Yoneda A_∞-algebras and lattices of vector spaces

Estanislao Herscovich

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Abstract

Generalizing some clever computations for monomial algebras by J. Chuang and A. King in an unpublished article, we provide a sufficient criterion that allows in some cases to explicitly compute the A_∞-algebra structure of the Yoneda algebra $\operatorname{Ext}^\bullet_A(K,K)$ of a nonnegatively graded connected algebra $A = TV/I$ in the case where the A_∞-algebra structure of the Yoneda algebra is given as the dual of (signed) inclusions appearing in the lattice of submodules of TV representing the Tor groups $\operatorname{Tor}^A_p(K,K)$ for $p \in \mathbb{N}$. This includes the case of monomial algebras A considered by Chuang and King, but also generalized Koszul algebras, and other nice algebras. The previous nice description of the A_∞-(co)algebra structure is however far from universal. Indeed, we also give an example of a nonnegatively graded connected algebra A where the A_∞-algebra structure of the Yoneda algebra can never be obtained by dualizing the (signed) inclusions of the Tor groups.

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1 Introduction

It is well known that, given an augmented K-algebra A, where K is a finite product of copies of a field $\Bbbk$, the Yoneda algebra $\operatorname{Ext}^\bullet_A(K,K)$ has a unique (up to non-canonical isomorphism) A_∞-algebra structure. Explicit computations are typically performed by using Homological Perturbation Theory (HPT), which is based on some homotopical information on a projective resolution of the trivial A-module K and typically requires a good deal of calculations. Under some further assumptions on the resolution, one can describe the required homotopical information by combinatorial data, which improves the computational efficiency, under what is usually known as Algebraic Discrete Morse Theory (ADMT) (see [17]). A drawback of this point of view is that in some cases the resolution one is dealing with does not naturally verify the extra assumptions of ADMT, and in many situations the level of computations is still rather high.

The goal of this article is to provide a general –albeit not universal– approach to the determination of the A_∞-algebra on $\operatorname{Ext}^\bullet_A(K,K)$ for a nonnegatively graded connected algebra $A = TV/I$ based on the lattice structure of $\operatorname{Tor}^A_p(K,K)$ when the latter are realized by subspaces of the tensor algebra TV. More precisely, we give a sufficient criterion that allows in some cases to explicitly compute the A_∞-algebra structure of the Yoneda algebra $\operatorname{Ext}^\bullet_A(K,K)$, where the latter is given by the dual of the (signed) inclusions appearing in the lattice of submodules of TV

*Institut Fourier, UMR 5582, Laboratoire de Mathématiques, Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France.

representing the Tor groups $\mathrm{Tor}_p^A(K,K)$ for $p \in \mathbb{N}$ (see Theorem 3.13). This can be considered as the simplest possible case of A_∞-algebra structure on the Yoneda algebra $\mathrm{Ext}_A^\bullet(K,K)$.

Our motivation comes from the unpublished article [5], where the authors computed the A_∞-algebra structure of the Yoneda algebra $\mathrm{Ext}_A^\bullet(K,K)$ for any monomial K-algebra A by cleverly studying the internal structure of the spaces involved, but also from the characterization of minimal projective resolutions in [4]. The aim of this work is to show that the results in that article can be extended to other families of nonnegatively graded connected algebras. Indeed, this is the case for N-Koszul algebras A, for which we immediately reobtain the A_∞-algebra structure of $\mathrm{Ext}_A^\bullet(K,K)$ computed in [9]. We also present an example for the case of an algebra that is neither monomial nor generalized Koszul. The previous nice description of the A_∞-(co)algebra structure of $\mathrm{Ext}_A^\bullet(K,K)$ (resp., $\mathrm{Tor}_\bullet^A(K,K)$) is however far from universal. Indeed, we provide an example of a nonnegatively graded connected algebra A where the A_∞-algebra structure of the Yoneda algebra can never be obtained by dualizing the (signed) inclusions of the Tor groups.

I would like to express my deep gratitude to Professor Chuang for recently sharing with me [5].

2 Preliminaries

2.1 Notation

We will denote by $\mathbb{N}_0$ (resp., $\mathbb{N}$) the set of nonnegative (resp., positive) integers $\{0,1,2,\dots\}$ (resp., $\{1,2,\dots\}$), and given $n',n'' \in \mathbb{N}_0$ we denote by $[\![n',n'']\!]$ the set $\{n \in \mathbb{N}_0 : n' \leq n \leq n''\}$. If $\bar{p} = (p_1,\dots,p_i) \in \mathbb{N}_0^i$ for $i \in \mathbb{N}$, denote $|\bar{p}| = p_1 + \cdots + p_i$. Given $\bar{p} = (p_1,\dots,p_i) \in \mathbb{N}^i$ for $i \in \mathbb{N}$, let $\mathscr{o}(\bar{p}) = |\bar{p}| + 2 - i$. It is clear that $\mathscr{o}(\bar{p}) \geq 2$. Given integers $1 \leq j < k \leq i$, let $\mathfrak{s}_{j,k} : \mathbb{N}_0^i \to \mathbb{N}_0^{k-j+1}$ be the map sending $\bar{p} = (p_1,\dots,p_i) \in \mathbb{N}_0^i$ to $(p_j,\dots,p_k)$.

2.2 Graded modules

Let $\Bbbk$ be a field and K be a $\Bbbk$-algebra such that $K^{\mathrm{e}} = K \otimes_{\Bbbk} K^{\mathrm{op}}$ is semisimple. By **graded module** we mean a left module V over K^{e} endowed with a direct sum decomposition $V = \oplus_{n\in\mathbb{Z}} V_n$ of left K^{e}-modules. Moreover, a **homogeneous morphism of graded modules** $f : V \to W$ of degree $d \in \mathbb{Z}$ is a morphism of left K^{e}-modules such that $f(V_n) \subseteq W_{n+d}$ for all $n \in \mathbb{Z}$. We will denote by ${}_{K^{\mathrm{e}}}\mathrm{Mod}$ the category of modules with homogeneous morphisms of degree zero, which is endowed with a natural tensor product $\otimes_K$ having the grading whose n-th homogeneous component is

$$(V \otimes_K W)_n = \bigoplus_{m\in\mathbb{Z}} V_m \otimes_K W_{n-m},$$

for all $n \in \mathbb{Z}$. We will denote the tensor product simply by $\otimes$. The category ${}_{K^{\mathrm{e}}}\mathrm{Mod}$ endowed with the previous tensor product and the unit given by K concentrated in degree zero is a symmetric monoidal category.

2.3 Graded submodules of the tensor algebra

From now on we assume that K is a finite product of copies of $\Bbbk$, and we fix a complete set $\{e_1,\dots,e_\ell\}$ of orthogonal idempotents of K. Let V be a positively graded module. Recall that the **tensor algebra** TV is the graded module $\oplus_{n\in\mathbb{N}_0} V^{\otimes n}$, with the product given by concatenation and the unit $K = V^{\otimes 0} \to TV$ given by the canonical inclusion. It is clear that TV is a graded algebra with the induced grading by that of V. To reduce the notation we will denote the concatenation (or tensor) product of TV simply by juxtaposition or a dot.

Let $U \subseteq TV$ be a graded submodule. We recall that U is **left** (**right**) **tensor-intersection faithful** if $(U.W_1) \cap (U.W_2) = U.(W_1 \cap W_2)$ (resp., $(W_1.U) \cap (W_2.U) = (W_1 \cap W_2).U$) for all graded submodules $W_1, W_2 \subseteq T(V)$. Moreover, U is said to be **tensor-intersection faithful** if it is left and right tensor-intersection faithful. It is immediate to see that, if $\{W_i : i \in I\}$ is an independent family of graded submodules of TV (*i.e.* given elements $w_{i_1} \in W_{i_1}, \dots, w_{i_n} \in W_{i_n}$, with $i_j \neq i_{j'}$ if $j \neq j'$, such that

$$w_{i_1} + \cdots + w_{i_n} = 0,$$

then $w_{i_1} = \cdots = w_{i_n} = 0$) and $U \subseteq TV$ is left (resp., right) tensor-intersection faithful, then $\{U.W_1, \dots, U.W_n\}$ (resp., $\{W_1.U, \dots, W_n.U\}$) is also an independent family (see [10], Cor. 3.8).

The following simple characterization of tensor-intersection faithfulness was proved in [12], Prop. 3.3 (see also [10], Prop. 3.2.). We recall first that, given two families $\{u_i : i \in I\}$ and $\{w_i : i \in I\}$ of elements of TV, we say that $\{u_i : i \in I\}$ is **left** (resp., **right**) **composable** with respect to $\{w_i : i \in I\}$ if, for all $i \in I$ and for all idempotents $e \in K$, $u_i.e \neq 0$ (resp., $e.u_i \neq 0$) holds whenever $e.w_i \neq 0$ (resp., $w_i.e \neq 0$).

Proposition 2.1. *Let $U \subseteq TV$ be a graded submodule. Then, the following conditions are equivalent:*

(i) U is left (resp., right) tensor-intersection faithful;

(ii) $U \cap (U.(TV)_{>0}) = 0$ (resp., $U \cap (TV)_{>0}.U) = 0$);

(iii) given a finite set of indices I and a linearly independent set $\{u_i : i \in I\} \subseteq U$ (for the underlying vector space structure of U over $\Bbbk$) that is left (resp., right) composable with respect to a collection of arbitrary $w_i \in TV$ for $i \in I$, if $\sum_{i \in I} u_i \otimes w_i$ (resp., $\sum_{i \in I} w_i \otimes u_i$) vanishes, then $w_i = 0$ for all $i \in I$.

As noted in [10], Cor. 3.7, the implication (ii) $\Rightarrow$ (iii) gives the following result.

Corollary 2.2. *Let $U \subseteq TV$ be left (resp., right) tensor-intersection faithful. If $\{U_i\}_{i \in I}$ is an arbitrary family of independent graded submodules of U, then the family of graded submodules of the tensor algebra given by $\{U_i.(TV)\}_{i \in I}$ (resp., $\{(TV).U_i\}_{i \in I}$) is also independent.*

2.4 Basics on graded algebras

A **nonnegatively graded connected algebra** over K is a unital associative algebra (A, μ_A, η_A) in the monoidal category ${}_{K^e}\mathrm{Mod}$ whose underlying graded module $A = \oplus_{n \in \mathbb{N}_0} A_n$ is concentrated in nonnegative degrees and $A_0 = K$. The grading of A and of all the graded modules considered in this subsection is called the **Adams grading**. We remark that when considering the Adams grading, the Koszul sign rule is trivial, *i.e.* all signs are $+1$. We call $\mu_A : A \otimes A \to A$ the **product** and $\eta_A : K \to A$ the **unit**. As usual, we will denote the nonnegatively graded connected algebra simply by A and the product of elements of A simply by a dot or juxtaposition.

If A is a nonnegatively graded connected algebra, set $A_{>0} = \oplus_{n \in \mathbb{N}} A_n$. Then, the graded vector space $A_{>0}/(A_{>0} \cdot A_{>0})$ is called the **space of generators** of A. Let $\bar{s} : A_{>0}/(A_{>0} \cdot A_{>0}) \to A_{>0}$ be a (homogeneous) section of the canonical projection $A_{>0} \to A_{>0}/(A_{>0} \cdot A_{>0})$ and V its image. It is easy to see that the space of generators is concentrated in positive degrees, *i.e.* $V = \oplus_{n \in \mathbb{N}} V_n$. The inclusion $V \subseteq A_{>0}$ induces a surjective homogeneous morphism of degree zero $\pi : TV \to A$ and the kernel $I = \mathrm{Ker}(\pi)$ is included in $(TV)_{>0} \cdot (TV)_{>0}$, where $(TV)_{>0} = \oplus_{n \in \mathbb{N}} V^{\otimes n}$. We will assume from now on that the space of generators $V = \oplus_{n\in\mathbb{N}} V_n$ is **locally finite dimensional** over $\Bbbk$, *i.e.* for every $n \in \mathbb{N}$ the underlying vector space V_n has finite dimension over $\Bbbk$.

The graded module $I/((TV)_{>0} \cdot I + I \cdot (TV)_{>0})$ is called the **space of relations** of A. Let $\tilde{s} : I/((TV)_{>0} \cdot I + I \cdot (TV)_{>0}) \to I$ be a (homogeneous) section of the canonical projection $I \to I/((TV)_{>0} \cdot I + I \cdot (TV)_{>0})$ and R its image. Then, it is easy to see that I is the ideal generated by R, so $TV/\langle R\rangle \simeq A$.

2.5 Tor and Ext groups of graded algebras

Assume that $A = TV/I$ is a nonnegatively graded connected algebra, where $V \subseteq A_{>0}$ and $I \subseteq (TV)_{>0} \cdot (TV)_{>0}$ are the graded modules introduced in the previous subsection. Then, $K = A/A_{>0}$ is a graded A-bimodule and we can consider $\mathrm{Tor}^A_\bullet(K, K)$ as well as $\mathrm{Ext}^\bullet_A(K, K)$, where K is regarded either as a left or right A-module. We note that for either choice of structure of A-module on K, both Ext groups coincide, and they are given as the dual of $\mathrm{Tor}^A_\bullet(K, K)$, either as a left K-module or a right K-module (see [13], Prop. 2.13).

Given $(q, \delta) \in \mathbb{N}_0 \times \{0, 1\}$ such that $(q, \delta) \neq (0, 0)$, we consider the graded submodules of $(TV)_{>0}$ given by

$$\begin{aligned} Y_{2q+\delta} &= \big((TV)^\delta_{>0} \cdot I^q\big) \cap \big(I^q \cdot (TV)^\delta_{>0}\big) \cap \big((TV)^{1-\delta}_{>0} \cdot I^{q-1} \cdot (TV)^{1-\delta}_{>0}\big), \\ X_{2q+\delta} &= (TV)_{>0} \cdot I^q \cdot (TV)^\delta_{>0} + (TV)^\delta_{>0} \cdot I^q \cdot (TV)_{>0} + I^{q+1}, \end{aligned} \tag{2.1}$$

where $(TV)^0_{>0} = K$ and $(TV)^1_{>0} = (TV)_{>0}$. Note that

$$Y_{2q+1} = \big((TV)_{>0} \cdot I^q\big) \cap \big(I^q \cdot (TV)_{>0}\big) \text{ and } X_{2q} = (TV)_{>0} \cdot I^q + I^q \cdot (TV)_{>0},$$

since $((TV)_{>0} \cdot I^q) \cap (I^q \cdot (TV)_{>0}) \subseteq I^{q-1}$, and $I^{q+1} \subseteq I^q \cdot (TV)_{>0}$, respectively. Moreover, it is easy to see that $Y_{p+1} \subseteq X_p \subseteq Y_p$ for all $p \in \mathbb{N}$. We remark that $Y_1 = (TV)_{>0}$, $X_1 = (TV)_{>0} \cdot (TV)_{>0}$, $Y_2 = I$ and $X_2 = (TV)_{>0} \cdot I + I \cdot (TV)_{>0}$.

Furthermore, V. Govorov proved that there are isomorphisms

$$\mathrm{Tor}^A_p(K, K) \simeq \frac{Y_p}{X_p}$$

of (Adams) graded modules for all $p \in \mathbb{N}$ (see [6], Lemma 1, or, more generally, [3], Thm. in Section 1). In particular, this gives the well-known isomorphisms $\mathrm{Tor}^A_1(K, K) \simeq V$ and $\mathrm{Tor}^A_2(K, K) \simeq R$.

Lemma 2.3. *Given $p \in \mathbb{N}$, let $T_p \subseteq Y_p$ be a graded submodule that is a complement to X_p inside of Y_p. Then,*

$$\begin{aligned} &T_{2q+\delta} \subseteq \big((TV)^\delta_{>0} \cdot I^q\big) \cap \big(I^q \cdot (TV)^\delta_{>0}\big) \subseteq I^q, \\ &\textit{and } T_{2q+\delta} \cap \big((TV)_{>0} \cdot I^q \cdot (TV)^\delta_{>0} + (TV)^\delta_{>0} \cdot I^q \cdot (TV)_{>0} + I^{q+1}\big) = 0, \end{aligned} \tag{2.2}$$

for all $q \in \mathbb{N}_0$ and $\delta \in \{0, 1\}$. In particular, T_p is a tensor-intersection faithful graded submodule of TV, for all $p \in \mathbb{N}$.

Proof. The identities (2.2) are an immediate consequence of (2.1). The identities (2.2) for $\delta = 0$ directly imply that T_{2q} is a tensor-intersection faithful graded submodule of TV, for all $q \in \mathbb{N}$. Moreover, the first inclusion in (2.2) for $\delta = 1$ and the definition of T_{2q+1} imply that

$$\big((TV)_{>0} \cdot T_{2q+1} + T_{2q+1} \cdot (TV)_{>0}\big) \subseteq (TV)_{>0} \cdot I^q \cdot (TV)_{>0} \subseteq X_{q+1},$$

which tells us that

$$T_{2q+1} \cap \big((TV)_{>0} \cdot T_{2q+1} + T_{2q+1} \cdot (TV)_{>0}\big) = 0,$$

so T_{2q+1} is tensor-intersection faithful for all $q \in \mathbb{N}_0$. ∎

2.6 A_∞-(co)algebras

The following notion was introduced by J. Stasheff in [18]. We recall that a **nonunitary A_∞-algebra** is a (cohomologically) graded module $E = \oplus_{n\in\mathbb{Z}} E^n$ together with a collection of maps $\{m_n\}_{n\in\mathbb{N}}$, where $m_n : E^{\otimes n} \to E$ is a homogeneous morphism of degree $2-n$, satisfying the equation

$$\sum_{(r,s,t)\in\mathcal{I}_N} (-1)^{r+st} m_{r+1+t} \circ (\mathrm{id}_E^{\otimes r} \otimes m_s \otimes \mathrm{id}_E^{\otimes t}) = 0, \qquad (\mathrm{SI}(N))$$

for all $N \in \mathbb{N}$, where $\mathcal{I}_N = \{(r,s,t) \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}_0 : r+s+t = N\}$. We will denote by $\mathrm{SI}(N)$ the homogeneous morphism of degree $3-N$ from $E^{\otimes N}$ to E given by the left hand side of ($\mathrm{SI}(N)$).

We also recall that an A_∞-algebra is (**strictly**) **unitary** if there exists a distinguished homogeneous map $\eta_E : K \to E$ such that η_E is a unit for the product m_2, and $m_n \circ (\mathrm{id}_E^{\otimes(i-1)} \otimes \eta_E \otimes \mathrm{id}_E^{\otimes(n-i)}) = 0$ for all $n \in \mathbb{N}\setminus\{2\}$ and $i \in [\![1,n]\!]$.

Analogously, a **noncounitary A_∞-coalgebra** is a (homologically) graded module $T = \oplus_{n\in\mathbb{Z}} T_n$ together with a locally finite collection of maps $\{\Delta_n\}_{n\in\mathbb{N}}$, where $\Delta_n : T \to T^{\otimes n}$ is a homogeneous morphism of degree $n-2$, satisfying the equation

$$\sum_{(r,s,t)\in\mathcal{I}_N} (-1)^{rs+t} (\mathrm{id}_T^{\otimes r} \otimes \Delta_s \otimes \mathrm{id}_T^{\otimes t}) \circ \Delta_{r+1+t} = 0, \qquad (\mathrm{cSI}(N))$$

for all $N \in \mathbb{N}$. We recall that a family of morphisms $\{f_n : W \to W_n\}_{n\in\mathbb{N}}$ is said to be **locally finite** if for every $w \in W$ there exists a finite set $S \subseteq \mathbb{N}$ such that $f_n(w)$ vanishes if $n \in \mathbb{N}\setminus S$. We will denote by $\mathrm{cSI}(N)$ the homogeneous morphism of degree $N-3$ from T to $T^{\otimes N}$ given by the left hand side of ($\mathrm{cSI}(N)$).

We finally recall that an A_∞-coalgebra T is (**strictly**) **counitary** if there exists a distinguished homogeneous map $\epsilon_T : T \to K$ such that ϵ_T is a unit for the coproduct Δ_2, and $(\mathrm{id}_T^{\otimes(i-1)} \otimes \epsilon_T \otimes \mathrm{id}_T^{\otimes(n-i)}) \circ \Delta_n = 0$ for all $n \in \mathbb{N}\setminus\{2\}$ and $i \in [\![1,n]\!]$. Recall that an A_∞-(co)algebra is said to be **minimal** if m_1 (resp., Δ_1) vanishes. For the definitions of (co)augmented A_∞-(co)algebras, morphisms of A_∞-(co)algebras, the reader can check for instance [11], Section 2.1.

It is easy to see that the graded dual $T^\# = \oplus_{n\in\mathbb{Z}} T_n^*$ of a (counitary) A_∞-coalgebra is a (unitary) A_∞-algebra (see [11], Section 2.3), where we remark the dual $(-)^* = \mathrm{Hom}_K(-,K)$ is taken with respect to either the left or right K-module structure. As noted in [13], Section 2.4, the previous two A_∞-algebra structures are strictly isomorphic.

We remark that for the (co)homological grading we apply the usual Koszul sign rule of homological algebra, *i.e.* $(f\otimes g)(v\otimes w) = (-1)^{|g||v|} f(v)\otimes g(w)$, for all homogeneous $f : V \to V'$, $g : W \to W'$, $v \in V$ and $w \in W$, where $|x|$ denotes the (co)homological degree of a homogeneous element x.

Let $T = \oplus_{n\in\mathbb{N}_0} T_n$ be a nonnegatively graded module with $T_0 = K$. Let $\bar{T} = \oplus_{n\in\mathbb{N}} T_n$, and $\eta : K \to T$ and $\bar{\iota} : \bar{T} \to T$ be the canonical inclusions. Given a noncounitary A_∞-coalgebra structure on $\bar{T}$ given by $\{\bar{\Delta}_n\}_{n\in\mathbb{N}}$, then, for $n \in \mathbb{N}$, define the maps $\Delta_n : T \to T^{\otimes n}$ as follows. If $n \neq 2$, let $\Delta_n \circ \eta = 0$ and $\Delta_n \circ \bar{\iota} = \bar{\iota}^{\otimes n} \circ \bar{\Delta}_n$, and if $n = 2$, let $\Delta_2 \circ \eta = \eta\otimes\eta$ and $\Delta_2 \circ \bar{\iota} = \bar{\iota}^{\otimes 2} \circ \bar{\Delta}_2 + \eta\otimes\bar{\iota} + \bar{\iota}\otimes\eta$. Then, it is easy to verify that T endowed with the maps $\{\Delta_n\}_{n\in\mathbb{N}}$ is an A_∞-coalgebra with counit $\epsilon_T : T \to T_0 = K$ given by the canonical projection. We will call such counitary A_∞-coalgebra $T = \oplus_{n\in\mathbb{N}_0} T_n$ **standard**.

Example 2.4. *Let $A = TV/I$ be a nonnegatively graded connected algebra and let $B^+(A)$ the augmented bar construction of A, which is a coaugmented dg coalgebra (see [16], Sections 1.2.2 and 2.3.3). By the dual Merkulov construction (see [14]), the homology of $B^+(A)$, which coincides with $\mathrm{Tor}_\bullet^A(K,K)$, has a unique (up to nonunique equivalence)* *standard A_∞-coalgebra structure. Furthermore, the graded dual of the homology of $B^+(A)$ is quasi-isomorphic to the augmented A_∞-algebra $\mathrm{Ext}_A^\bullet(K,K)$, where $\mathrm{Ext}_A^\bullet(K,K)$ is endowed with a Merkulov model coming from the cohomology of the graded dual of the augmented bar construction of A.*

2.7 A_∞-(co)algebras and resolutions

We briefly recall the relation between a A_∞-coalgebra structure on $\mathrm{Tor}_\bullet^A(K,K)$ and the minimal projective resolution of the trivial graded A-module K. For more details on the definitions and notation we refer the reader to [11], Thm. 4.2, that we present below in our particular setting.

Let $T = \oplus_{n\in\mathbb{N}_0} T_n$ be a standard A_∞-coalgebra and $A = \oplus_{n\in\mathbb{N}_0} A_n$ nonnegatively Adams graded connected algebra. Let $\epsilon_A : A \to A_0 = K$ be the canonical projection and $\eta_T : K = T_0 \to T$ be the canonical inclusion. A (**generalized or homotopical**) **twisting cochain** from a T to A is a morphism $\tau \in \mathcal{H}om(C,A)$ of cohomological degree 1 and zero Adams degree such that both $\epsilon_A \circ \tau$ and $\tau \circ \eta_T$ vanish, and

$$\sum_{i\in\mathbb{N}} (-1)^{i(i+1)/2+1} \mu_A^{(i)} \circ \tau^{\otimes i} \circ \Delta_i = 0, \tag{2.3}$$

where $\mu_A^{(i)} : A^{\otimes i} \to A$ is the iterative application of the product of A, and $\Delta_i : T \to T^{\otimes i}$ is the i-th higher comultiplication of T. Given a twisting cochain τ from T to A, define the **twisted tensor product** $A \otimes_\tau T$ as the complex with nth-homogeneous component given by the A-module $A \otimes_\tau T_n$ and the differential

$$\partial^\tau = \mathrm{id}_A \otimes \Delta_1 + \sum_{i\in\mathbb{N}} (-1)^{\frac{i(i+1)}{2}} (\mu_A^{(i+1)} \otimes \mathrm{id}_T) \circ (\mathrm{id}_A \otimes \tau^{\otimes i} \otimes \mathrm{id}_T) \circ (\mathrm{id}_A \otimes \Delta_{i+1}).$$

We recall the following theorem from [13], that was originally announced by B. Keller at the X ICRA of Toronto, Canada, in 2002 (see however [15]).

Theorem 2.5. *Let T be a minimal (i.e. $\Delta_1 = 0$) standard A_∞-coalgebra and A be a nonnegatively (Adams) graded connected algebra, which we regard in zero (co)homological degree, locally finite dimensional over $\Bbbk$. Then, the following are equivalent:*

(i) There is a quasi-isomorphism of minimal coaugmented A_∞-coalgebras

$$T \to \mathrm{Tor}_\bullet^A(K,K).$$

(ii) There is a twisting cochain $\tau : T \to A$ such that the twisted tensor product $A \otimes_\tau T$ is a minimal projective resolution of the trivial graded left A-module K.

3 Main result

3.1 General setting and definitions

All along this subsection, we consider a fixed positively graded module $V = \oplus_{n\in\mathbb{N}} V_n$. A family $\bar{T} = \{T_p : p \in \mathbb{N}\}$ of independent tensor-intersection faithful graded submodules of $(TV)_{>0}$ is said to be **based** if, for every $p \in \mathbb{N}$, T_p is further endowed with a direct sum decomposition $T_p = \oplus_{\alpha\in\mathcal{J}_p} T_{p,\alpha}$ such that $T_{p,\alpha} \neq 0$ for all $\alpha \in \mathcal{J}_p$, where $\mathcal{J}_p$ is a set of indices. Note that the previous condition means in particular that $\mathcal{J}_p = \emptyset$ if $T_p = 0$. We will also write $\bar{T} = \oplus_{p\in\mathbb{N}} T_p$ and $\mathfrak{p}_{p,\alpha} : \bar{T} \to T_{p,\alpha}$ the canonical projection.

Let $\bar{T}$ be a based family of graded submodules of $(TV)_{>0}$. Given an integer $i \geq 2$, $\bar{p} = (p_1,\dots,p_i) \in \mathbb{N}^i$ and $\alpha \in \mathcal{J}_{o(\bar{p})}$, define

$$\mathcal{K}_\alpha^{\bar{p}} = \Big\{ (\alpha_1,\dots,\alpha_i) \in \mathcal{J}_{p_1} \times \cdots \times \mathcal{J}_{p_i} : T_{o(\bar{p}),\alpha} \subseteq T_{p_1,\alpha_1} \cdots T_{p_i,\alpha_i} \Big\},$$

where we recall that $\phi(\bar{p}) = |\bar{p}| + 2 - i$ and we denote the tensor product simply by a dot.

Lemma 3.1. *Let $\bar{T}$ be a based family of graded submodules of $(TV)_{>0}$. Given an integer $i \geq 2$ and $\bar{p} = (p_1, \dots, p_i) \in \mathbb{N}^i$, then*

$$\{T_{p_1,\alpha_1} \cdots T_{p_i,\alpha_i} : (\alpha_1, \dots, \alpha_i) \in \mathcal{J}_{p_1} \times \cdots \times \mathcal{J}_{p_i}\} \tag{3.1}$$

is an independent family of graded submodules of $(TV)_{>0}$. In consequence, $\#(\mathcal{K}^{\bar{p}}_{\bar{\alpha}}) \leq 1$.

Proof. Since $\{T_{p_1,\alpha_1} : \alpha_1 \in \mathcal{J}_{p_1}\}$ is an independent family of graded submodules of the tensor-intersection faithful graded submodule T_{p_1}, Corollary 2.2 tells us that (3.1) is an independent family. □

Remark 3.2. *Lemma 3.1 cannot hold if we further allow $\bar{p}$ to vary in (3.1). For instance, the family $\{V \otimes R, R \otimes V\}$ obtained from a Koszul algebra $TV/\langle R\rangle$ of global dimension at least 3 is not independent in TV.*

Definition 3.3. *Recall the notation in Subsection 2.5. A family $\bar{T} = \{T_p : p \in \mathbb{N}\}$ of graded submodules of $(TV)_{>0}$ is said to be **compatible** with a homogeneous ideal $I \subseteq (TV)_{>0} \cdot (TV)_{>0}$ of the tensor algebra TV if $T_p \subseteq Y_p$ is a complement to X_p inside of Y_p, for all $p \in \mathbb{N}$. We will further assume without loss of generality that $T_1 = V$ and $T_2 = R$.*

Note that, by Lemma 2.3, any family $\bar{T} = \{T_p : p \in \mathbb{N}\}$ of graded submodules of $(TV)_{>0}$ compatible with a homogeneous ideal is automatically independent and every T_p is tensor-intersection faithful.

Example 3.4. *Let $s \geq 2$ be an integer and $A = TV/\langle R\rangle$ an **s-homogeneous algebra**, i.e. $R \subseteq V^{\otimes s}$. Recall that $\phi_s(2m) = sm$ and $\phi_s(2m+1) = sm+1$, for all $m \in \mathbb{N}_0$. Define the family $\bar{T} = \{T_p : p \in \mathbb{N}\}$ of graded submodules of $(TV)_{>0}$ by $T_1 = V$ and*

$$T_p = \bigcap_{i=0}^{\phi_s(p-2)} V^{\otimes i} \otimes R \otimes V^{\otimes\left(\phi_s(p-2)-i\right)}$$

*for $p \geq 2$. Moreover, for $p \in \mathbb{N}$, define $\mathcal{J}_p = \{p\}$ if $T_p \neq 0$. If A is **generalized Koszul** (or **s-Koszul** if we want to emphasize the Adams degree of the relations of A), i.e. the minimal projective resolution $P_\bullet$ of the trivial left A-module K satisfies that P_n is (a graded free left A-module) generated in degree $\phi_s(n)$, for all $n \in \mathbb{N}_0$ (see [2, 7]), then $\bar{T}$ is a based family compatible with the ideal $\langle R\rangle$ (see [2], eq. (2.5)).*

Example 3.5. *Let V be a module, and let $\mathcal{B} \subseteq \cup_{i,j=1}^{\ell} e_i.V.e_j$ be a basis of the underlying vector space over $\Bbbk$, which we assume is finite. Recall that a submodule W of $(TV)_{>0}$ is said to be **monomial** (with respect to $\mathcal{B}$) if its underlying vector space over $\Bbbk$ has a basis included in $\cup_{n\in\mathbb{N}}\mathcal{B}^n$. The latter basis is then unique and it is called the **monomial basis** of W and their elements are called **monomials**. Moreover, a nonnegatively graded algebra $A = TV/I$ is said to be **monomial** if I is a monomial submodule of $(TV)_{>0}$.*

*If $A = TV/I$ is a monomial algebra, then the submodules in (2.1) are clearly monomial, since intersections and sums of monomial submodules of $(TV)_{>0}$ are monomial. Hence, given $p \in \mathbb{N}$, there is a unique family $\bar{T} = \{T_p : p \in \mathbb{N}\}$ of monomial submodules of $(TV)_{>0}$ compatible with I. The unique monomial basis $V^{(p-1)}$ of T_p is called the **set of (Anick) $(p-1)$-chains** of A (see [1]). For $p \in \mathbb{N}$, define $\mathcal{J}_p = V^{(p-1)}$ and the decomposition*

$$T_p = \bigoplus_{\omega \in V^{(p-1)}} \Bbbk.\omega,$$

where $\Bbbk.\omega$ has the obvious module structure. Then, $\bar{T}$ is a based family compatible with I.

3.2 Some properties

The two lemmas in this subsection are generalizations of some results in [5].

Lemma 3.6. *Let $\bar{T}$ be a based family of graded submodules of $(TV)_{>0}$ compatible with a homogeneous ideal I. Let $i \geq 2$ be an integer, $\bar{p} = (p_1, \dots, p_i) \in \mathbb{N}^i$ and $\alpha \in \mathscr{J}_{\mathscr{o}(\bar{p})}$ such that $\mathscr{K}^{\bar{p}}_{\alpha} \neq \emptyset$, where we recall that $\mathscr{o}(\bar{p}) = |\bar{p}|+2-i$. Then, $p_2, \dots, p_{i-1}$ are odd. Moreover, if we write $\bar{p} = 2\bar{q} + \bar{\delta}$, with $\bar{q} = (q_1, \dots, q_i) \in \mathbb{N}_0^i$ and $\bar{\delta} = (\delta_1, \dots, \delta_i) \in \{0,1\}^i$, then $\mathscr{o}(\bar{p})$ has the same parity as $\mathscr{o}(\bar{\delta}) \in \{0,1,2\}$ and the latter is precisely $\delta_1 + \delta_i$, the number of odd entries in (p_1, p_i).*

Proof. Note first that $\mathscr{o}(\bar{\delta}) = |\bar{\delta}| + 2 - i \leq 2$ and $\mathscr{o}(\bar{p}) = 2|\bar{p}| + 2 - i = 2|\bar{q}| + \mathscr{o}(\bar{\delta})$, so $\mathscr{o}(\bar{p})$ and $\mathscr{o}(\bar{\delta})$ have the same parity. Since $0 \neq T_{\mathscr{o}(\bar{p}),\alpha}$ and $\mathscr{K}^{\bar{p}}_{\alpha} \neq \emptyset$,

$$0 \neq T_{\mathscr{o}(\bar{p}),\alpha} \subseteq T_{p_1,\alpha_1} \cdots T_{p_i,\alpha_i} \subseteq I^{q_1} \cdots I^{q_i} = I^{|\bar{q}|}$$

where $\bar{\alpha} = (\alpha_1, \dots, \alpha_i) \in \mathscr{K}^{\bar{p}}_{\alpha}$ and the last inclusion follows from the first line in (2.2). Since $T_{2|\bar{q}|+\mathscr{o}(\bar{\delta})} \cap I^{|\bar{q}|} = 0$ if $\mathscr{o}(\bar{\delta}) < 0$, by the last identity in (2.2), we get $\mathscr{o}(\bar{\delta}) \in \{0,1,2\}$. If $\mathscr{o}(\bar{\delta}) = 2$, then $\bar{\delta} = (1, \dots, 1)$, *i.e.* all the entries of $\bar{p}$ are odd. It remains to consider the cases $\mathscr{o}(\bar{\delta}) \in \{0,1\}$.

Assume that there exists $j \in [\![2, i-1]\!]$ such that p_j is even. We will show that this leads to a contradiction. Since $\mathscr{o}(\bar{\delta}) \in \{0,1\}$, then either p_1 or p_i is odd, so $\delta_1 + \delta_i \geq 1$. The first inclusion in (2.2) tells us that

$$\begin{aligned} 0 &\neq T_{2|\bar{q}|+\mathscr{o}(\bar{\delta}),\alpha} \subseteq T_{2q_1+\delta_1,\alpha_1} \cdot T_{p_2,\alpha_2} \cdots T_{p_{i-1},\alpha_{i-1}} \cdot T_{2q_i+\delta_i,\alpha_i} \\ &\subseteq (TV)^{\delta_1}_{>0} \cdot I^{q_1} \cdot I^{q_2} \cdots I^{q_{i-1}} \cdot I^{q_i} \cdot (TV)^{\delta_i}_{>0} = (TV)^{\delta_1}_{>0} \cdot I^{|\bar{q}|} \cdot (TV)^{\delta_i}_{>0}, \end{aligned}$$

which is absurd by the second identity of (2.2), since $\mathscr{o}(\bar{\delta}) = 1$ if $\delta_1 = \delta_i = 1$, and $\mathscr{o}(\bar{\delta}) = 0$ if $\{\delta_1, \delta_i\} = \{0,1\}$.

Finally, since $p_2, \dots, p_{i-1}$ are odd, $\mathscr{o}(\bar{\delta}) = \delta_1 + \delta_i$, which clearly coincides with the number of odd entries in (p_1, p_i). □

Lemma 3.7. *Assume the same hypotheses as in Lemma 3.6. Let $1 \leq j < k \leq i$ be integers with $k - j < i - 1$ and $\alpha_{j,k} \in \mathscr{J}_{\mathscr{o}(\mathfrak{s}_{j,k}(\bar{p}))}$ such that $\mathscr{K}^{\mathfrak{s}_{j,k}(\bar{p})}_{\alpha_{j,k}} \neq \emptyset$. Then,*

$$T_{\mathscr{o}(\bar{p}),\alpha} \cap \left(T_{p_1,\alpha_1} \cdots T_{p_{j-1},\alpha_{j-1}} \cdot T_{\mathscr{o}(\mathfrak{s}_{j,k}(\bar{p})),\alpha_{j,k}} \cdot T_{p_{k+1},\alpha_{k+1}} \cdots T_{p_i,\alpha_i} \right) = 0. \tag{3.2}$$

Proof. We will use the same terminology as in Lemma 3.6. Since $1 \leq j < k \leq i$, $\mathscr{o}(\mathfrak{s}_{j,k}(\bar{\delta})) \in \{1,2\}$, *i.e.* all the entries of $\mathfrak{s}_{j,k}(\bar{p}) \in \mathbb{N}^{k-j+1}$ are odd or there is exactly one entry that is even. In the latter case, either $j = 1$ and p_1 is even, or $k = i$ and p_i is even. Note that the first line of (2.2) tells us that

$$T_{\mathscr{o}(\mathfrak{s}_{j,k}(\bar{p}))} \subseteq \left((TV)^{2-\delta_{j,k}}_{>0} \cdot I^{|\mathfrak{s}_{j,k}(\bar{q})|+\delta_{j,k}-1} \right) \cap \left(I^{|\mathfrak{s}_{j,k}(\bar{q})|+\delta_{j,k}-1} \cdot (TV)^{2-\delta_{j,k}}_{>0} \right), \tag{3.3}$$

where we have written $\delta_{j,k} = \mathscr{o}(\mathfrak{s}_{j,k}(\bar{\delta}))$.

Suppose now that $\mathscr{o}(\mathfrak{s}_{j,k}(\bar{\delta})) = 2$, *i.e.* the entries of $\mathfrak{s}_{j,k}(\bar{p}) \in \mathbb{N}^{k-j+1}$ are odd. If $j > 1$, then (3.3) implies that

$$\begin{aligned} &T_{p_1,\alpha_1} \cdots T_{p_{j-1},\alpha_{j-1}} \cdot T_{\mathscr{o}(\mathfrak{s}_{j,k}(\bar{p})),\alpha_{j,k}} \cdot T_{p_{k+1},\alpha_{k+1}} \cdots T_{p_i,\alpha_i} \\ &\subseteq (TV)^{\delta_1}_{>0} \cdot I^{q_1} \cdots I^{q_{j-1}} \cdot I^{|\mathfrak{s}_{j,k}(\bar{q})|+1} \cdot I^{q_{k+1}} \cdots I^{q_i} \subseteq (TV)^{\delta_1}_{>0} \cdot I^{|\bar{q}|+1}, \end{aligned} \tag{3.4}$$

where we used the first line of (2.2). The analogous argument tells us that, if $k < i$, the first member of (3.4) is included in $I^{|\bar{q}|+1} \cdot (TV)^{\delta_i}_{>0}$. In either case, the second identity of (2.2) implies (3.2). Indeed, if $\mathscr{o}(\bar{\delta}) = 2$, then (3.2) is an immediate of the second identity of (2.2), if $\mathscr{o}(\bar{\delta}) = 1$ and either $j > 1$ or $k < i$ then (3.2) follows from the second identity of (2.2) and the fact that the left member of (3.4) is included in $I^{|\bar{q}|+1}$, and if $\mathscr{o}(\bar{\delta}) = 0$ then (3.2) follows from $I^{|\bar{q}|+1} \subseteq I^{|\bar{q}|} \cdot (TV)_{>0}$ and the second identity of (2.2).

Finally, assume that $\mathscr{o}(\mathfrak{d}_{j,k}(\delta)) = 1$, *i.e.* either $j = 1$ and p_1 is even, or $k = i$ and p_i is even. Note that in this case $\mathscr{o}(\bar{\delta}) \in \{0, 1\}$. Then, (3.3) tells us that the left member of (3.4) is included in $(TV)_{>0} \cdot I^{|\bar{q}|} \cdot (TV)^{\delta_i}_{>0}$ if $j = 1$, and in $(TV)^{\delta_1}_{>0} \cdot I^{|\bar{q}|} \cdot (TV)_{>0}$ if $k = i$. In either case, the second identity of (2.2) implies (3.2). Indeed, if $\mathscr{o}(\bar{\delta}) = 1$ and $j > 1$ (resp., $k < i$), then p_i (resp., p_1) is odd, so the left member of (3.4) is included in $(TV)_{>0} \cdot I^{|\bar{q}|} \cdot (TV)_{>0}$ and (3.2) follows from the second identity of (2.2). If $\mathscr{o}(\bar{\delta}) = 0$, (3.2) follows from the fact that the left member of (3.4) is included in $(TV)_{>0} \cdot I^{|\bar{q}|}$ if $j = 1$ or $I^{|\bar{q}|} \cdot (TV)_{>0}$ if $k = i$, together with the second identity of (2.2). □

3.3 The A_∞-coalgebra structure

For the rest of the section $\bar{T}$ **will be a based family of graded submodules of** $(TV)_{>0}$ **compatible with a homogeneous ideal** I.

3.3.1 Comultiplications as signed inclusions

Given an integer $i \geq 2$, $\bar{p} = (p_1, \ldots, p_i) \in \mathbb{N}^i$, $\alpha \in \mathcal{J}_{\mathscr{o}(\bar{p})}$ and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_i) \in \mathcal{J}_{p_1} \times \cdots \times \mathcal{J}_{p_i}$, define

$$\Delta^{\bar{p}}_{\alpha,\bar{\alpha}} : T_{\mathscr{o}(\bar{p}),\alpha} \longrightarrow T_{p_1,\alpha_1} \otimes \cdots \otimes T_{p_i,\alpha_i} \subseteq \bar{T}^{\otimes i} \tag{3.5}$$

as $(-1)^{i(p_1-1)}$ times the inclusion if $\bar{\alpha} \in \mathcal{K}^{\bar{p}}_{\alpha}$, and zero else. Note that, given $\mathscr{p} \in \mathbb{N}$ and $\alpha \in \mathcal{J}_{\mathscr{p}}$, the family

$$\left\{ \Delta^{\bar{p}}_{\alpha,\bar{\alpha}} : \bar{p} \in \mathbb{N}^i, i \geq 2, \mathscr{o}(\bar{p}) = \mathscr{p} \right\} \tag{3.6}$$

of maps whose domain is $T_{\mathscr{p},\alpha}$ is locally finite, since each module T_{p_j,α_j} is concentrated in strictly positive degrees. Given an integer $i \geq 2$, we finally define $\Delta_i : \bar{T} \to \bar{T}^{\otimes i}$ as

$$\Delta_i = \sum_{\bar{p} \in \mathbb{N}^i} \sum_{\alpha \in \mathcal{J}_{\mathscr{o}(\bar{p})}} \sum_{\bar{\alpha} \in \mathcal{J}_{p_1} \times \cdots \times \mathcal{J}_{p_i}} \Delta^{\bar{p}}_{\alpha,\bar{\alpha}}. \tag{3.7}$$

The locally finiteness property of (3.6) tells us that Δ_i is well defined.

3.3.2 The Stasheff identities

Let $N \geq 3$ be an integer, $\bar{p} \in \mathbb{N}^N$ and $\bar{\alpha} \in \mathcal{J}_{p_1} \times \cdots \times \mathcal{J}_{p_N}$. Given $(r, s, t) \in \mathcal{I}_N$, consider the maps

$$\begin{aligned} &(-1)^{rs+t}(\mathfrak{p}_{p_1,\alpha_1} \otimes \cdots \otimes \mathfrak{p}_{p_N,\alpha_N}) \circ (\mathrm{id}^{\otimes r}_{\bar{T}} \otimes \Delta_s \otimes \mathrm{id}^{\otimes t}_{\bar{T}}) \circ \Delta_{r+1+t}, \\ &(\mathfrak{p}_{p_1,\alpha_1} \otimes \cdots \otimes \mathfrak{p}_{p_N,\alpha_N}) \circ \mathrm{cSI}(N) : \bar{T} \to T_{p_1,\alpha_1} \otimes \cdots \otimes T_{p_N,\alpha_N}, \end{aligned} \tag{3.8}$$

where $\mathrm{cSI}(N) : \bar{T} \to \bar{T}^{\otimes N}$ is defined by the left member of (cSI(N)). By cohomological degree reasons, the restriction of either map in (3.8) to T_q vanishes if $q \neq \mathscr{o}(\bar{p}) + 1$. Given $\alpha \in \mathcal{J}_{\mathscr{o}(\bar{p})+1}$, define the maps

$$\mathrm{cSI}(N)^{r,s,t}_{\bar{p},\bar{\alpha},\alpha}, \mathrm{cSI}(N)_{\bar{p},\bar{\alpha},\alpha} : T_{\mathscr{o}(\bar{p})+1,\alpha} \to T_{p_1,\alpha_1} \otimes \cdots \otimes T_{p_N,\alpha_N}.$$

as the composition of the inclusion of $T_{\mathscr{o}(\bar{p})+1,\alpha}$ inside of $\bar{T}$ and the corresponding map in (3.8). Note that

$$\mathrm{cSI}(N)_{\bar{p},\bar{\alpha},\alpha} = \sum_{(r,s,t) \in \mathcal{I}_N} \mathrm{cSI}(N)^{r,s,t}_{\bar{p},\bar{\alpha},\alpha}, \tag{3.9}$$

and Δ_s is evaluated at $T_{\mathscr{o}(\mathfrak{z}_{r+1,r+s}(\bar{p}))}$ in the summand $\mathrm{cSI}(N)^{r,s,t}_{\bar{p},\bar{\alpha},\alpha}$ of $\mathrm{cSI}(N)_{\bar{p},\bar{\alpha},\alpha}$. We remark for later use that, if $\mathrm{cSI}(N)^{r,s,t}_{\bar{p},\bar{\alpha},\alpha} \neq 0$, it is the inclusion of $T_{\mathscr{o}(\bar{p})+1,\alpha}$ inside of $T_{p_1,\alpha_1} \otimes \cdots \otimes T_{p_N,\alpha_N}$ times a sign $\square^{r,s,t}_{\bar{p}} \in \{\pm 1\}$ given by

$$\square^{r,s,t}_{\bar{p}} = \begin{cases} -(-1)^{(N+1)(p_1+p_s)+s(p_s-1)}, & \text{if } r = 0, \\ -(-1)^{p_1(N-1)+sp_{r+1}-r}, & \text{if } r > 0, \end{cases} \tag{3.10}$$

where $N = r + s + t$. Indeed, if $r > 0$,

$$\begin{aligned} \square^{r,s,t}_{\bar{p}} &= (-1)^{rs+t}(-1)^{s(p_1+\cdots+p_r)}(-1)^{s(p_{r+1}-1)}(-1)^{(r+1+t)(p_1-1)} \\ &= (-1)^{rs+N-s-r}(-1)^{s(p_1+r-1)}(-1)^{s(p_{r+1}-1)}(-1)^{(N-s+1)(p_1-1)} \\ &= -(-1)^{p_1(N+1)+sp_{r+1}-r}, \end{aligned}$$

where the second sign of the second member comes from the application of the Koszul sign rule to Δ_s, and we have used used that $p_2, \ldots, p_r$ are odd by Lemma 3.6. On the other hand, if $r = 0$,

$$\begin{aligned} \square^{r,s,t}_{\bar{p}} &= (-1)^{N-s}(-1)^{s(p_1-1)}(-1)^{(N-s+1)(p_1+\cdots+p_s+1-s)} \\ &= -(-1)^{N-s+1}(-1)^{s(p_1-1)}(-1)^{(N-s+1)(p_1+p_s+1)} \\ &= -(-1)^{s(p_1-1)+(N-s+1)(p_1+p_s)} = -(-1)^{(N+1)(p_1+p_s)+s(p_s-1)}, \end{aligned}$$

where we have used that $t = N - s$ and that $p_2, \ldots, p_{s-1}$ are odd by Lemma 3.6. We also note that, putting $r = N - s$ in the second line of (3.10), *i.e.* if $t = 0$, we get that

$$\square^{r,s,0}_{\bar{p}} = (-1)^{(p_1+1)(N-1)+s(p_{r+1}+1)}. \tag{3.11}$$

On the other hand, if $t > 0$ and p_{r+s} is even, then (3.10) gives

$$\square^{r,s,t}_{\bar{p}} = -(-1)^{(p_1+1)(N-1)+N-t}. \tag{3.12}$$

for all $(r, s, t) \in \mathcal{I}_N$, where we used that $s + r = N - t$, and that Lemma 3.6 implies that p_{r+1} is odd if $r > 0$.

Using Lemma 3.6 in (3.9) we see that $\mathrm{cSI}(N)_{\bar{p},\bar{\alpha},\alpha}$ trivially vanishes if the set $(p_1, \ldots, p_N)$ has at least four different even entries. Moreover, the same result tells us that, if $\mathrm{cSI}(N)_{\bar{p},\bar{\alpha},\alpha} \neq 0$, then

(O.1) $p_1, \ldots, p_N$ are odd integers;

(O.2) there exists a unique $j \in [\![1, N]\!]$ such that p_j is even;

(O.3) $(p_1, \ldots, p_N)$ has exactly 2 different even entries, one of which is p_1 or p_N;

(O.4) p_1 and p_N are even, and there is a unique $j \in [\![2, N-1]\!]$ such that p_j is even.

Note that $\mathscr{o}(\bar{p}) + 1$ is odd in cases (O.1) and (O.3), and it is even in cases (O.2) and (O.4). We will also regroup (O.1)-(O.4) into the following cases:

(GO.1) either (O.1), or (O.2) and $j \in \{1, N\}$, or (O.3) and p_1 and p_N are even;

(GO.2) either (O.2) and $j \in [\![2, N-1]\!]$, or (O.3) and there is $j \in [\![2, N-1]\!]$ such that p_j is even, or (O.4).

In short, (GO.1) consists of all $(p_1, \ldots, p_N) \in \mathbb{N}^n$ such that $p_2, \ldots, p_{N-1}$ are odd, whereas (GO.2) consists of all $(p_1, \ldots, p_N) \in \mathbb{N}^n$ such that $(p_2, \ldots, p_{N-1})$ contains exactly one even entry.

We then have the following direct application of Lemmas 3.6 and 3.7.

(C.1) In case (GO.1), Lemma 3.6 applied to Δ_{r+1+s} tells us that $\mathrm{cSI}(N)^{r,s,t}_{\bar{p},\bar{\alpha},\alpha} \neq 0$ implies either $r = 0$ or $t = 0$. Moreover, by Lemma 3.7 there exist $s_\ell, s_r \in [\![2, N-1]\!]$ such that $\mathrm{cSI}(N)^{0,s',N-s'}_{\bar{p},\bar{\alpha},\alpha} = \mathrm{cSI}(N)^{N-s'',s'',0}_{\bar{p},\bar{\alpha},\alpha} = 0$ for all $s', s'' \in [\![2, N-1]\!]$ such that $s' \neq s_\ell$ and $s'' \neq s_r$.

(C.2) In case (GO.2), Lemma 3.6 applied to $\Delta_{r_j+1+s_j}$ tells us that $\mathrm{cSI}(N)^{r_j,s_j,t_j}_{\bar{p},\bar{\alpha},\alpha} \neq 0$ implies either $r_j = j-1$ or $t_j = N-j$. Furthermore, Lemma 3.7 tells us that there exist integers $s_\ell \in [\![2, N-j+1]\!]$ and $s_r \in [\![2, j]\!]$ such that $\mathrm{cSI}(N)^{r_j,s',N-s'-r_j}_{\bar{p},\bar{\alpha},\alpha} = \mathrm{cSI}(N)^{N-t_j-s'',s'',t_j}_{\bar{p},\bar{\alpha},\alpha} = 0$ for all $s' \in [\![2, N-j+1]\!]$ and $s'' \in [\![2, j]\!]$ such that $s' \neq s_\ell$ and $s'' \neq s_r$.

Definition 3.8. *Let $\bar{T}$ be a based family of graded submodules of $(TV)_{>0}$ compatible with a homogeneous ideal I. Using the notation of (C.1) and (C.2), we say that $\bar{T}$ is* ***balanced*** *if, for every integer $N \geq 3$, $\bar{p} \in \mathbb{N}^N$ and $\bar{\alpha} \in \mathcal{J}_{p_1} \times \cdots \times \mathcal{J}_{p_N}$,*

(N.1) there exists $s_\ell \in [\![2, N-1]\!]$ such that $\mathrm{cSI}(N)^{0,s_\ell,N-s_\ell}_{\bar{p},\bar{\alpha},\alpha} \neq 0$ if and only if there exists $s_r \in [\![2, N-1]\!]$ such that $\mathrm{cSI}(N)^{N-s_r,s_r,0}_{\bar{p},\bar{\alpha},\alpha} \neq 0$, if $\bar{p}$ is of type (GO.1);

(N.2) there exists $s_\ell \in [\![2, N-j+1]\!]$ such that $\mathrm{cSI}(N)^{j-1,s_\ell,N-s_\ell-j+1}_{\bar{p},\bar{\alpha},\alpha} \neq 0$ if and only if there exists $s_r \in [\![2, j]\!]$ such that $\mathrm{cSI}(N)^{j-s_r,s_r,N-j}_{\bar{p},\bar{\alpha},\alpha} \neq 0$, if $\bar{p}$ is of type (GO.2) with parameter $j \in [\![2, N-1]\!]$.

Proposition 3.9. *Let $\bar{T}$ be a balanced based family of graded submodules of $(TV)_{>0}$ compatible with a homogeneous ideal I. Then, $\bar{T} = \oplus_{p\in\mathbb{N}} T_p$ endowed with the maps $\{\Delta_i\}_{i\geq 2}$ defined in (3.7) is a minimal noncounitary A_∞-coalgebra.*

Proof. Note first that the family of maps $\{\Delta_i\}_{i\geq 2}$ is locally finite, by degree reasons. It remains to show that (3.9) vanishes for all integers $N \geq 3$ and all N-tuples $\bar{p} = (p_1, \dots, p_N) \in \mathbb{N}^N$, $\bar{\alpha} \in \mathcal{J}_{p_1} \times \cdots \times \mathcal{J}_{p_N}$, and $\alpha \in \mathcal{J}_{\phi(\bar{p})+1}$. As noted in the beginning of this subsubsection, if $\mathrm{cSI}(N)_{\bar{p},\bar{\alpha},\alpha} \neq 0$, then $\bar{p}$ is either of type (GO.1) or (GO.2). By the balanced assumption on $\bar{T}$ and the definition of the maps $\{\Delta_i\}_{i\geq 2}$ in (3.7), it suffices to show that

$$\square^{0,s_\ell,N-s_\ell}_{\bar{p}} = -\square^{N-s_r,s_r,0}_{\bar{p}}$$

in case (N.1), and

$$\square^{r_j,s_\ell,N-s_\ell-r_j}_{\bar{p}} = -\square^{N-t_j-s_r,s_r,t_j}_{\bar{p}}$$

in case (N.2). Following the notation explained in (C.1), we see that the first line of (3.10) and (3.11) yield

$$\square^{0,s_\ell,N-s_\ell}_{\bar{p}} = -(-1)^{(N+1)(p_1+1)} = -\square^{N-s_r,s_r,0}_{\bar{p}}$$

in case (GO.1), since p_{s_ℓ} and p_{N-s_r+1} are odd by assumption. On the other hand, following the notation explained in (C.2), the second line of (3.10) and (3.12) also tell us that

$$\square^{r_j,s_\ell,N-s_\ell-r_j}_{\bar{p}} = (-1)^{j+p_1(N+1)} = -\square^{N-t_j-s_r,s_r,t_j}_{\bar{p}}$$

in case (GO.2), since $p_{r_j+1} = p_j$ is even and $p_{r_j+1} = p_j$ is even by assumption. The proposition is thus proved. $\square$

3.4 A resolution associated with the A_∞-coalgebra structure

Definition 3.10. *We say that a based family $\bar{T} = \{T_p : p \in \mathbb{N}\}$ is* ***left resolutive****, if, given $p \in \mathbb{N}$ and $\alpha \in \mathcal{J}_p$, there exists an integer $i \geq 2$ such that*

$$\mathcal{K}^{(1,\dots,1,p-1)}_{\alpha} \neq \emptyset,$$

with $(1, \dots, 1, p-1) \in \mathbb{N}_0^i$. The ***right resolutive*** *condition is defined analogously.*

Theorem 3.11. *Assume the same hypotheses as in Proposition 3.9. Let $A = TV/I$, $T = K \oplus \bar{T}$ be the standard counitary A_∞-coalgebra obtained from the A_∞-coalgebra structure on $\bar{T}$ as explained in the last paragraph of Subsection 2.6, and let $\tau : T \to A$ be the map given as minus the composition of the canonical projection $T \to T_1 = V$ and the inclusion $V \to A$.*

If $\bar{T}$ is left resolutive, then τ is a twisting cochain. Moreover, the twisted tensor product $A \otimes_\tau T$ is a minimal projective resolution of the trivial graded left A-module K. Conversely, if there is a twisting cochain $\tau : T \to A$ such that $A \otimes_\tau T$ is a minimal projective resolution of the trivial graded left A-module K, then $\bar{T}$ is left resolutive.

Proof. We will prove the first implication of the statement. We first note that, by cohomological degree reasons, (2.3) is trivially verified if the left member of (2.3) is evaluated at an element of T_i for $i \neq 2$. It suffices to show that (2.3) holds when the left member is restricted to T_2. By the left resolutive assumption of $\bar{T}$, given any $\alpha \in \mathcal{J}_2$, there is an integer $j \geq 2$ such that $T_{2,\alpha} \subseteq V^{\otimes j}$. Since $T_2 \subseteq I$, we conclude that the left member of (2.3) restricted to $T_{2,\alpha}$ vanishes, as was to be shown. Let us now prove the second part of the first implication. Note that the complex $A \otimes_\tau T$ is minimal since $\mathrm{id}_K \otimes_A \partial^\tau$ vanishes. It suffices to show that it is exact. We further notice that $A \otimes_\tau T$ is exact up to homological degree 1, since it coincides with the usual expression of the minimal projective resolution of the trivial graded left A-module K up to homological degree 2. Moreover, we also note that, since $\bar{T}$ is compatible with I, (2.1) immediately implies that $I \cdot W_k \subseteq Z_k$ and $I \cdot Y_k \subseteq X_{k+1}$ for all $k \in \mathbb{N}$, which in turn tells us that $T_p \cap I \cdot T_{p+1} = 0$, for all $k \in \mathbb{N}$. As a consequence, the restriction to the submodule $K \otimes T_{p,\alpha} \simeq T_{p,\alpha}$ of $A \otimes T_{p,\alpha}$ of the differential ∂^τ, which is induced by the inclusion $T_{p,\alpha} \subseteq (TV)_{>0} \otimes T_{p-1}$, is injective. Assume that $A \otimes_\tau T$ is exact up to homological degree $p \in \mathbb{N}$, *i.e.* the kernel of $\partial^\tau_i : A \otimes T_i \to A \otimes T_{i-1}$ coincides with the image of $\partial^\tau_{i+1} : A \otimes T_{i+1} \to A \otimes T_i$ for all $i \in [\![1, p]\!]$. We will prove it is exact in homological degree $p+1$. The expression of the differential ∂^τ gives us the canonical isomorphism

$$\mathrm{Ker}(\partial^\tau_{\mathrm{i}}) \simeq \frac{\big(TV \cdot T_i \cap I \cdot T_{i-1}\big) + I \cdot T_i}{I \cdot T_i},$$

which in turn gives us the isomorphism

$$K \otimes_A \mathrm{Ker}(\partial^\tau_{\mathrm{i}}) \simeq \frac{\big(TV \cdot T_i \cap I \cdot T_{i-1}\big) + I \cdot T_i}{(TV)_{>0} \cdot \big(TV \cdot T_i \cap I \cdot T_{i-1}\big) + (TV)_{>0} \cdot I \cdot T_i},$$

for all $i \in \mathbb{N}$. Note moreover that

$$(TV)_{>0} \cdot \big(TV \cdot T_i \cap I \cdot T_{i-1}\big) + (TV)_{>0} \cdot I \cdot T_i \subseteq (TV)_{>0} \cdot I \cdot T_{i-1} + (TV)_{>0} \cdot I \cdot T_i, \quad (3.13)$$

for all $i \in \mathbb{N}$. We claim that the intersection of T_{p+1} and the right member of (3.13) is trivial. Indeed, if we write $p = 2q + \delta$ with $\delta \in \{0, 1\}$, the first line in (2.2) tells us that the right member of (3.13) is included in

$$(TV)_{>0} \cdot I^{q+\delta} \cdot (TV)^{1-\delta}_{>0} + (TV)_{>0} \cdot I^{q+1},$$

which has trivial intersection with T_{p+1} by the second identity of (2.2). Since the intersection of T_{p+1} and the right member of (3.13) is trivial, the composition of $\partial^\tau_{p+1}|_{K \otimes T_{p+1}}$ and the canonical projection $\mathrm{Ker}(\partial^\tau_{\mathrm{p}}) \to K \otimes_A \mathrm{Ker}(\partial^\tau_{\mathrm{p}})$ is injective. Since both T_{p+1} and $K \otimes_A \mathrm{Ker}(\partial^\tau_{\mathrm{p}}) \simeq \mathrm{Tor}^A_{p+1}(K, K)$ have the same Hilbert series as Adams graded vector spaces over $\Bbbk$, by the recursive assumption on the exactness of $A \otimes_\tau T$, the previous composition map is an isomorphism, which tells us that $A \otimes_\tau T$ is exact in homological degree p.

We finally prove the converse. By cohomological and Adams degree reasons, we see that $\tau : T \to A$ is the composition of the canonical projection $T \to T_1$, a map $\bar{\tau} : T_1 \to V$ and the minus canonical inclusion $V \to A$. Since $A \otimes_\tau T$ is the minimal projective resolution of the trivial graded left A-module K, by comparing it with the standard description of the minimal projective resolution of the trivial graded left A-module K up to homological degree 2, we conclude that $\bar{\tau}$ is injective, so bijective since both spaces have the same Hilbert series as Adams graded vector spaces over $\Bbbk$. Since $\Delta_n|_R : R \to V^{\otimes n}$ is the inclusion for $n \geq 2$, $\mu_A \circ \bar{\tau}^{\otimes n} \circ \Delta_n|_R = 0$ is then tantamount to $\bar{\tau}^{\otimes n}(R) \subseteq R$, which means that the unique isomorphism of algebras $TV \to TV$ induced by $\bar{\tau}$ gives an isomorphism of algebras $g_\tau : A \to A$. As a consequence, by composing with g_τ^{-1}, we may assume without loss of generality that $\bar{\tau}$ is the canonical inclusion of V inside of A. Now, the fact that $A \otimes_\tau T$ is the minimal projective resolution of the trivial graded left A-module K implies in particular that the restriction to the submodule $K \otimes T_{p,\alpha} \simeq T_{p,\alpha}$ of $A \otimes T_{p,\alpha}$ of the differential ∂^τ, which is induced by the inclusion $T_{p,\alpha} \subseteq (TV)_{>0} \otimes T_{p-1}$, is injective. This implies that $\bar{T}$ is left resolutive, as was to be shown. The theorem is thus proved. □

The following result follows immediately from Theorem 3.13 by replacing A by its opposite algebra.

Corollary 3.12. *Let $\bar{T}$ be a balanced based family of graded submodules of $(TV)_{>0}$ compatible with a homogeneous ideal I and let $A = TV/I$. Then $\bar{T}$ is left resolutive if and only if it is right resolutive.*

Combining Theorem 2.5, Proposition 3.9 and Theorem 3.11, we directly obtain the main result of this article.

Theorem 3.13. *Let $\bar{T}$ be a left resolutive balanced based family of graded submodules of $(TV)_{>0}$ compatible with a homogeneous ideal I and let $A = TV/I$. Then, the A_∞-coalgebra $T = K \oplus \bar{T}$ considered in Proposition 3.9 is quasi-isomorphic to the A_∞-coalgebra $\mathrm{Tor}^A_\bullet(K,K)$. As a consequence, the corresponding A_∞-algebra structure on the graded dual $T^\#$ of T is quasi-isomorphic to the augmented A_∞-algebra $\mathrm{Ext}^\bullet_A(K,K)$.*

4 Some examples

Finally, as an application, we show how the results of the previous section allow to easily compute (and in many cases reobtain) the A_∞-(co)algebra of $\mathrm{Ext}^\bullet_A(K,K)$ (resp., $\mathrm{Tor}^A_\bullet(K,K)$) for several classes of nonnegatively graded connected algebras A.

4.1 Generalized Koszul algebras

We continue here with the assumptions of Example 3.4, where A is an s-Koszul algebra for $s \geq 2$. We will show that the based family of graded submodules of $(TV)_{>0}$ considered there is balanced and left resolutive. This follows easily from the definitions but we explain it for the reader's convenience. The left resolutive condition is trivially verified, provided the balanced condition holds, since $T_{2q} \subseteq V^{\otimes(s-1)} \otimes T_{2q-1}$ and $T_{2q+1} \subseteq V \otimes T_{2q}$ for all $q \in \mathbb{N}$.

Let us finally verify that the family $\bar{T}$ is balanced. Since the indices $\alpha \in \mathcal{J}_p$ are trivial in this case, we will omit them. Note first that the Adams degree of T_p is $\phi_s(p)$. Using this and a direct Adams degree argument, a nonzero map $\Delta^{\bar{p}}_{\alpha,\bar{\alpha}}$ of the form (3.5) preserving the Adams degree exists only for $i \in \{2, s\}$. Moreover, using again the Adams degree of the graded submodules of the family $\bar{T}$, we conclude that

(K.1) $(\mathfrak{p}_{p_1} \otimes \mathfrak{p}_{p_2}) \circ \Delta_2|_{T_p} = 0$ if $p = p_1 + p_2$ is even and $p_1, p_2 \in \mathbb{N}$ are odd;

(K.2) $(\mathfrak{p}_{p_1} \otimes \cdots \otimes \mathfrak{p}_{p_s}) \circ \Delta_N|_{T_{\mathscr{o}(\bar{p})}} = 0$ unless all entries of $\bar{p} = (p_1, \ldots, p_s)$ are odd.

Note that $\mathscr{o}(\bar{p})$ is even if all entries of $\bar{p} = (p_1, \ldots, p_s)$ are odd. This implies in particular that, if $N \in \mathbb{N} \setminus \{3, s+1\}$, every term of the Stasheff identity $\mathrm{cSI}(N)$ trivially vanishes. Hence, it suffices to show that $\mathrm{cSI}(N)$ vanishes for $N \in \{3, s+1\}$. The case $N = 3$ is immediate, since Δ_2 is the usual inclusion, which also concludes the case $s = 2$.

It remains to consider $N = s + 1 > 3$, for which the only possible cases among (O.1)-(O.4) are (O.1) and (O.2), by properties (K.1) and (K.2). In case (O.1), with $\bar{p} = (p_1, \ldots, p_{s+1})$ having odd entries, we clearly have $\mathrm{cSI}(s+1)^{0,s,1}_{\bar{p}} \neq 0$ and $\mathrm{cSI}(s+1)^{1,s,0}_{\bar{p}} \neq 0$ for (N.1), *i.e.* $s_\ell = s_r = s$ in (N.1). In case (O.2), all the entries $\bar{p} = (p_1, \ldots, p_{s+1})$ are odd except for p_j which is even, with $j \in [\![1, s+1]\!]$. If $j = 1$ (resp., N), we clearly have $\mathrm{cSI}(N)^{0,2,s-1}_{\bar{p}} \neq 0$ (resp., $\mathrm{cSI}(N)^{0,s,1}_{\bar{p}} \neq 0$) and $\mathrm{cSI}(N)^{1,s,0}_{\bar{p}} \neq 0$ (resp., $\mathrm{cSI}(N)^{s-1,2,0}_{\bar{p}} \neq 0$) for (N.1), *i.e.* $s_\ell = 2$ (resp., $s_\ell = s$) and $s_r = s$ (resp., $s_r = 2$) in (N.1). Finally, if $j \in [\![2, s]\!]$, we clearly have $\mathrm{cSI}(N)^{j-1,2,s-j}_{\bar{p}} \neq 0$ and $\mathrm{cSI}(N)^{j-2,2,s+1-j}_{\bar{p}} \neq 0$ for (N.2), *i.e.* $s_\ell = s_r = 2$ in (N.2). As a consequence, the family $\bar{T}$ is balanced.

By applying Theorem 3.13 we reobtain the A_∞-coalgebra $\mathrm{Tor}^A_\bullet(K, K)$ mentioned in [11], Section 4.3, and the A_∞-algebra structure on $\mathrm{Ext}^\bullet_A(K, K)$ obtained in [9], Thm. 6.5.

4.2 Monomial algebras

We continue here with the assumptions of Example 3.5, where $A = TV/I$ is a monomial algebra. In the unpublished article [5], the authors essentially proved that the based family $\bar{T}$ of Example 3.5 is balanced. We will explain the results of [5], proving that $\bar{T}$ is balanced. In this case, the definition of Anick chains directly implies that $\bar{T}$ is left resolutive, so Theorem 3.13 gives the A_∞-coalgebra structure on $\mathrm{Tor}^A_\bullet(K, K)$ and the corresponding A_∞-algebra structure on $\mathrm{Ext}^\bullet_A(K, K)$.

Note first that, if $u \in T_p$ is a monomial element, the index $\alpha \in \mathscr{J}_p$ such that $u \in T_{p,\alpha}$ is given by the same monomial element u. For this reason, we will omit the indices α in what follows.

First, remark the following nice property. Let $\bar{p} = (p_1, \ldots, p_n) = 2\bar{q} + \bar{\delta}$, with $\bar{q} = (q_1, \ldots, q_n) \in \mathbb{N}_0^n$ and $\bar{\delta} = (\delta_1, \ldots, \delta_n) \in \{0, 1\}^n$, be such that $p_2, \ldots, p_{n-1}$ are odd. If the monomials $u_i \in T_{p_i}$ for $i \in [\![1, n]\!]$ satisfy that

$$u = u_1 \ldots u_n \in (TV)^{1-\delta_1}_{>0} \cdot I^{|\bar{q}|-1+\mathscr{o}(\bar{\delta})} \cdot (TV)^{1-\delta_n}_{>0} \tag{4.1}$$

and that $u_i \ldots u_j \notin T_{\mathscr{o}(\mathscr{s}_{i,j}(\bar{p}))}$ for all $1 \leq i < j \leq n$ such that $j - i < n - 1$, then $u \in T_{\mathscr{o}(\bar{p})}$. To prove this, note that the first line of (2.2) tells us that $u \in (TV_{>0})^{\delta_1} \cdot I^{|\bar{q}|} \cdot (TV_{>0})^{\delta_n}$, which together with (4.1) implies that $u \in Y_{\mathscr{o}(\bar{p})}$, by (2.1). The last assumption and a recursive argument show that $u_\ell = u_1 \ldots u_{n-1} \notin (TV_{>0})^{1-\delta_1} \cdot I^{|\mathscr{s}_{1,n-1}(\bar{q})|+\delta_1}$ and $u_r = u_2 \ldots u_n \notin I^{|\mathscr{s}_{2,n}(\bar{q})|+\delta_n} \cdot (TV_{>0})^{1-\delta_n}$. Combining this with $u_1 \notin (TV)^{\delta_1}_{>0} \cdot I^{q_1}$ and $u_n \notin I^{q_n} \cdot (TV)^{\delta_n}_{>0}$, which follow from $u_1 \in J_{p_1}$ and $u_n \in J_{p_n}$ together with (2.2), we get that

$$u \notin \left((TV)^{1-\delta_1}_{>0} \cdot I^{|\bar{q}|+\delta_1} \cdot (TV)^{\delta_n}_{>0}\right) \cup \left((TV)^{\delta_1}_{>0} \cdot I^{|\bar{q}|+\delta_n} \cdot (TV)^{1-\delta_n}_{>0}\right). \tag{4.2}$$

Using that u is a monomial and comparing (4.2) with the second identity of (2.1), we see that $u \notin X_{\mathscr{o}(\bar{p})}$, which implies that $u \in T_{\mathscr{o}(\bar{p})}$.

Let us finally show that $\bar{T}$ is balanced. We will use the previous terminology. Let $\bar{p} = (p_1, \ldots, p_N) \in \mathbb{N}^N$ be such that $(p_2, \ldots, p_{N-1})$ has at most one even entry, and let $u_i \in T_{p_i}$ be monomials for $i \in [\![1, N]\!]$ such that $u = u_1 \ldots u_N \in T_{\mathscr{o}(\bar{p})+1}$.

(1) If $\bar{p}$ is of type (GO.1), then

$$u \in Y_{\mathscr{o}(\bar{p})+1} \subseteq \big((TV_{>0})^{1-\delta_1} \cdot I^{|\bar{q}|+\delta_1} \cdot (TV_{>0})^{\delta_N}\big) \cap \big((TV)^{\delta_1}_{>0} \cdot I^{|\bar{q}|+\delta_N} \cdot (TV)^{1-\delta_N}_{>0}\big),$$

so $u_\ell = u_1 \dots u_{N-1} \in (TV_{>0})^{1-\delta_1} \cdot I^{|\mathscr{d}_{1,N-1}(\bar{q})|+\delta_1}$ and $u_{\mathscr{r}} = u_2 \dots u_N \in I^{|\mathscr{d}_{2,N}(\bar{q})|+\delta_N} \cdot (TV)^{1-\delta_N}_{>0}$. Applying the result in the previous paragraph to u_ℓ and $u_{\mathscr{r}}$ we deduce that there exist $s_\ell, s_{\mathscr{r}} \in [\![2, N-1]\!]$ such that $\mathrm{cSI}(N)^{0,s_\ell,N-s_\ell}_{\bar{p},\bar{\alpha},\alpha} \neq 0$ and $\mathrm{cSI}(N)^{N-s_{\mathscr{r}},s_{\mathscr{r}},0}_{\bar{p},\bar{\alpha},\alpha} \neq 0$, proving (N.1).

(2) If $\bar{p}$ is of type (GO.2), let $j \in [\![2, N-1]\!]$ be such that p_j is even. Then,

$$u \in Y_{\mathscr{o}(\bar{p})+1} \subseteq (TV_{>0})^{|\delta_N-\delta_1|} \cdot I^{|\bar{q}|+\delta_1\delta_N} \cdot (TV_{>0})^{|\delta_N-\delta_1|}. \tag{4.3}$$

By (C.2), we know that $u_1 \dots u_{j-k} \notin T_{\mathscr{o}(\mathscr{d}_{1,j-k}(\bar{p}))}$ and $u_{j+k} \dots u_N \notin T_{\mathscr{o}(\mathscr{d}_{j+k,N}(\bar{p}))}$ for any $k \geq 1$, so the result in the previous paragraph implies that

$$\begin{aligned} u'_\ell &= u_1 \dots u_{j-1} \notin (TV_{>0})^{|\delta_N-\delta_1|} \cdot I^{|\mathscr{d}_{1,j-1}(\bar{q})|+\delta_1\delta_N}, \\ u'_{\mathscr{r}} &= u_{j+1} \dots u_N \notin I^{|\mathscr{d}_{j+1,N}(\bar{q})|+\delta_1\delta_N} \cdot (TV_{>0})^{|\delta_N-\delta_1|}. \end{aligned} \tag{4.4}$$

Hence, (4.3) and (4.4) give $u_\ell = u'_\ell u_j \in (TV_{>0})^{|\delta_N-\delta_1|} \cdot I^{|\mathscr{d}_{1,j}(\bar{q})|+\delta_1\delta_N} \cdot (TV)_{>0}$ and $u_{\mathscr{r}} = u_j u'_\ell \in (TV)_{>0} \cdot I^{|\mathscr{d}_{j,N}(\bar{q})|+\delta_1\delta_N} \cdot (TV_{>0})^{|\delta_N-\delta_1|}$. Applying the result in the previous paragraph to u_ℓ and $u_{\mathscr{r}}$ we deduce that there exist $s_\ell \in [\![2, N-j+1]\!]$ and $s_{\mathscr{r}} \in [\![2, j]\!]$ such that $\mathrm{cSI}(N)^{j-1,s_\ell,N-s_\ell-j+1}_{\bar{p},\bar{\alpha},\alpha} \neq 0$ and $\mathrm{cSI}(N)^{j-s_{\mathscr{r}},s_{\mathscr{r}},N-j}_{\bar{p},\bar{\alpha},\alpha} \neq 0$, proving (N.2).

4.3 Another example

To illustrate other possible uses of Theorem 3.13, we will also utilize it to compute the A_∞-coalgebra structure on $\mathrm{Tor}^A_\bullet(K, K)$ for an algebra A which is neither generalized Koszul nor monomial.

Consider the quiver Q given by

$$\begin{tikzcd} & & & \overset{4}{\bullet} \arrow[dr, "e"] & & & \\ \overset{1}{\bullet} \arrow[r, "a"] & \overset{2}{\bullet} \arrow[r, "b"] & \overset{3}{\bullet} \arrow[ur, "c"] \arrow[dr, "d"'] & & \overset{6}{\bullet} \arrow[r, "g"] & \overset{7}{\bullet} \arrow[r, "h"] & \overset{8}{\bullet} \\ & & & \overset{5}{\bullet} \arrow[ur, "f"'] & & & \end{tikzcd}$$

the ring K given by the product of 8 copies of the base field $\Bbbk$, identified with the vertices of Q, and the module V generated by the arrows $\mathscr{B}_1$ of Q, so $\Bbbk Q = TV$. Let $A = \Bbbk Q/I$, where I is the ideal generated by the submodule $R \subseteq TV$ generated by $\mathscr{B}_2 = \{abc, abd, egh, fgh, ce - df\}$. Let $T_1 = V$, $T_2 = R$, $T_3 \subseteq TV$ be the submodule generated by $\mathscr{B}_3 = \{\beta_{38} = (ce - df)gh, \beta_{16} = ab(ce - df)\}$, $T_4 \subseteq TV$ be the submodule generated by $\mathscr{B}_4 = \{\beta_{18} = ab(ce - df)gh\}$ and $T_p = 0$ if $p \geq 5$. It is easy to verify that (see [8]) that $T_p \simeq \mathrm{Tor}^A_p(K, K)$ for all $p \in \mathbb{N}$, and thus $\bar{T} = \{T_p : p \in \mathbb{N}\}$ is a family compatible with I. We will consider that it is based for $\mathscr{J}_p = \mathscr{B}_p$ and the direct sum decomposition induced by the latter bases. Applying the definitions in Subsection 3.3, we get that $\Delta_n = 0$ for $n \geq 4$, $\Delta_3(xyz) = x \otimes y \otimes z$ and $\Delta_2(xyz) = 0$ for all $xyz \in \{abc, abd, egh, fgh\}$, $\Delta_2(ce - df) = c \otimes e - d \otimes f$ and $\Delta_3(ce - df) = 0$, as well as

$$\begin{aligned} &\Delta_2(\beta_{16}) = abc \otimes e - abd \otimes f, && \Delta_2(\beta_{38}) = c \otimes egh - d \otimes fgh, \\ &\Delta_2(\beta_{18}) = abc \otimes egh - abd \otimes fgh, && \Delta_3(\beta_{16}) = a \otimes b \otimes (ce - df), \\ &\Delta_3(\beta_{38}) = -(ce - df) \otimes g \otimes h, && \Delta_3(\beta_{18}) = a \otimes b \otimes (ce - df)gh + ab(ce - df) \otimes g \otimes h. \end{aligned}$$

It is easy to verify that $\bar{T}$ is balanced and left resolutive, so Theorem 3.13 implies that the previous structure is quasi-isomorphic to the A_∞-coalgebra $\mathrm{Tor}^A_\bullet(K,K)$, so its dual gives a an A_∞-algebra quasi-isomorphic to $\mathrm{Ext}^\bullet_A(K,K)$.

4.4 A non-example

We give here a rather simple example of algebra for which the A_∞-algebra structure on the Yoneda algebra $\mathrm{Ext}^\bullet_A(K,K)$ cannot be given as the dual of the lattice of inclusions for any choice of submodules of TV representing the Tor groups. Consider the quiver Q given by

$$\begin{tikzcd}
 & & \overset{3}{\bullet} \arrow[dr, "d"] & & & \\
\overset{1}{\bullet} \arrow[r, "a"] & \overset{2}{\bullet} \arrow[ur, "b"] \arrow[dr, "c"'] & & \overset{5}{\bullet} \arrow[r, "f"] & \overset{6}{\bullet} \arrow[r, "g"] & \overset{7}{\bullet} \\
 & & \overset{4}{\bullet} \arrow[ur, "e"'] & & &
\end{tikzcd}$$

the ring K given by the product of 7 copies of the base field $\Bbbk$, identified with the vertices of Q, and the module V generated by the arrows $\mathscr{B}_1$ of Q, so $\Bbbk Q = TV$. Let $A = \Bbbk Q/I$, where I is the ideal generated by the submodule $R \subseteq TV$ generated by $\mathscr{B}_2 = \{ac, df, efg, bd - ce\}$. Let $T_1 = V$ and $T_2 = R$.

Given $\lambda = (\lambda_1 : \lambda_2)$ in $\mathbb{P}^1(\Bbbk)$, let $T_3^\lambda \subseteq TV$ be the submodule generated by $\mathscr{B}_3^\lambda = \{\beta_\lambda = \lambda_1 acef + \lambda_2 abdf, (bd - ce)fg\}$. It is straightforward to verify that, given any family $\bar{T} = \{T_p : p \in \mathbb{N}\}$ compatible with I, then $T_3 = T_3^{\lambda_0}$ for some $\lambda^0 = (\lambda_1^0 : \lambda_2^0) \neq (1 : -1)$, since $a(ce - bd)f \in (TV)_{>0} \cdot I \cdot (TV)_{>0}$ and this is impossible if T_3 is part of a compatible family, by (2.2). Assume that the family $\bar{T}$ is based and let $\alpha \in \mathscr{J}_3$ such that $\beta_{\lambda^0} \in T_{3,\alpha}$. It is easy to see that the maps Δ_n defined in Subsection 3.3 for all integers $n \geq 2$ satisfy $\Delta_n(\beta_{\lambda_0}) = 0$ for all $n \neq 3$, by degree reasons. Moreover, $\Delta_3(\beta_{\lambda^0}) \in V \otimes V \otimes R$ if and only if $\lambda_1^0 = 0$, whereas $\Delta_3(\beta_{\lambda_0}) \in R \otimes V \otimes V$ if and only if $\lambda_2^0 = 0$. This implies that $\bar{T}$ is either left or right resolutive, but not both. By Corollary 3.12, the maps Δ_n defined in Subsection 3.3 can never give a A_∞-coalgebra structure quasi-isomorphic to $\mathrm{Tor}^A_\bullet(K,K)$.

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