Categories of Graph Rewrite Rules

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Abstract. What are the legitimate morphisms between graph rewrite rules? The question is complicated by the diversity of approaches. From the familiar Double-Pushout (DPO) to the more recent PBPO and many others, the rules have different shapes, semantics (defined by direct transformations) and even matchings. We propose to represent these approaches by categories of rules, direct transformations and matchings related by functors in *Rewriting Environments with Matchings.* From these we extract a so-called X-functor whose properties are key to make rule morphisms meaningful. We show that these properties are preserved by combining approaches and by restricting them to strict matchings.

1 Introduction

An "approach" to graph rewriting is characterized by a particular notion of *rules* in a category $\mathcal{C}$, whose objects are those we wish to rewrite (generally graph-like data structures), together with a notion of *direct transformations* that defines how an individual rule transforms an input object G.

It is generally expected that such transformations proceed by local replacement of a part of G by something else, as in term rewriting. This replacement is usually decomposed as the deletion of a part of G followed by the addition of a new part. However, the deletion step may also involve some duplications of parts of G, as in the Sesqui-Pushout (or SqPO, see [6]) and the Pullback-Pushout (or PBPO, see [5]) approaches.

All in all there is no general agreement on what is a graph rewrite rule and how it should be applied. A closer look reveals that the disagreement concerns the deletion/duplication step, while most approaches agree that the addition step is a pushout (that may also merge parts of G), though in the SPO approach both steps are performed by one pushout in a category of partial morphisms, see [10]. But partial pushouts are also obtained as pushouts of $\mathcal{C}$-spans.

This observation leads to a natural notion of *partial transformations* of $\mathcal{C}$-objects (see [4] and Section 2.2 below) and morphisms between them, thus defining a category $\mathcal{C}_{\text{pt}}$. These morphisms can be understood as subsumptions. Informally, a transformation is subsumed by another one if it performs fewer modifications to G; that is both fewer deletions/duplications and fewer additions/mergings. It is therefore based on comparing *how* the results are obtained and not on comparing the results themselves (earning less and spending less can make one either poorer or richer in the end).

The category $\mathcal{C}_{\text{pt}}$ can be used to provide a semantics to morphisms between direct transformations of most approaches, through functors $\mathsf{P} : \mathcal{D} \to \mathcal{C}_{\text{pt}}$ where $\mathcal{D}$ is any category of such morphisms, that can therefore be understood as subsumptions (in particular a discrete category $\mathcal{D}$ of direct transformations is valid).

But how can we link morphisms between rules (or any category $\mathcal{R}$ whose objects are the rules of a given approach) to P? We can easily assume that any direct transformation contains the rule that it uses, hence a functor $\mathsf{R} : \mathcal{D} \to \mathcal{R}$ (the two functors make a *Rewriting Environment* in [4], RE for short). But this is clearly not sufficient to provide a semantics for rule morphisms: if $\mathcal{D}$ is discrete then non trivial $\mathcal{R}$-morphisms are meaningless.

We therefore need a constraint on $\mathcal{R}$-morphisms (w.r.t. to $\mathcal{D}$-morphisms); it will be given by a *Correctness Condition*. Informally, this condition means that any rule morphism yields a corresponding subsumption between direct transformations provided that they are obtained by overlapping matchings. The overlap should be consistent with the subsumption.

For this reason we need to enhance REs with a notion of matchings of left-hand sides of rules, and of morphisms (representing overlaps) between matchings. This notion should be general enough to encompass the standard situation, where matchings are $\mathcal{C}$-morphisms, but also the non standard matchings used in PBPO direct transformations (that involve two consecutive $\mathcal{C}$-morphisms, a match and a co-match), and possibly others. This is the subject of Section 3, where it is also shown how different notions of matchings can be combined, and how the standard notions of monic matches and identity matching may be generalized.

In Section 4 this general notion of matching is plugged into REs, and it is shown how it can contribute to analyze their properties. The analysis involves the definition of a category of *redexes* (i.e., pairs of a rule and a matching of its left-hand side). This allows to define a so-called X-*functor* whose properties are shown to characterize some key properties of REs, since in particular it is fully faithful iff the Correctness Condition holds. It is also shown how different REs can be combined and how they can be restricted to strict matchings.

But first we need to compile some definitions and results, mostly concerning functors.

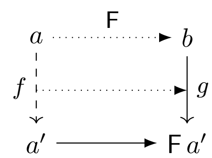
2 Background

The standard notions of Category Theory are assumed, see [11]. For any category $\mathcal{C}$, we write $G \in \mathcal{C}$ to indicate that G is a $\mathcal{C}$-object, and $|\mathcal{C}|$ is the discrete category on $\mathcal{C}$-objects. Then G may also denote the functor from the terminal category $\mathbf{1}$ to $\mathcal{C}$ or to $|\mathcal{C}|$ (as specified in the context) that maps the object of $\mathbf{1}$ to G.

2.1 Functors

We will use the following notion from [4].

Definition 1 (right-full). *A functor* $\mathsf{F} : \mathcal{A} \to \mathcal{B}$ *is* right-full *if for all* $a' \in \mathcal{A}$*, all* $b \in \mathcal{B}$ *and all* $\mathcal{B}$*-morphism* $g : b \to \mathsf{F}\, a'$*, there exist* $a \in \mathcal{A}$ *and an* $\mathcal{A}$*-morphism* $f : a \to a'$ *such that* $\mathsf{F}\, f = g$*.*



Note that a full functor may not be right-full (since b may have no preimage) and a right-full functor may not be full (since a depends on g). It is obvious that right-fullness is closed by composition and that all isomorphisms are right-full.

Lemma 1. *For any functors* $\mathsf{F} : \mathcal{A} \to \mathcal{B}$ *and* $\mathsf{G} : \mathcal{B} \to \mathcal{C}$ *such that* G *is faithful then* F *is faithful iff* $\mathsf{G} \circ \mathsf{F}$ *is faithful. If* G *is faithful and right-full then* F *is right-full iff* $\mathsf{G} \circ \mathsf{F}$ *is right-full.*

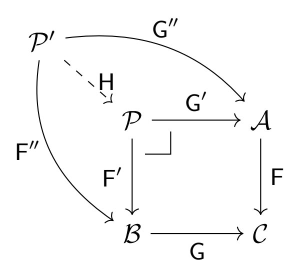
Proof. Suppose $\mathsf{G} \circ \mathsf{F}$ is right-full and let $a' \in \mathcal{A}$, $b \in \mathcal{B}$ and $g : b \to \mathsf{F}\, a'$ in $\mathcal{B}$, then $\mathsf{G}\, g : \mathsf{G}\, b \to \mathsf{G}\,\mathsf{F}\, a'$ is a $\mathcal{C}$-morphism, hence there exists $a \in \mathcal{A}$ and $f : a \to a'$ in $\mathcal{A}$ such that $\mathsf{G}\,(\mathsf{F}\, f) = \mathsf{G}\, g$, hence such that $\mathsf{F}\, f = g$, which shows that F is right-full. The other claims are proven similarly (or well known). □

Definition 2 (embedding, meet, inverse image). *We call* embedding *a functor that is both faithful and injective on objects (equivalently, that is left-cancellable). If* $\mathcal{A}$ *is a subcategory of* $\mathcal{B}$ *then the* canonical embedding of $\mathcal{A}$ into $\mathcal{B}$ *is the functor* $\mathsf{J} : \mathcal{A} \to \mathcal{B}$ *defined by* $\mathsf{J}\, a \coloneqq a$ *for all* $a \in \mathcal{A}$ *and* $\mathsf{J}\, f \coloneqq f$ *for all* $\mathcal{A}$*-morphisms* f*.*

A meet *of two functors* $\mathsf{F} : \mathcal{A} \to \mathcal{C}$ *and* $\mathsf{G} : \mathcal{B} \to \mathcal{C}$ *consists in a category* $\mathcal{P}$ *and two functors* $\mathsf{G}' : \mathcal{P} \to \mathcal{A}$ *and* $\mathsf{F}' : \mathcal{P} \to \mathcal{B}$ *such that* $\mathsf{F} \circ \mathsf{G}' = \mathsf{G} \circ \mathsf{F}'$ *and for all functors* F'', G'' *such that* $\mathsf{F} \circ \mathsf{G}'' = \mathsf{G} \circ \mathsf{F}''$ *there is a unique functor* H *such that* $\mathsf{F}'' = \mathsf{F}' \circ \mathsf{H}$ *and* $\mathsf{G}'' = \mathsf{G}' \circ \mathsf{H}$.

If F *is an embedding then* F' *is called an* inverse image of F along G.

Since meets have the universal property of pullbacks (they are pullbacks in the "category" of categories) they will be pictured as are standard pullbacks:



It is well-known (see [9]) that meets always exist: take for $\mathcal{P}$ the subcategory of $\mathcal{A} \times \mathcal{B}$ with objects (a, b) such that $\mathsf{F}\, a = \mathsf{G}\, b$ and with morphisms (f, g) such that $\mathsf{F}\, f = \mathsf{G}\, g$, and take for G' the projection $\pi_{\mathcal{A}} : \mathcal{P} \to \mathcal{A}$ on the first coordinate and for F' the projection $\pi_{\mathcal{B}} : \mathcal{P} \to \mathcal{B}$ on the second coordinate. All meets are isomorphic to this one, that we may therefore call *the* meet of F and G.

If F is an embedding of $\mathcal{A}$ into $\mathcal{C}$, then $\mathsf{F}(\mathcal{A})$ is a subcategory of $\mathcal{C}$ and (a restriction of) F is an isomorphism from $\mathcal{A}$ to its image $\mathsf{F}(\mathcal{A})$, with inverse, say, $\mathsf{F}^{-1} : \mathsf{F}(\mathcal{A}) \to \mathcal{A}$. Let $\mathcal{B}'$ be the subcategory of $\mathcal{B}$ with all $b \in \mathcal{B}$ such that $\mathsf{G}\,b \in \mathsf{F}(\mathcal{A})$ and all $\mathcal{B}$-morphisms g such that $\mathsf{G}\,g$ is a $\mathsf{F}(\mathcal{A})$-morphism. It is easy to see that the second projection $\pi_2 : \mathcal{P} \to \mathcal{B}'$ is an isomorphism, that $\pi_{\mathcal{B}} \circ {\pi_2}^{-1}$ is the canonical embedding of $\mathcal{B}'$ into $\mathcal{B}$ and that $\pi_{\mathcal{A}} \circ {\pi_2}^{-1} = \mathsf{F}^{-1} \circ \mathsf{G}'$ where $\mathsf{G}' : \mathcal{B}' \to \mathsf{F}(\mathcal{A})$ is a restriction of G. Hence all inverse images of F along G are isomorphic to $\mathcal{B}'$, that we may therefore call *the* inverse image of F along G.

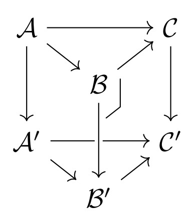
Besides, the following properties of functors are preserved by meets:

Lemma 2. *If F is right-full (resp. full, resp. faithful, resp. an embedding, resp. surjective on objects) then so is F'.*

Proof. Suppose F is right-full and let (a', b') be an object in *the* meet of F and G, let $b \in \mathcal{B}$ and $g : b \to b'$ in $\mathcal{B}$ (since $\pi_{\mathcal{B}}(a', b') = b'$), then $\mathsf{G}\,g : \mathsf{G}\,b \to \mathsf{F}\,a'$ is a $\mathcal{C}$-morphism (since $\mathsf{F}\,a' = \mathsf{G}\,b'$), hence there exists $a \in \mathcal{A}$ and an $\mathcal{A}$-morphism $f : a \to a'$ such that $\mathsf{F}\,f = \mathsf{G}\,g$ (and hence $\mathsf{F}\,a = \mathsf{G}\,b$), so that $(f, g) : (a, b) \to (a', b')$ is a morphism such that $\pi_{\mathcal{B}}(f, g) = g$, hence $\pi_{\mathcal{B}}$ is right-full. This holds for F' since it is obtained by composing $\pi_{\mathcal{B}}$ with an isomorphism. The other claims are proven similarly. □

We will also use an instance of the well-known pullback composition and decomposition lemma (see [1, Proposition 11.10]), applied to meets of functors:

Lemma 3. *If the following diagram of functors*



commutes and the right face is a meet, then the left face is a meet iff the back face is a meet.

Similarly we will use the fact that meets are mono-sources (see [1, Proposition 11.6]):

Lemma 4. *If $\mathcal{P}$ with F' and G' is a meet of F and G, and $\mathsf{H}, \mathsf{H}' : \mathcal{P}' \to \mathcal{P}$ are such that $\mathsf{F}' \circ \mathsf{H} = \mathsf{F}' \circ \mathsf{H}'$ and $\mathsf{G}' \circ \mathsf{H} = \mathsf{G}' \circ \mathsf{H}'$ then $\mathsf{H} = \mathsf{H}'$.*

Definition 3 (sum $\mathcal{A}_1 + \mathcal{A}_2$, injections $\mathsf{I}_{\mathcal{A}_i}$, functors $[\mathsf{F}_1, \mathsf{F}_2]$, $\mathsf{F}_1 + \mathsf{F}_2$). *Given two categories $\mathcal{A}_1$ and $\mathcal{A}_2$, their* sum *is the category $\mathcal{A}_1 + \mathcal{A}_2$ whose objects are pairs (i, a) where $i \in \{1, 2\}$ and $a \in \mathcal{A}_i$, and morphisms $f : (i, a) \to (i, a')$ are the $\mathcal{A}_i$-morphisms $f : a \to a'$, with the obvious composition (the union of the compositions in $\mathcal{A}_1$ and $\mathcal{A}_2$). The* injections *are the two embeddings $\mathsf{I}_{\mathcal{A}_i} : \mathcal{A}_i \to \mathcal{A}_1 + \mathcal{A}_2$ defined by $\mathsf{I}_{\mathcal{A}_i}\,a \coloneqq (i, a)$ for all $a \in \mathcal{A}_i$ and $\mathsf{I}_{\mathcal{A}_i}\,f \coloneqq f$ for all $\mathcal{A}_i$-morphisms f.*

For any functors $\mathsf{F}_i : \mathcal{A}_i \to \mathcal{C}$ *let* $[\mathsf{F}_1, \mathsf{F}_2] : \mathcal{A}_1 + \mathcal{A}_2 \to \mathcal{C}$ *be the functor defined by* $[\mathsf{F}_1, \mathsf{F}_2]\,(i, a) \coloneqq \mathsf{F}_i\, a$ *for all* $(i, a) \in \mathcal{A}_1 + \mathcal{A}_2$ *and* $[\mathsf{F}_1, \mathsf{F}_2]\, f \coloneqq \mathsf{F}_i\, f$ *for all* $\mathcal{A}_i$*-morphisms* f*.*

For any functors $\mathsf{G}_i : \mathcal{A}_i \to \mathcal{B}_i$ *let* $\mathsf{G}_1 + \mathsf{G}_2 : \mathcal{A}_1 + \mathcal{A}_2 \to \mathcal{B}_1 + \mathcal{B}_2$ *be the functor defined by* $\mathsf{G}_1 + \mathsf{G}_2 \coloneqq [\mathsf{I}_{\mathcal{B}_1} \circ \mathsf{G}_1, \mathsf{I}_{\mathcal{B}_2} \circ \mathsf{G}_2]$*.*

Injections have the universal property of coproducts, i.e., that $[\mathsf{F}_1, \mathsf{F}_2]$ is the unique functor such that $[\mathsf{F}_1, \mathsf{F}_2] \circ \mathsf{I}_{\mathcal{A}_i} = \mathsf{F}_i$ for $i = 1, 2$. From this it is easy to deduce that $[\mathsf{I}_{\mathcal{A}_1}, \mathsf{I}_{\mathcal{A}_2}] = 1_{\mathcal{A}_1 + \mathcal{A}_2}$, $[\mathsf{F} \circ \mathsf{F}_1, \mathsf{H} \circ \mathsf{F}_2] = \mathsf{F} \circ [\mathsf{F}_1, \mathsf{F}_2]$, $[\mathsf{F}_1 \circ \mathsf{G}_1, \mathsf{F}_2 \circ \mathsf{G}_2] = [\mathsf{F}_1, \mathsf{F}_2] \circ (\mathsf{G}_1 + \mathsf{G}_2)$ and $(\mathsf{H}_1 + \mathsf{H}_2) \circ (\mathsf{G}_1 + \mathsf{G}_2) = (\mathsf{H}_1 \circ \mathsf{G}_1) + (\mathsf{H}_2 \circ \mathsf{G}_2)$.

As above the following properties of functors are preserved by sums.

Lemma 5. *If* $\mathsf{F}_i : \mathcal{A}_i \to \mathcal{B}_i$ *are right-full (resp. full, resp. faithful, resp. embeddings) for* $i = 1, 2$ *then so is* $\mathsf{F}_1 + \mathsf{F}_2$*.*

Proof. Suppose F_1 and F_2 are right-full and let $(i', a') \in \mathcal{A}_1 + \mathcal{A}_2$, $(i, b) \in \mathcal{B}_1 + \mathcal{B}_2$ and $g : (i, b) \to (\mathsf{F}_1 + \mathsf{F}_2)\,(i', a')$ a $\mathcal{B}_1 + \mathcal{B}_2$-morphism, since $(\mathsf{F}_1 + \mathsf{F}_2)\,(i', a') = (i', \mathsf{F}_{i'}\, a')$ then $i = i'$ and $g : b \to \mathsf{F}_i\, a'$ is a $\mathcal{B}_i$-morphism, hence there exist $a \in \mathcal{A}_i$ and an $\mathcal{A}_i$-morphism $f : a \to a'$ such that $\mathsf{F}_i\, f = g$, and therefore $(\mathsf{F}_1 + \mathsf{F}_2)\, f = (\mathsf{I}_{\mathcal{B}_i} \circ \mathsf{F}_i)\, f = \mathsf{F}_i\, f = g$, hence $\mathsf{F}_1 + \mathsf{F}_2$ is right-full. The other claims are proven similarly. □

We will also use the fact that meets are preserved by sums.

Lemma 6. *If*
$$\begin{array}{ccc} \mathcal{P}_i & \xrightarrow{\mathsf{G}'_i} & \mathcal{A}_i \\ {\scriptstyle \mathsf{F}'_i}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{F}_i} \\ \mathcal{B}_i & \xrightarrow[\mathsf{G}_i]{} & \mathcal{C}_i \end{array}$$
for $i = 1, 2$ *then*
$$\begin{array}{ccc} \mathcal{P}_1 + \mathcal{P}_2 & \xrightarrow{\mathsf{G}'_1 + \mathsf{G}'_2} & \mathcal{A}_1 + \mathcal{A}_2 \\ {\scriptstyle \mathsf{F}'_1 + \mathsf{F}'_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{F}_1 + \mathsf{F}_2} \\ \mathcal{B}_1 + \mathcal{B}_2 & \xrightarrow[\mathsf{G}_1 + \mathsf{G}_2]{} & \mathcal{C}_1 + \mathcal{C}_2 \end{array}$$

2.2 Rewriting Environments

All considered approaches to graph rewriting produce a $\mathcal{C}$-span $D \xleftarrow{k} K \xrightarrow{r} R$, where D is called the *context*, K the *interface* and R the *right-hand side*. The context D is obtained from the input G by deletions/duplications so that, equivalently, we can see G as being obtained from D by additions/mergings (inverse to deletions/duplications) hence that there is a morphism $f : D \to G$. Hence the notion of partial transformations in Definition 4 below.

Following the informal description of a subsumption, we say that a partial transformation p is subsumed by p' if the latter's context D' is be obtained from the former's D (from the same input G) by further deletions/duplications, so that there must be a morphism from D' to D. Similarly, p' should make further additions/mergings than p, hence there should ba a morphism from R to R' and also from K to K' (the interface glues the new part to the old, and further additions may require a bigger interface).

Definition 4 (category $\mathcal{C}_{\mathrm{pt}}$, functor In, Rewriting Environments). *A* partial transformation *p in $\mathcal{C}$ is a diagram*

$$G \xleftarrow{\;f\;} D \xleftarrow{\;k\;} K \xrightarrow{\;r\;} R$$

For any category $\mathcal{C}$, let $\mathcal{C}_{\mathrm{pt}}$ be the category whose objects are partial transformations and morphisms $s : p \to p'$ are triples (s_1, s_2, s_3) of $\mathcal{C}$-morphisms such that

$$\begin{array}{ccccccc}
G & \xleftarrow{\;f\;} & D & \xleftarrow{\;k\;} & K & \xrightarrow{\;r\;} & R \\
= \big| & & \big\uparrow s_1 & & s_2 \big\downarrow & & \big\downarrow s_3 \\
G' & \xleftarrow[\;f'\;]{} & D' & \xleftarrow[\;k'\;]{} & K' & \xrightarrow[\;r'\;]{} & R'
\end{array}$$

commutes in $\mathcal{C}$, with the obvious composition $(s'_1, s'_2, s'_3) \circ (s_1, s_2, s_3) \coloneqq (s_1 \circ s'_1, s'_2 \circ s_2, s'_3 \circ s_3)$.

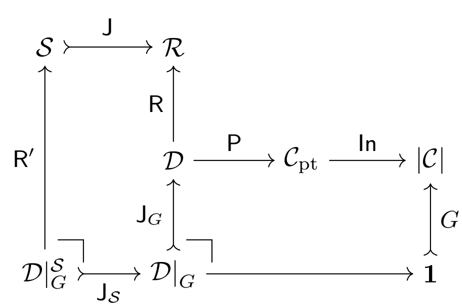
Let $\mathsf{In} : \mathcal{C}_{\mathrm{pt}} \to |\mathcal{C}|$ be the input functor *defined as $\mathsf{In}\, p \coloneqq G$.*

A Rewriting Environment *(or* RE*) $\mathscr{R}$ for $\mathcal{C}$ consists of a category $\mathcal{D}$ of* direct transformations*, a category $\mathcal{R}$ of* rules *and two functors*

$$\mathcal{R} \xleftarrow{\;\mathsf{R}\;} \mathcal{D} \xrightarrow{\;\mathsf{P}\;} \mathcal{C}_{\mathrm{pt}}$$

A rule system *in $\mathscr{R}$ is a category $\mathcal{S}$ with an embedding $\mathsf{J} : \mathcal{S} \to \mathcal{R}$ (alternately, $\mathcal{S}$ is a subcategory of $\mathcal{R}$ and J is the canonical embedding).*

Given a rule system and an input *$\mathcal{C}$-object G, we build the categories $\mathcal{D}|_G$, $\mathcal{D}|_G^{\mathcal{S}}$ and functors J_G, $\mathsf{J}_{\mathcal{S}}$ as inverse images of the embeddings G and J.*



(1)

Note that by our construction of inverse images and by Lemma 2 (since $G : \mathbf{1} \to |\mathcal{C}|$ is full), $\mathcal{D}|_G$ is a full subcategory of $\mathcal{D}$. Similarly, $\mathcal{D}|_G^{\mathcal{S}}$ is a subcategory of $\mathcal{D}$ that may not be full if J is not full. Hence $\mathcal{D}|_G^{\mathcal{S}}$ contains all direct transformations of G by the rules in $\mathcal{S}$, and all the subsumptions between these whose image by R also belongs to $\mathcal{S}$ (or its image by J). In this way rule systems are used to specify subcategories of direct transformations of an input object G.

Neither R nor R' are generally full, for the simple reason that if a rule can be applied with two unrelated (say disjoint) matchings in G, then the corresponding direct transformations (say d_1 and d_2) may not subsume each other. Indeed, their contexts (i.e. the contexts of $\mathsf{P}\, d_1$ and $\mathsf{P}\, d_2$) may be obtained by deleting disjoint parts of G, and then none can be obtained by further deletions from the other. Hence, not every morphism in $\mathcal{S}$ is reflected by a morphism in $\mathcal{D}|_G^{\mathcal{S}}$.

Another important property is whether R' is right-full, for this means that for any direct transformation $d' \in \mathcal{D}|_G^{\mathcal{S}}$ and any rule morphism $f : r \to \mathsf{R}'\, d'$ in the rule system $\mathcal{S}$, there exists a transformation $d \in \mathcal{D}|_G^{\mathcal{S}}$ subsumed by d' (with subsumption s s.t. $\mathsf{R}'\, s = f$), i.e., when a rule $\mathsf{R}'\, d'$ applies any "subrule" r necessarily applies. This is a property that seems all too natural but it does not always hold, e.g., in the DPO approach with unrestricted matchings. It is shown in [4, Proposition 6.3] that R' is right-full whenever R is right-full, hence we will focus on R.

One important feature of REs is that they can be combined:

Definition 5. *Given two REs* $\mathscr{R}_i = \mathcal{R}_i \xleftarrow{\mathsf{R}_i} \mathcal{D}_i \xrightarrow{\mathsf{P}_i} |\mathcal{C}|$ *for* $\mathcal{C}$ *for* $i = 1, 2$*, their* sum $\mathscr{R}_1 + \mathscr{R}_2$ *is the RE*

$$\mathcal{R}_1 + \mathcal{R}_2 \xleftarrow{\mathsf{R}_1 + \mathsf{R}_2} \mathcal{D}_1 + \mathcal{D}_2 \xrightarrow{[\mathsf{P}_1, \mathsf{P}_2]} \mathcal{C}_{\mathrm{pt}}$$

By Lemma 5 we obviously have:

Proposition 1. *If* R_1 *and* R_2 *are right-full (resp. faithful) then so is* $\mathsf{R}_1 + \mathsf{R}_2$*.*

3 Matching Environments

A matching is generally understood as a kind of relation between a source object, that we call a *pattern* p, and a target object, in our case the input object $G \in \mathcal{C}$. In most approaches this is simply a $\mathcal{C}$-morphism, so that matchings may be composed. In Term Rewriting a matching is a substitution together with a position in G, but once again matchings can be composed since G can be understood also as a pattern (a term with variables). However, in the PBPO approach a matching in G is a commuting diagram

$$\begin{array}{c} L \\ \downarrow m \\ G \\ \downarrow c \\ T_L \end{array} \qquad (t_L = c \circ m : L \to T_L)$$

where t_L belongs to the PBPO rule. More precisely, the pattern appears to be the $\mathcal{C}$-morphism t_L, and the matching that relates it to G is the pair (m, c) that factors t_L through G. Anyway, the pattern and the target have different natures and it therefore seems difficult to consider a target as a pattern. Thus we have to drop the possibility to compose matchings.

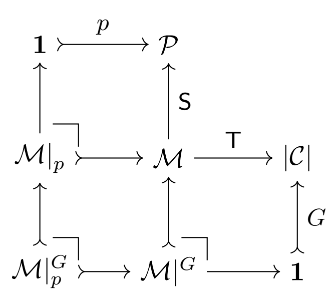
For this reason we treat matchings and patterns as objects each in their respective category. Informally, a matching designates the location in G of its pattern. Pattern morphisms are generalizing in the sense that a matching of the codomain pattern induces a matching of the domain pattern. Matching morphisms describe overlaps in the sense that the domain pattern and location are overlapped by a more general pattern at a nearby location (hence in the same G). This yields a definition similar to REs.

Definition 6 (Matching Environment $\mathscr{M}$, categories $\mathcal{M}|^G$, $\mathcal{M}|_p$, $\mathcal{M}|_p^G$). *A* Matching Environment *(or* ME*)* $\mathscr{M}$ *for* $\mathcal{C}$ *consists of a category* $\mathcal{M}$ *whose objects are called* matchings*, a category* $\mathcal{P}$ *whose objects are called* patterns*, and two* source *and* target *functors*

$$\mathcal{P} \xleftarrow{\mathsf{S}} \mathcal{M} \xrightarrow{\mathsf{T}} |\mathcal{C}|$$

An object $m \in \mathcal{M}$ *is a matching* of $\mathsf{S}\,m$ into $\mathsf{T}\,m$*.*

As in REs we extract from $\mathcal{M}$ *the category* $\mathcal{M}|_p$ *of matchings* of *a pattern* $p \in \mathcal{P}$*, the category* $\mathcal{M}|^G$ *of matchings* into *an object* $G \in \mathcal{C}$*, and the category* $\mathcal{M}|_p^G$ *of matchings* of p into G*, by taking the inverse image of the corresponding embeddings* $p : \mathbf{1} \to \mathcal{P}$ *and* $G : \mathbf{1} \to |\mathcal{C}|$ *along* S *and* T *as pictured below (so that we obtain subcategories of* $\mathcal{M}$*).*



This general definition is easily illustrated by the standard notion of matchings, where patterns are $\mathcal{C}$-objects and matchings are $\mathcal{C}$-morphisms to G and are connected by pattern morphisms restricted by a trivial commuting condition. For PBPO rules the notions of matchings and patterns should be clear, but their morphisms may not. The notion of subsumption morphism between PBPO rules given in [4, Definition 6.8] spills the beans: a morphism from t_L to $t_{L'}$ is a factorization of t_L through $t_{L'}$.

Definition 7 (Matching Environments $\mathscr{M}_{\mathrm{std}}$, $\mathscr{M}_{\mathrm{fct}}$). *Let* $\mathcal{M}_{\mathrm{std}}$ *be the category whose objects are the* $\mathcal{C}$*-morphisms and whose morphisms* $f : m \to m'$ *are* $\mathcal{C}$*-morphisms such that*

$$
\begin{array}{ccc}
L & \xrightarrow{f} & L' \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\
G & \overset{=}{\rule[0.5ex]{2em}{0.4pt}} & G'
\end{array}
$$

commutes, with the same composition as $\mathcal{C}$. The functor $\mathsf{S}_{\mathrm{std}} : \mathcal{M}_{\mathrm{std}} \to \mathcal{C}$ is defined by $\mathsf{S}_{\mathrm{std}}\, m \coloneqq L$ where L is the domain of m, and $\mathsf{S}_{\mathrm{std}}\, f \coloneqq f$. The functor $\mathsf{T}_{\mathrm{std}} : \mathcal{M}_{\mathrm{std}} \to |\mathcal{C}|$ is defined by $\mathsf{T}_{\mathrm{std}}\, m \coloneqq G$ where G is the codomain of m, and $\mathsf{T}_{\mathrm{std}}\, f \coloneqq 1_G$ for all $f : m \to m'$ in $\mathcal{M}_{\mathrm{std}}$ and G is the common codomain of m and m'. Let $\mathscr{M}_{\mathrm{std}}$ be the Matching Environment $\mathcal{C} \xleftarrow{\mathsf{S}_{\mathrm{std}}} \mathcal{M}_{\mathrm{std}} \xrightarrow{\mathsf{T}_{\mathrm{std}}} |\mathcal{C}|$.

Let $\mathcal{P}_{\mathrm{fct}}$ be the category whose objects are $\mathcal{C}$-morphisms and whose morphisms are pairs of $\mathcal{C}$-morphisms $(f, g) : p \to p'$ such that

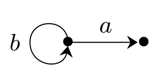
$$
\begin{array}{ccc}
L & \xrightarrow{f} & L' \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\
T & \xleftarrow[g]{} & T'
\end{array}
$$

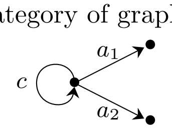
commutes, with obvious composition $(f', g') \circ (f, g) \coloneqq (f' \circ f, g \circ g')$. The category of matchings $\mathcal{M}_{\mathrm{fct}}$ has as objects pairs (m, c) of consecutive $\mathcal{C}$-morphisms, i.e., such that $c \circ m$ exists, and as morphisms pairs of $\mathcal{C}$-morphisms $(f, g) : (m, c) \to (m', c')$ such that

$$
\begin{array}{ccc}
L & \xrightarrow{f} & L' \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\
G & \overset{=}{\rule[0.5ex]{2em}{0.4pt}} & G' \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\
T & \xleftarrow[g]{} & T'
\end{array}
$$

commutes, with composition as in $\mathcal{P}_{\mathrm{fct}}$. The functor $\mathsf{S}_{\mathrm{fct}} : \mathcal{M}_{\mathrm{fct}} \to \mathcal{P}_{\mathrm{fct}}$ is defined by $\mathsf{S}_{\mathrm{fct}}(m, c) \coloneqq c \circ m$ and $\mathsf{S}_{\mathrm{fct}}(f, g) \coloneqq (f, g)$. The functor $\mathsf{T}_{\mathrm{fct}} : \mathcal{M}_{\mathrm{fct}} \to |\mathcal{C}|$ is defined by $\mathsf{S}_{\mathrm{fct}}(m, c) \coloneqq G$ and $\mathsf{S}_{\mathrm{fct}}(f, g) \coloneqq 1_G$ for all $(f, g) : (m, c) \to (m', c')$, where G is the common codomain of m, m' and domain of c, c'. Let $\mathscr{M}_{\mathrm{fct}}$ be the Matching Environment $\mathcal{P}_{\mathrm{fct}} \xleftarrow{\mathsf{S}_{\mathrm{fct}}} \mathcal{M}_{\mathrm{fct}} \xrightarrow{\mathsf{T}_{\mathrm{fct}}} |\mathcal{C}|$.

Example 1. In $\mathscr{M}_{\mathrm{std}}$ where $\mathcal{C}$ is the category of graphs, we consider the pattern

$p =$ 

and the graph $G =$ 

There are two monic matchings $m_i \coloneqq \{b \mapsto c,\, a \mapsto a_i\}$ of p into G for $i = 1, 2$, and one non monic matching $m_0 \coloneqq \{a, b \mapsto c\}$. The $\mathcal{C}$-morphism $f \coloneqq \{a, b \mapsto b\} : p \to p$ is an $\mathcal{M}_{\text{std}}$-morphism from m_0 to m_1 since $m_1 \circ f = m_0$, and also a $\mathcal{M}_{\text{std}}$-morphism from m_0 to m_2 since $m_2 \circ f = m_0$. Hence the category $\mathcal{M}_{\text{std}}|_p^G$ is a span $m_1 \leftarrow m_0 \rightarrow m_2$.

We have the same problem with MEs as with REs since the notion allows $\mathcal{M}$ to be discrete even when $\mathcal{P}$ is not. One may hope to constrain $\mathcal{M}$-morphisms by requiring S to be full, but neither S_{std} nor S_{fct} are full. Fortunately they share the following properties.

Proposition 2. *The functors* S_{std} *and* S_{fct} *are right-full and faithful.*

Proof. Faithfulness is obvious. To show that S_{std} is right-full, take $m' : L' \to G'$ an object of $\mathcal{M}_{\text{std}}$ and $f : L \to \mathsf{S}_{\text{std}}\, m'$ a morphism in $\mathcal{C}$, since $\mathsf{S}_{\text{std}}\, m' = L'$ then $f : m' \circ f \to m'$ is a morphism in $\mathcal{M}_{\text{std}}$ such that $\mathsf{S}_{\text{std}}\, f = f$.

Let $(m', c') \in \mathcal{M}_{\text{fct}}$ and $(f, g) : p \to c' \circ m'$ in $\mathcal{P}_{\text{fct}}$, then $(f, g) : (m' \circ f, g \circ c') \to (m', c')$ is a morphism in $\mathcal{M}_{\text{fct}}$, hence S_{fct} is also right-full. □

It therefore seems reasonable to require, at least, that S be right-full, which obviously imposes a constraint on $\mathcal{M}$-morphisms w.r.t. $\mathcal{P}$-morphisms. It also embodies the informal description of pattern morphisms as ways of obtaining a matching of the domain pattern from a matching of the codomain pattern. Requiring further that S be faithful means that such ways are unique once the location of the patterns (i.e., the matchings) are fixed.

Next we see that, as in Definition 5 for REs, MEs can easily be combined while preserving the properties of the source functors.

Definition 8 (sum of ME). *The* sum $\mathscr{M}_1 + \mathscr{M}_2$ *of two Matching Environments* $\mathscr{M}_i = \mathcal{P}_i \xleftarrow{\mathsf{S}_i} \mathcal{M}_i \xrightarrow{\mathsf{T}_i} |\mathcal{C}|$ *for* $i = 1, 2$ *is the Matching Environment*

$$\mathcal{P}_1 + \mathcal{P}_2 \xleftarrow{\mathsf{S}_1 + \mathsf{S}_2} \mathcal{M}_1 + \mathcal{M}_2 \xrightarrow{[\mathsf{T}_1, \mathsf{T}_2]} |\mathcal{C}|$$

By Lemma 5 we have:

Proposition 3. *If* S_1 *and* S_2 *are right-full (resp. faithful) then so is* $\mathsf{S}_1 + \mathsf{S}_2$.

3.1 Strict and trivial matchings

We now try to generalize the notion of monic matchings in $\mathscr{M}_{\text{std}}$. Since matchings do not compose, we cannot rely on the standard notion of regularity (or cancellability) on the left. Intuitively, we wish to avoid matchings that relate distinct items (vertices, edges or whatever) of a pattern to the same item in G, and hence to avoid that locations that exist in patterns should be confused by matchings. If we see a matching as a map, we want every part of the map to have a unique location. The location of a part of a map within the whole map being given by a morphism, we are asking for a 1-1 correspondence between such morphisms and the parts of the map. This idea leads to a simple definition.

Definition 9 (strict matchings). *A matching $s \in \mathcal{M}$ is* strict *if for all morphisms $f : m \to s$ and $f' : m' \to s$ in $\mathcal{M}$, $m = m'$ entails $f = f'$.*

We first check that this notion corresponds to monic standard matchings.

Proposition 4. *A matching in $\mathcal{M}_{\mathrm{std}}$ is strict iff it is monic.*

Proof. Obvious since for all $f : m \to s$ and $f' : m' \to s$ in $\mathcal{M}_{\mathrm{std}}$ we have $m = m'$ iff $s \circ f = s \circ f'$. □

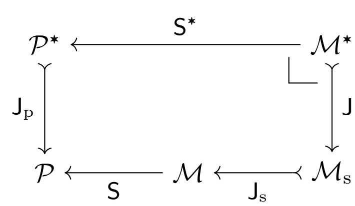
We then see that strict factor matchings appear as a quite natural extension of the standard case.

Proposition 5. *A matching $(m, c) \in \mathcal{M}_{\mathrm{fct}}$ is strict iff m is monic and c is epic.*

Proof. Suppose (m, c) is strict, for all $\mathcal{C}$-morphisms f_1, f_2, g_1, g_2 such that $m \circ f_1 = m \circ f_2$ and $g_1 \circ c = g_2 \circ c$, since $(f_1, g_1) : (m \circ f_1, g_1 \circ c) \to (m, c)$ and $(f_2, g_2) : (m \circ f_2, g_2 \circ c) \to (m, c)$ are $\mathcal{M}_{\mathrm{fct}}$-morphisms with identical domains, then $f_1 = f_2$ and $g_1 = g_2$, hence m is monic and c is epic. The converse is similar. □

We now devise a way to build a ME that only contains strict matchings and monic overlaps.

Definition 10 (strict restriction $\mathscr{M}^\star$ of $\mathscr{M}$). *Given a ME $\mathscr{M}$ as in Definition 6, let $\mathcal{P}^\star$ be the subcategory of monomorphisms of $\mathcal{P}$ and $\mathsf{J}_{\mathrm{p}} : \mathcal{P} \to \mathcal{P}^\star$ the canonical embedding. Let $\mathcal{M}_{\mathrm{s}}$ be the full subcategory of strict matchings of $\mathcal{M}$ and $\mathsf{J}_{\mathrm{s}} : \mathcal{M}_{\mathrm{s}} \to \mathcal{M}$ the (full) canonical embedding. Finally, let $\mathsf{S}^\star : \mathcal{M}^\star \to \mathcal{P}^\star$ be the inverse image of J_{p} along $\mathsf{S} \circ \mathsf{J}_{\mathrm{s}}$.*



The strict restriction *$\mathscr{M}^\star$ of $\mathscr{M}$ is the Matching Environment*

$$\mathcal{P}^\star \xleftarrow{\mathsf{S}^\star} \mathcal{M}^\star \xrightarrow{\mathsf{T}^\star} |\mathcal{C}|$$

where $\mathsf{T}^\star \coloneqq \mathsf{T} \circ \mathsf{J}_{\mathrm{s}} \circ \mathsf{J}$ (the restriction of T to $\mathcal{M}^\star$).

Example 2. The pattern category of the standard ME $\mathscr{M}_{\mathrm{std}}$ is $\mathcal{C}$, hence its restriction $\mathcal{C}^\star$ is the category with all $\mathcal{C}$-objects and whose morphisms are the $\mathcal{C}$-monomorphisms. Similarly, the $\mathcal{M}^\star_{\mathrm{std}}$-objects are the $\mathcal{C}$-monomorphisms, and the $\mathcal{M}^\star_{\mathrm{std}}$-morphisms $f : m \to m'$ are the $\mathcal{C}$-monomorphisms such that $m = m' \circ f$. The functors $\mathsf{S}^\star_{\mathrm{std}}$ and $\mathsf{T}^\star_{\mathrm{std}}$ are the obvious restrictions of $\mathsf{S}_{\mathrm{std}}$ and $\mathsf{T}_{\mathrm{std}}$ to monic matchings and monomorphisms. Note that $\mathcal{M}^\star_{\mathrm{std}}$ is a full subcategory of $\mathcal{M}_{\mathrm{std}}$ (since $m' \circ f$ monic entails f monic).

We can then prove that the relevant property of the source functor is preserved by this restriction. We first need an easy lemma.

Lemma 7. *For any $\mathcal{M}$-monomorphism $f : m \rightarrowtail m'$, if m' is strict then so is m.*

Proof. Let $g : n \to m$ and $g' : n' \to m$, if $n = n'$ then $f \circ g = f \circ g' : n \to m'$, hence $g = g'$. □

Proposition 6. *If S is right-full and faithful then so is $\mathsf{S}^\star$.*

Proof. Since $\mathsf{S} \circ \mathsf{J}_\mathrm{s}$ is faithful then so is $\mathsf{S}^\star$ by Lemma 2. Assuming furthermore that S is right-full, let $m' \in \mathcal{M}^\star$ and $g : p \rightarrowtail \mathsf{S}^\star\, m'$, since $\mathsf{S}^\star\, m' = \mathsf{S}\, m'$ there exists $f : m \to m'$ in $\mathcal{M}$ such that $\mathsf{S}\, f = g$. Monomorphisms are reflected by faithful functors [7, Proposition 12.8] hence f is monic and by Lemma 7 m is strict, so that f is an $\mathcal{M}_\mathrm{s}$-morphism. Hence by construction of inverse images f is a $\mathcal{M}^\star$-morphism and $\mathsf{S}^\star\, f = \mathsf{S}\, f = g$, thus $\mathsf{S}^\star$ is right-full. □

It is easy to see that $(\mathscr{M}^\star)^\star = \mathscr{M}^\star$, hence this restriction is one-shot. Similarly we see that restrictions and sums commute, i.e., $(\mathscr{M}_1 + \mathscr{M}_2)^\star = \mathscr{M}_1^\star + \mathscr{M}_2^\star$.

Another notion that seems difficult to generalize in absence of composable matchings is that of an identity matching, that first requires an identity between a pattern and the input object G, which is out of reach. Obviously the notion can only be generalized up to a point, and only up to $\mathcal{M}$-isomorphisms. The idea is to say that a matching into G is trivial if it is wider than all other matchings into G, in the sense that they are all a part of it.

Definition 11 (trivial matchings). *For any $g \in \mathcal{C}$, a matching into G is* trivial *if it is terminal in $\mathcal{M}|^G$.*

Hence a matching m into G is trivial if for all matchings m' into G there is a unique $f : m' \to m$. This unicity obviously entails that m is strict.

We see that trivial standard matchings are only an approximation of identity matchings, but the best we can expect.

Proposition 7. *A matching in $\mathcal{M}_\mathrm{std}$ is trivial iff it is an isomorphism.*

Proof. If $m \in \mathcal{M}_\mathrm{std}$ is a trivial matching into G, since 1_G is a matching into G then there exists $f : 1_G \to m$, so that $m \circ f = 1_G$, i.e., m is a retraction, and since m is monic by Proposition 4, then m is an isomorphism (by [7, Proposition 6.7]). The converse is obvious. □

Proposition 8. *A matching $(m, c) \in \mathcal{M}_\mathrm{fct}$ is trivial iff m and c are isomorphisms.*

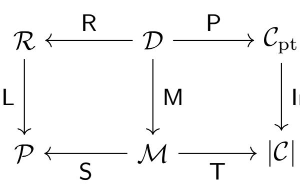
Proof. If (m, c) is a strict matching into G then there exists $(f, g) : (1_G, 1_G) \to (m, c)$, hence $m \circ f = 1_G$ and $g \circ c = 1_G$, i.e., m is a retraction and c is a section, and since m is monic and c is epic by Proposition 5 then m and c are isomorphisms. The converse is obvious. □

This obviously means that a pattern $p \in \mathcal{P}_\mathrm{fct}$ admits trivial matchings (in $\mathcal{M}|_p$) iff p is a $\mathcal{C}$-isomorphism (and then only into $\mathcal{C}$-objects isomorphic to p's domain and codomain).

4 Rewriting Environments with Matchings

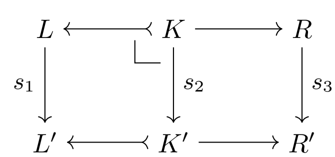
We can now enhance Rewriting Environments by stating that every direct transformation has a matching of the left-hand side of its rule into its input object. This matching and the left-hand side are accessed by means of two new functors, and their relations are expressed as commuting conditions.

Definition 12 (REM). *A* Rewriting Environment with Matchings *(or* REM*)* $\mathscr{E}$ *consists of a Rewriting Environment* $\mathscr{R}$ *as in Definition 4, a Matching Environment* $\mathscr{M}$ *as in Definition 6 and two functors* L *and* M *such that*

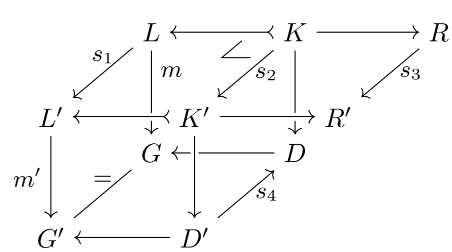


commutes. For every rule $r \in \mathcal{R}$*, the pattern* $\mathsf{L}\, r$ *is the* left-hand side *of* r*.*

Example 3. In [4] the category $\mathcal{R}_{\mathrm{DPO}}$ of DPO rules (in $\mathcal{C}$) is defined with morphisms as triples (s_1, s_2, s_3) of $\mathcal{C}$-morphisms such that



commutes and the left square is a pullback. Hence there is an obvious functor $\mathsf{L}_{\mathrm{DPO}} : \mathcal{R}_{\mathrm{DPO}} \to \mathcal{C}$ where $\mathsf{L}_{\mathrm{DPO}}\,(s_1, s_2, s_3) = s_1$. Similarly, the category $\mathcal{D}_{\mathrm{DPO}}$ has DPO diagrams as objects, and morphisms (s_1, s_2, s_3, s_4) such that



commutes and the top left square is a pullback. Again there is an obvious functor $\mathsf{M}_{\mathrm{DPO}} : \mathcal{D}_{\mathrm{DPO}} \to \mathcal{M}_{\mathrm{std}}$ where $\mathsf{M}_{\mathrm{DPO}}\,(s_1, s_2, s_3, s_4) = s_1$. The DPO RE together with $\mathscr{M}_{\mathrm{std}}$ and the two functors $\mathsf{L}_{\mathrm{DPO}}$ and $\mathsf{M}_{\mathrm{DPO}}$ constitute a REM. These functors are neither full nor faithful. If $\mathcal{C}$ does not have pullbacks they are not right-full either.

Using these functors we can now express the relevant condition for rule morphisms to specify subsumptions. Following Diagram (1), we expect every $f : r \to r'$ in $\mathcal{S}$ to be uniquely reflected through R' by a subsumption s in $\mathcal{D}|_G^{\mathcal{S}}$ whenever there exist two direct transformations $d, d' \in \mathcal{D}|_G^{\mathcal{S}}$ using rules $\mathsf{J}\,r$ and $\mathsf{J}\,r'$ respectively, whose matchings $\mathsf{M}\,d$ and $\mathsf{M}\,d'$ overlap consistently with f (i.e., with a $g : \mathsf{M}\,d \to \mathsf{M}\,d'$ such that $\mathsf{S}\,g = \mathsf{L}\,(\mathsf{J}\,f)$) with of course $s : d \to d'$ and $\mathsf{M}\,s = g$. Dispensing with r and r' we can express this *Correctness Condition* as

> for all rule system $\mathcal{S}$, input $G \in \mathcal{C}$, direct transformations $d, d' \in \mathcal{D}|_G^{\mathcal{S}}$, $\mathcal{S}$-morphism $f : \mathsf{R}'\,d \to \mathsf{R}'\,d'$ and $\mathcal{M}$-morphism $g : \mathsf{M}\,d \to \mathsf{M}\,d'$ such that $\mathsf{L}\,(\mathsf{J}\,f) = \mathsf{S}\,g$, there exists a unique $\mathcal{D}|_G^{\mathcal{S}}$-morphism $s : d \to d'$ such that $\mathsf{R}'\,s = f$ and $\mathsf{M}\,s = g$. (2)

We can further dispense with $\mathcal{S}$ and G with the equivalent formulation

> for all direct transformations $d, d' \in \mathcal{D}$, $\mathcal{R}$-morphism $f : \mathsf{R}\,d \to \mathsf{R}\,d'$ and $\mathcal{M}$-morphism $g : \mathsf{M}\,d \to \mathsf{M}\,d'$ such that $\mathsf{L}\,f = \mathsf{S}\,g$, there exists a unique $\mathcal{D}$-morphism $s : d \to d'$ such that $\mathsf{R}\,s = f$ and $\mathsf{M}\,s = g$. (3)

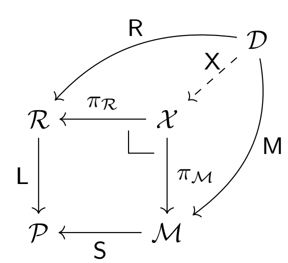
Proposition 9. *(2) and (3) are equivalent.*

Proof. (2) $\Rightarrow$ (3) Let $d, d' \in \mathcal{D}$, $f : \mathsf{R}\,d \to \mathsf{R}\,d'$ and $g : \mathsf{M}\,d \to \mathsf{M}\,d'$ such that $\mathsf{L}\,f = \mathsf{S}\,g$. Let $\mathcal{S} \coloneqq \mathcal{R}$ and $G \coloneqq (\mathsf{P} \circ \mathsf{In})\,d = (\mathsf{T} \circ \mathsf{M})\,d$, since $\mathsf{T}\,g = 1_G$ then $(\mathsf{P} \circ \mathsf{In})\,d' = (\mathsf{T} \circ \mathsf{M})\,d' = G$ so that $d, d' \in \mathcal{D}|_G^{\mathcal{S}}$. We have $\mathsf{R}'\,d = \mathsf{R}\,d$ and similarly for d', hence by (2) a unique $s : d \to d'$ exists.

(3) $\Rightarrow$ (2) Let $d, d' \in \mathcal{D}|_G^{\mathcal{S}}$, $f : \mathsf{R}'\,d \to \mathsf{R}'\,d'$ and $g : \mathsf{M}\,d \to \mathsf{M}\,d'$ such that $(\mathsf{L} \circ \mathsf{J})\,f = \mathsf{S}\,g$, then $\mathsf{J}\,f : \mathsf{R}\,d \to \mathsf{R}\,d'$ hence by (3) there exists a unique $\mathcal{D}$-morphism $s : d \to d'$ such that $\mathsf{R}\,s = \mathsf{J}\,f$ and $\mathsf{M}\,s = g$. Since $(\mathsf{P} \circ \mathsf{In})\,s = 1_G$ then s is a $\mathcal{D}|_G$-morphism, and a $\mathcal{D}|_G^{\mathcal{S}}$-morphism by $\mathsf{R}\,s = \mathsf{J}\,f$. Since $\mathsf{J}\,(\mathsf{R}'\,s) = \mathsf{R}\,s = \mathsf{J}\,f$ then $\mathsf{R}'\,s = f$. □

A much simpler formulation can be obtained with the following tools.

Definition 13 (redex category $\mathcal{X}$, X-functor). *In a REM $\mathscr{E}$ as in Definition 12, the* redex category $\mathcal{X}$ *of $\mathscr{E}$ and the projection functors $\pi_{\mathcal{R}} : \mathcal{X} \to \mathcal{R}$, $\pi_{\mathcal{M}} : \mathcal{X} \to \mathcal{M}$ are obtained as the meet of L and S. We call* X-functor *of $\mathscr{E}$ the unique functor $\mathsf{X} : \mathcal{D} \to \mathcal{X}$ such that $\mathsf{R} = \pi_{\mathcal{R}} \circ \mathsf{X}$ and $\mathsf{M} = \pi_{\mathcal{M}} \circ \mathsf{X}$.*



Proposition 10. *(3) holds iff the* X*-functor is fully faithful.*

Proof. Suppose X is fully faithful. Let d, d', f and g as in (3), then $(f, g) : (\mathsf{R}\,d, \mathsf{M}\,d) \to (\mathsf{R}\,d', \mathsf{M}\,d')$ is an $\mathcal{X}$-morphism, hence there exists a unique $\mathcal{D}$-morphism $s : d \to d'$ such that $\mathsf{X}\,s = (f, g)$, i.e., such that $\mathsf{R}\,s = f$ and $\mathsf{M}\,s = g$. Thus property (3) holds.

Suppose that (3) holds. Let $d, d' \in \mathcal{D}$ and $(f, g) : \mathsf{X}\,d \to \mathsf{X}\,d'$ be any $\mathcal{X}$-morphism, then $f = \pi_{\mathcal{R}}\,(f, g) : \mathsf{R}\,d \to \mathsf{R}\,d'$ is a $\mathcal{R}$-morphism and $g = \pi_{\mathcal{M}}\,(f, g) : \mathsf{M}\,d \to \mathsf{M}\,d'$ is an $\mathcal{M}$-morphism such that $\mathsf{L}\,f = \mathsf{S}\,g$, hence by (3) there exists $s : d \to d'$ in $\mathcal{D}$ such that $\mathsf{R}\,s = f$ and $\mathsf{M}\,s = g$, i.e., such that $\mathsf{X}\,s = (\mathsf{R}\,s, \mathsf{M}\,s) = (f, g)$. Hence X is fully faithful. □

Other properties of the X-functor are relevant. As explained in Section 2.2 we are interested in whether R is right-full and faithful. We easily see that these are equivalent to the same properties of the X-functor.

Proposition 11. *If* S *is faithful then* R *is faithful iff* X *is faithful. If* S *is right-full and faithful then* R *is right-full iff* X *is right-full.*

Proof. By Lemma 2 $\pi_{\mathcal{R}}$ is faithful (resp. right-full and faithful), hence by Lemma 1, X is faithful (resp. right-full) iff so is $\pi_{\mathcal{R}} \circ \mathsf{X} = \mathsf{R}$. □

Example 4. It is shown in [4, Proposition 6.4] that if $\mathcal{C}$ is adhesive [8] then for all $d, d' \in \mathcal{D}_{\mathrm{DPO}}$, all $(s_1, s_2, s_3) : \mathsf{R}_{\mathrm{DPO}}\,d \to \mathsf{R}_{\mathrm{DPO}}\,d'$ in $\mathcal{R}_{\mathrm{DPO}}$ such that $m = m' \circ s_1$ there exists a unique $s : d \to d'$ in $\mathcal{D}_{\mathrm{DPO}}$ such that $\mathsf{R}_{\mathrm{DPO}}\,s = (s_1, s_2, s_3)$. Since m and m' refer to the corresponding morphisms in the DPO diagrams d and d', the equation $m = m' \circ s_1$ means that $s_1 : \mathsf{M}_{\mathrm{DPO}}\,d \to \mathsf{M}_{\mathrm{DPO}}\,d'$ is a morphism in $\mathcal{M}_{\mathrm{std}}$, and since $\mathsf{L}_{\mathrm{DPO}}\,(s_1, s_2, s_3) = s_1 = \mathsf{S}_{\mathrm{std}}\,s_1$, it is equivalent to $\mathsf{L}_{\mathrm{DPO}}\,(s_1, s_2, s_3) = \mathsf{S}_{\mathrm{std}}\,g$ for any $g : \mathsf{M}_{\mathrm{DPO}}\,d \to \mathsf{M}_{\mathrm{DPO}}\,d'$. Hence property (3) holds for the DPO REM, and its X-functor is therefore fully faithful (in adhesive categories).

We leave it to the reader to check that this also holds for the SPO REM (in categories of presheaves) by [12, Theorem 3.5]), for the SqPO REM by [4, Proposition 6.6] and for the PBPO REM by [4, Proposition 6.9].

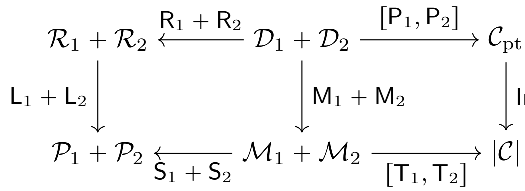
4.1 Combining REMs

We can easily combine REMs as we did with REs and MEs. This can be done by preserving the properties of the X-functors.

Definition 14 (sum of REMs). *The* sum $\mathscr{E}_1 + \mathscr{E}_2$ *of two REMs* $\mathscr{E}_i$ *given by*

$$\begin{array}{ccccc}
\mathcal{R}_i & \xleftarrow{\mathsf{R}_i} & \mathcal{D}_i & \xrightarrow{\mathsf{P}_i} & \mathcal{C}_{\mathrm{pt}} \\
\Big\downarrow{\scriptstyle \mathsf{L}_i} & & \Big\downarrow{\scriptstyle \mathsf{M}_i} & & \Big\downarrow{\scriptstyle \mathsf{In}} \\
\mathcal{P}_i & \xleftarrow[\mathsf{S}_i]{} & \mathcal{M}_i & \xrightarrow[\mathsf{T}_i]{} & |\mathcal{C}|
\end{array}$$

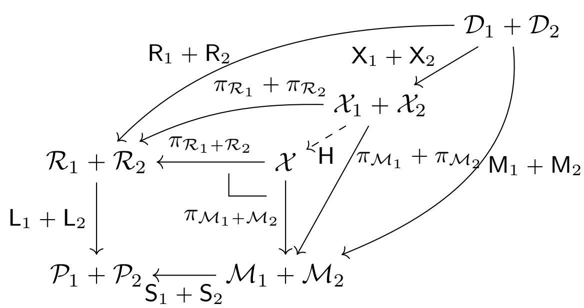
for $i = 1, 2$ is the REM



Note that this diagram is easily seen to commute.

Proposition 12. *If the X_1-functor and the X_2-functor are right-full (resp. full, resp. faithful, resp. embeddings) then so is the X-functor of the sum.*

Proof. Let $\mathcal{X}$ be the redex category of the sum, i.e., the meet of $\mathsf{L}_1 + \mathsf{L}_2$ and $\mathsf{S}_1 + \mathsf{S}_2$, and $\mathsf{X} : \mathcal{D}_1 + \mathcal{D}_2 \to \mathcal{X}$ be the unique functor such that $\mathsf{R}_1 + \mathsf{R}_2 = \pi_{\mathcal{R}_1+\mathcal{R}_2} \circ \mathsf{X}$ and $\mathsf{M}_1 + \mathsf{M}_2 = \pi_{\mathcal{M}_1+\mathcal{M}_2} \circ \mathsf{X}$. By Lemma 6 $\mathcal{X}_1 + \mathcal{X}_2$ with $\pi_{\mathcal{R}_1} + \pi_{\mathcal{R}_2}$ and $\pi_{\mathcal{M}_1} + \pi_{\mathcal{M}_2}$ is also a meet of $\mathsf{L}_1 + \mathsf{L}_2$ and $\mathsf{S}_1 + \mathsf{S}_2$, hence there exists a unique isomorphism $\mathsf{H} : \mathcal{X}_1 + \mathcal{X}_2 \to \mathcal{X}$ such that $\pi_{\mathcal{R}_1} + \pi_{\mathcal{R}_2} = \pi_{\mathcal{R}_1+\mathcal{R}_2} \circ \mathsf{H}$ and $\pi_{\mathcal{M}_1} + \pi_{\mathcal{M}_2} = \pi_{\mathcal{M}_1+\mathcal{M}_2} \circ \mathsf{H}$.



But then we see that

$$\begin{aligned}\pi_{\mathcal{R}_1+\mathcal{R}_2} \circ \mathsf{H} \circ (\mathsf{X}_1 + \mathsf{X}_2) &= (\pi_{\mathcal{R}_1} + \pi_{\mathcal{R}_2}) \circ (\mathsf{X}_1 + \mathsf{X}_2) \\ &= (\pi_{\mathcal{R}_1} \circ \mathsf{X}_1) + (\pi_{\mathcal{R}_2} \circ \mathsf{X}_2) = \mathsf{R}_1 + \mathsf{R}_2\end{aligned}$$

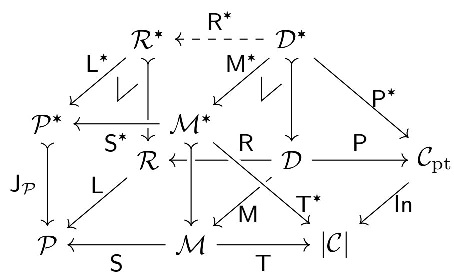
and similarly that $\pi_{\mathcal{M}_1+\mathcal{M}_2} \circ \mathsf{H} \circ (\mathsf{X}_1 + \mathsf{X}_2) = \mathsf{M}_1 + \mathsf{M}_2$. By Lemma 4 we therefore have $\mathsf{X} = \mathsf{H} \circ (\mathsf{X}_1 + \mathsf{X}_2)$ and we conclude with Lemma 5. □

4.2 The strict restriction of a REM

We finally see that it is possible to construct a REM that contains only direct transformations obtained with strict matchings. The restriction also concerns subsumption morphisms since only monic overlaps are available, and therefore also rule morphisms in order to preserve the Correctness Condition.

Definition 15 (strict restriction $\mathscr{E}^\star$ of REM $\mathscr{E}$). *Given a REM $\mathscr{E}$ as in Definition 12 and its strict restriction $\mathscr{M}^\star$ as in Definition 10, let $\mathsf{J}_\mathcal{R} : \mathcal{R}^\star \to \mathcal{R}$ be the inverse image of the canonical embedding $\mathsf{J}_\mathcal{P} : \mathcal{P}^\star \to \mathcal{P}$ along L, and $\mathsf{L}^\star : \mathcal{R}^\star \to \mathcal{P}^\star$ be the corresponding restriction of L.*

Similarly, let $\mathsf{J}_\mathcal{D} : \mathcal{D}^\star \to \mathcal{D}$ be the inverse image of the canonical embedding $\mathsf{J}_\mathcal{M} : \mathcal{M}^\star \to \mathcal{M}$ along M, and $\mathsf{M}^\star : \mathcal{D}^\star \to \mathcal{M}^\star$ be the corresponding restriction of M. Let $\mathsf{P}^\star = \mathsf{P} \circ \mathsf{J}_\mathcal{D}$.



It is easy to see that $\mathsf{J}_\mathcal{P} \circ \mathsf{S}^\star \circ \mathsf{M}^\star = \mathsf{L} \circ \mathsf{R} \circ \mathsf{J}_\mathcal{D}$, and since $\mathcal{R}^\star$ with $\mathsf{J}_\mathcal{R}, \mathsf{L}^\star$ is a meet of L and $\mathsf{J}_\mathcal{P}$, then there exists a unique $\mathsf{R}^\star : \mathcal{D}^\star \to \mathcal{R}^\star$ such that $\mathsf{L}^\star \circ \mathsf{R}^\star = \mathsf{S}^\star \circ \mathsf{M}^\star$ and $\mathsf{J}_\mathcal{R} \circ \mathsf{R}^\star = \mathsf{R} \circ \mathsf{J}_\mathcal{D}$. Hence we get a REM $\mathscr{E}^\star$ (the top two faces in the diagram above), called the strict restriction *of $\mathscr{E}$.*

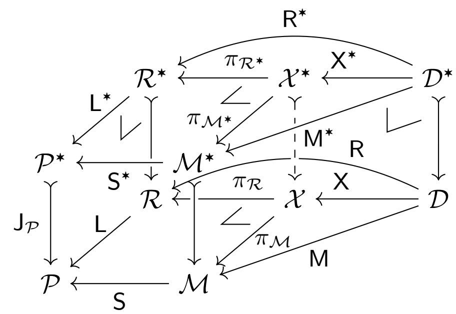
Note that by our special construction of inverse images, the functors $\mathsf{J}_\mathcal{R}$ and $\mathsf{J}_\mathcal{D}$ are canonical embeddings, so that $\mathsf{L}^\star$, $\mathsf{M}^\star$, $\mathsf{R}^\star$ and $\mathsf{P}^\star$ are restrictions of L, M, R and P respectively. Besides, since $\mathcal{P}^\star$ and $\mathcal{P}$ have the same objects ($\mathsf{J}_\mathcal{P}$ is surjective on objects) then so do $\mathcal{R}^\star$ and $\mathcal{R}$ ($\mathsf{J}_\mathcal{R}$ is surjective on objects by Lemma 2), so that all rules are preserved by the restriction, i.e., all the rules are liable to be applied with strict matchings.

Example 5. We have build $\mathscr{M}^\star_{\mathrm{std}}$ in Example 2. We now build the strict restriction of the DPO REM. We first see that $\mathcal{R}^\star_{\mathrm{DPO}}$ is obtained as the inverse image of the canonical embedding of $\mathcal{C}^\star$ in $\mathcal{C}$ along $\mathsf{L}_{\mathrm{DPO}}$, hence the $\mathcal{R}^\star_{\mathrm{DPO}}$-objects are all the $\mathcal{R}_{\mathrm{DPO}}$-objects (all DPO-rules), and the $\mathcal{R}^\star_{\mathrm{DPO}}$-morphisms are all the $(s_1, s_2, s_3) : r \to r'$ such that $\mathsf{L}_{\mathrm{DPO}}\,(s_1, s_2, s_3)$ is in $\mathcal{C}^\star$, i.e., such that s_1 is monic. This is the category $\mathcal{R}_{\mathrm{mDPO}}$ in [4, Definition 3.3].

Similarly, the $\mathcal{D}^\star_{\mathrm{DPO}}$-objects are the direct DPO-transformations $d \in \mathcal{D}_{\mathrm{DPO}}$ such that $\mathsf{M}_{\mathrm{DPO}}\, d$ is monic, and the $\mathcal{D}^\star_{\mathrm{DPO}}$-morphisms are the $\mathcal{D}_{\mathrm{DPO}}$-morphisms $(s_1, s_2, s_3, s_4) : d \to d'$ such that s_1 is monic. This is a full subcategory of $\mathcal{D}_{\mathrm{DPO}}$, denoted $\mathcal{D}_{\mathrm{mDPO}}$ in [4, Definition 3.8].

Proposition 13. *If X is right-full (resp. full, resp. faithful, resp. an embedding) then so is $\mathsf{X}^\star$.*

Proof. We consider the following diagram where both X-functors are depicted, together with their construction.



It is easy to see that $\mathsf{S} \circ \mathsf{J}_{\mathcal{M}} \circ \pi_{\mathcal{M}^\star} = \mathsf{L} \circ \mathsf{J}_{\mathcal{R}} \circ \pi_{\mathcal{R}^\star}$, and since $\mathcal{X}$ with $\pi_{\mathcal{R}}$, $\pi_{\mathcal{M}}$ is a meet of L and S then there exists a unique $\mathsf{J}_{\mathcal{X}} : \mathcal{X}^\star \to \mathcal{X}$ such that $\pi_{\mathcal{M}} \circ \mathsf{J}_{\mathcal{X}} = \mathsf{J}_{\mathcal{M}} \circ \pi_{\mathcal{M}^\star}$ and $\pi_{\mathcal{R}} \circ \mathsf{J}_{\mathcal{X}} = \mathsf{J}_{\mathcal{R}} \circ \pi_{\mathcal{R}^\star}$.

We therefore have a commuting cube from $\mathcal{X}^\star$ to $\mathcal{P}$. Since its left and top faces are meets, then by Lemma 3 the diagonal square $(\pi_{\mathcal{M}^\star}, \mathsf{J}_{\mathcal{P}} \circ \mathsf{S}^\star, \mathsf{J}_{\mathcal{R}} \circ \pi_{\mathcal{R}^\star}, \mathsf{L})$ is a meet, and since its bottom face is also a meet then again by Lemma 3 its right face is a meet. Hence $\mathsf{J}_{\mathcal{X}}$ is an embedding by Lemma 2.

We also have a square of X-functors, and we now show that it commutes. Indeed, it is easy to see that $\pi_{\mathcal{M}} \circ \mathsf{X} \circ \mathsf{J}_{\mathcal{D}} = \pi_{\mathcal{M}} \circ \mathsf{J}_{\mathcal{X}} \circ \mathsf{X}^\star$ and that $\pi_{\mathcal{R}} \circ \mathsf{X} \circ \mathsf{J}_{\mathcal{D}} = \pi_{\mathcal{R}} \circ \mathsf{J}_{\mathcal{X}} \circ \mathsf{X}^\star$, hence by Lemma 4 we get $\mathsf{X} \circ \mathsf{J}_{\mathcal{D}} = \mathsf{J}_{\mathcal{X}} \circ \mathsf{X}^\star$.

We can thus apply again Lemma 3 to get that the square of X-functors is a meet, and we conclude with Lemma 2. □

5 Conclusion

We conclude that any combination of the REMs for DPO, SPO, SqPO and PBPO approaches and their strict restrictions yields a REM with a fully faithful X-functor. This means that in the Global Coherent Transformation presented in [4], it is possible to apply simultaneously, say, a DPO-rule restricted to strict matchings and a PBPO-rule.

This way of defining rule morphisms through REMs that satisfy the Correctness Condition has yet been pursued only for algebraic approaches to graph rewriting. It is however perfectly possible to follow the same method with algorithmic approaches; the objects of the category $\mathcal{D}$ do not have to be defined algebraically. It is then possible that such approaches would turn out to require non algebraic constructions to define subsumption morphisms that satisfy the Correctness Condition. This of course assumes that direct transformations can be transformed into partial transformations (Definition 4) and therefore that their result can be obtained as a pushout. But considering that any $H \in \mathcal{C}$ is the pushout of $\varnothing \xleftarrow{!} \varnothing \xrightarrow{!} H$, this is always possible provided that $\mathcal{C}$ has an initial object $\varnothing$.

In particular this abstract representation of rules and rule morphisms yields a general understanding of the notion of symmetries of rules, as in [2,3]. The specific construction of the automorphism group of a rule in [3] should then appear as an instance of a general construction, provided the corresponding category of rules is defined.

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