$ and morphisms $(f, 1_{\mathcal{A}})$.

The *forgetful functor* $\mathsf{U} : \mathbf{ATM}(T, \mathcal{A}) \to \mathbf{Set}$ is defined by $\mathsf{U}(a, \mathcal{A}) \stackrel{\text{def}}{=} \mathsf{E}A$, where $a : A \to T$ and $\mathsf{U}(f, 1_{\mathcal{A}}) \stackrel{\text{def}}{=} \mathsf{E}f$ (usually denoted f).

By abuse of notation we write $\mathcal{A}$ for the set $\bigcup_{s \in S \cap \mathsf{E}T} \mathcal{A}_s$. A function f *stabilizes* $\mathcal{A}$ if $f^{-1}[x] = \{x\}$ for all $x \in \mathcal{A}$.

The proof that the categories $\mathbf{ATM}(T, \mathcal{A})$ are adhesive will only be sketched below. The key point is the following lemma.

Lemma 9.7. *For all objects $(a, \mathcal{A})$, $(b, \mathcal{A})$ of $\mathbf{ATM}(T, \mathcal{A})$ and morphism $f : a \to b$ of $\mathbf{Monogr}\backslash T$, we have*

$$(f, 1_{\mathcal{A}}) : (a, \mathcal{A}) \to (b, \mathcal{A}) \text{ is a morphism in } \mathbf{ATM}(T, \mathcal{A}) \text{ iff } f \text{ stabilizes } \mathcal{A}.$$

Proof. For all $s \in S \cap \mathsf{E}T$ we have $\mathcal{A}_s = (\mathsf{A}_T a)_s = a^{-1}[s]$ and $\mathcal{A}_s = b^{-1}[s]$. Since $b \circ f = a$ then $f^{-1}[\mathcal{A}_s] = f^{-1}[b^{-1}[s]] = a^{-1}[s] = \mathcal{A}_s$, hence $f^{-1}[\mathcal{A}] = \mathcal{A}$. Thus f stabilizes $\mathcal{A}$ iff $f(x) = x$ for all $x \in \mathcal{A}$ iff $(\mathsf{A}_T f)_s = f|_{\mathcal{A}_s}^{\mathcal{A}_s} = \mathrm{Id}_{\mathcal{A}_s} = (1_{\mathcal{A}})_s$ for all $s \in S \cap \mathsf{E}T$ iff $(f, 1_{\mathcal{A}})$ is a morphism in $\mathbf{ATM}(T, \mathcal{A})$. □

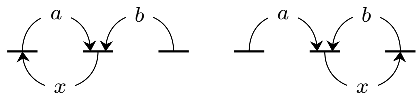
Hence the property of stabilization characterizes the difference between morphisms in $\mathbf{Monogr}\backslash T$ and morphisms in $\mathbf{ATM}(T, \mathcal{A})$. Besides, it is well-known how pushouts and pullbacks in $\mathbf{Monogr}\backslash T$ can be constructed from those in $\mathbf{Monogr}$, and we have seen that these can be constructed from those in $\mathbf{Set}$.

But then it is quite obvious that in $\mathbf{Set}$, starting from a span of functions that stabilize $\mathcal{A}$, it is always possible to find as pushout a cospan of functions that stabilize $\mathcal{A}$. Hence not only does $\mathbf{ATM}(T, \mathcal{A})$ have pushouts, but these are preserved by the functor U. A similar result holds for pullbacks, and a construction similar to Corollary 4.7 yields that U also preserves monomorphisms. Finally, we see that U reflects isomorphisms since f^{-1} stabilizes $\mathcal{A}$ whenever f does. We conclude as in Theorem 4.17.

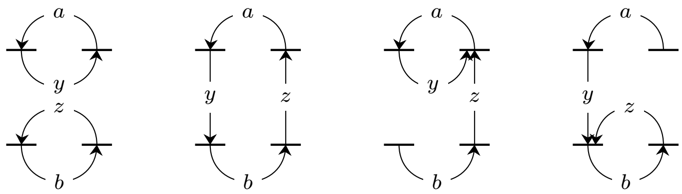
Theorem 9.8. $\mathbf{ATM}(T, \mathcal{A})$ *is adhesive.*

This result does not mean that all edges that are not attributes can be freely transformed. Their adjacencies to or from attributes may impose constraints that only few morphisms are able to satisfy.

Example 9.9. Let Σ be the signature with no operation name and one sort $\mathtt{s}$, and $\mathcal{A}$ be the Σ-algebra defined by $\mathcal{A}_{\mathtt{s}} = \{a, b\}$. We consider the type monograph $T = \{(e, \mathtt{s}), (\mathtt{s}, e)\}$. A monograph typed by T has any number (but at least one) of edges typed by e that must be adjacent either to a or b, and two edges typed by $\mathtt{s}$, namely a and b, that must be adjacent to either the same edge x typed by e, which yields two classes of monographs



(to which may be added any number of edges typed by e and adjacent to either a or b), or a and b are adjacent to y and z respectively, and we get four more classes:



The function $y, z \mapsto x$ is a morphism from these last two monographs to the two monographs above (respectively). There are no other morphisms between monographs from distinct classes. We therefore see that in the category $\mathbf{ATM}(T, \mathcal{A})$ it is possible to add or remove edges typed by e to which a or b are not adjacent, but there is no way to remove the edges y and z (because this would require a rule with a left morphism from an ATM without y and z to an ATM with y and z, and there is no such morphism), though they are not attributes.

Besides, we see that this category has no initial object, no terminal object, no products nor coproducts.

10 Conclusion

Monographs generalize standard notions of directed graphs by allowing edges of any length with free adjacencies. An edge of length zero represents a node, and if it has greater length it can be adjacent to any edge, including itself. In "monograph" the prefix mono- is justified by this unified view of nodes as edges and of edges with unrestricted adjacencies that provide formal conciseness (morphisms are functions characterized by a single equation); the suffix -graph is justified by the correspondence (up to isomorphism) between finite ω-monographs and their drawings.

Monographs are universal with respect to graph structures and the corresponding algebras, in the sense that monographs are equivalent to graph structures extended with suitable ordering conventions on their operator names, and that categories of typed monographs are equivalent to the corresponding categories of algebras (i.e., to the presheaf toposes). Since many standard or exotic notions of directed graphs can be represented as monadic algebras, they can also be represented as typed monographs, but these have two advantages over graph structures: they provide an orientation of edges and they (consequently) dispense with operator names.

Algebraic transformations of monographs are similar to those of standard graphs. Typed monographs may therefore be simpler to handle than graph structured algebras, as illustrated by the results of Section 9. The representation of oriented edges as sequences seems more natural than their standard representation as unstructured objects that have images by a bunch of functions. Thus type monographs emerge as a natural way of specifying graph structures.

References

[1] T. Boy de la Tour, R. Echahed, Parallel rewriting of attributed graphs, Theoretical Computer Science 848 (2020) 106–132.

[2] H. Ehrig, K. Ehrig, U. Prange, G. Taentzer, Fundamentals of Algebraic Graph Transformation, Monographs in Theoretical Computer Science. An EATCS Series, Springer, 2006.

[3] M. Löwe, Algebraic approach to single-pushout graph transformation, Theoretical Computer Science 109 (1993) 181–224.

[4] T. Boy de la Tour, Monographs, a category of graph structures, in: Recent Trends in Algebraic Development Techniques, 25th International Workshop, WADT 2020, Revised Selected Papers, Vol. 12669 of LNCS, Springer, 2021, pp. 54–74.

[5] P. Suppes, Axiomatic Set Theory, Dover Publications, Inc., 1972.

[6] D. Sannella, A. Tarlecki, Foundations of Algebraic Specification and Formal Software Development, Monographs in Theoretical Computer Science. An EATCS Series, Springer, 2012.

[7] H. Herrlich, G. E. Strecker, Category Theory, 3rd Edition, Heldermann Verlag, Berlin, 2007.

[8] S. Lack, P. Sobocinski, Adhesive and quasiadhesive categories, Informatique Théorique et Applications 39 (3) (2005) 511–545.

[9] P. Johnstone, S. Lack, P. Sobociński, Quasitoposes, quasiadhesive categories and artin glueing, in: T. Mossakowski, U. Montanari, M. Haveraaen (Eds.), CALCO 2007, Vol. 4624 of LNCS, Springer, 2007, pp. 312–326.

[10] D. Plump, Term graph rewriting, in: H. Ehrig, G. Engels, H.-J. Kreowski, G. Rozenberg (Eds.), Handbook of Graph Grammars and Computing by Graph Transformation, Volume 2: Applications, Languages and Tools, World Scientific, 1999, pp. 3–61.

[11] A. Burroni, Higher-dimensional word problems with applications to equational logic, Theoretical Computer Science 115 (1) (1993) 43–62.

[12] E. Robinson, G. Rosolini, Categories of partial maps, Information and Computation 79 (2) (1988) 95–130.

[13] A. Corradini, T. Heindel, F. Hermann, B. König, Sesqui-pushout rewriting, in: ICGT 2006, Vol. 4178 of LNCS, Springer, 2006, pp. 30–45.

[14] A. Corradini, D. Duval, R. Echahed, F. Prost, L. Ribeiro, AGREE - algebraic graph rewriting with controlled embedding, in: 8th ICGT, Vol. 9151 of LNCS, Springer, 2015, pp. 35–51.

[15] A. Corradini, D. Duval, R. Echahed, F. Prost, L. Ribeiro, The PBPO graph transformation approach, J. Log. Algebr. Meth. Program. 103 (2019) 213–231.

[16] T. Boy de la Tour, R. Echahed, Parallel coherent graph transformations, in: Recent Trends in Algebraic Development Techniques, 25th International Workshop, WADT 2020, Revised Selected Papers, Vol. 12669 of LNCS, Springer, 2021, pp. 75–97.

[17] H. Ehrig, M. Pfender, H. J. Schneider, Graph-grammars: An algebraic approach, in: 14th Annual Symposium on Switching and Automata Theory, Iowa City, Iowa, USA, October 15-17, 1973, 1973, pp. 167–180.

[18] A. Corradini, U. Montanari, F. Rossi, H. Ehrig, R. Heckel, M. Löwe, Algebraic approaches to graph transformation - part I: basic concepts and double pushout approach, World Scientific, 1997, pp. 163–246.

[19] A. Habel, J. Müller, D. Plump, Double-pushout graph transformation revisited, Math. Struct. Comput. Sci 11 (5) (2001) 637–688.

[20] J.-C. Raoult, On graph rewritings, Theoretical Computer Science 32 (1,2) (1984) 1–24.

[21] M. Löwe, Extended algebraic graph transformation, Ph.D. thesis, Technical University of Berlin, Germany (1991).

[22] M. Löwe, M. Korff, A. Wagner, An algebraic framework for the transformation of attributed graphs, in: R. Sleep, R. Plasmeijer, M. van Eekelen (Eds.), Term Graph Rewriting: Theory and Practice, John Wiley, New York, 1993, pp. 185–199.

[23] R. Heckel, J. M. Küster, G. Taentzer, Confluence of typed attributed graph transformation systems, in: A. Corradini, H. Ehrig, H. Kreowski, G. Rozenberg (Eds.), Graph Transformation, First International Conference ICGT 2002, Vol. 2505 of LNCS, Springer, 2002, pp. 161–176.

[24] H. Ehrig, K. Ehrig, U. Prange, G. Taentzer, Fundamental theory for typed attributed graphs and graph transformation based on adhesive HLR categories, Fundam. Informaticae 74 (1) (2006) 31–61.

[25] H. Ehrig, J. Padberg, U. Prange, A. Habel, Adhesive high-level replacement systems: A new categorical framework for graph transformation, Fundam. Informaticae 74 (1) (2006) 1–29.