Categories of Algebraic Rewrite Rules

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Abstract. What are the legitimate morphisms between algebraic graph rewrite rules? The question is complicated by the diversity of approaches. From the familiar Double-Pushout (DPO) to the more recent PBPO and many others, the rules have different shapes, semantics (defined by direct transformation diagrams) and even matchings. We propose to represent these approaches by categories of rules, direct transformations and matchings related by functors in *Rewriting Environments with Matchings*. From these we extract a so-called X-functor whose properties are key to make rule morphisms meaningful. We show that these properties are preserved by combining approaches and by restricting them to strict matchings.

1 Introduction

Many "approaches" to algebraic (graph) rewriting have been developed. The most familiar and oldest one is the Double-Pushout (DPO) approach [5], that has a property unknown to term rewriting: a matching of a rule in the input object G is not sufficient to apply the rule. Indeed, this approach imposes a strict semantics of replacement where a matched vertex (say) cannot be removed and replaced unless all adjacent edges are similarly matched and removed. Another semantics exists that tolerates the silent removal of such edges: the Single-Pushout (SPO) approach [9]. It is based on pushouts of partial morphisms and has been defined in a restricted class of categories compared to DPO. Other approaches, namely the Sesqui-Pushout (SqPO) [4] and the Pullback-Pushout (PBPO) [3], provide the possibility to duplicate matched parts of the input.

In this diversity it is difficult to isolate common features. An obvious one is that they all end with a pushout, either in the category $\mathcal{C}$ whose objects are considered for computing (generally graph-like data structures), or an extension of $\mathcal{C}$ to partial morphisms. A closer look reveals that the result of the transformation is always obtained as a pushout of a $\mathcal{C}$-span $D \xleftarrow{k} K \xrightarrow{r} R$, where D is called the *context* and K the *interface*. The object R may or may not (for SqPO transformations) be the right-hand side of the rule. Besides, all approaches define a morphism $G \xleftarrow{f} D$, though in different ways. Hence all approaches are based on specific rule-based transformations, from which a diagram $G \xleftarrow{f} D \xleftarrow{k} K \xrightarrow{r} R$ can always be extracted (by some mapping). Such diagrams are called *partial transformations* in [2].

Hence in order to develop general methods related to rule-based algebraic transformations, methods that are not committed to a specific approach, one can certainly rely on partial transformations. In [2] a transformation is defined that applies algebraic rewrite rules simultaneously to the input object G. This transformation is not restricted to a particular approach to algebraic rewriting, and can even be applied by mixing rules from different approaches.

One important feature that enhances the expressiveness of this transformation is the use of morphisms between rules. The idea is inspired by [11], where the overlap of two matchings in a graph can be represented as a common subgraph of the corresponding left-hand sides, or more generally as morphisms between left-hand sides. But since the transformation in [2] is based on partial transformations, it relies on a notion of morphisms between such diagrams, and hence of a category $\mathcal{C}_{\mathrm{pt}}$ of partial transformations (see Definition 4 below). A morphism $s : p \to p'$ can be understood as a subsumption (of p by p') due to the following property: the simultaneous application of partial transformations p and p' yields the same result as p' [2, Proposition 5.12].

Hence we should also be able to find subsumption morphisms between the rule-based transformations (usually called *direct transformations*) of any given approach, hence the map from these to partial transformations should involve morphisms; in other words there should be a functor from a category $\mathcal{D}$ of direct transformations of the given approach, to the category $\mathcal{C}_{\mathrm{pt}}$. Similarly, there should be a functor from $\mathcal{D}$ to a category $\mathcal{R}$ of rules whose morphisms can then be understood as subsumptions between rules. This constitutes a *Rewriting Environment (RE)* $\mathcal{R} \xleftarrow{\mathsf{R}} \mathcal{D} \xrightarrow{\mathsf{P}} \mathcal{C}_{\mathrm{pt}}$. The simplicity of this model is very convenient as it encompasses many different situations. In particular, the fact that $\mathcal{C}_{\mathrm{pt}}$ does not involve any notion of left-hand side and matching of rules makes it easy for the PBPO approach to fit in, despite its non-standard left-hand sides and matchings (see [2, Section 6.3]).

But this simplicity makes it difficult to understand, and even formulate, certain aspects of algebraic rewriting. In particular, this model is too tolerant on what should be the morphisms in $\mathcal{R}$ and $\mathcal{D}$, as it allows $\mathcal{D}$ to be discrete even if $\mathcal{R}$ is not. This model lacks the possibility to express in a general way how the morphisms in $\mathcal{D}$ must depend on the morphisms in $\mathcal{R}$. Intuitively, we understand that if a rule r is subsumed by r', and if they are applied at suitably overlapping positions, then the corresponding transformation d should be subsumed by the corresponding d'. But if r and r' are applied at unrelated positions then d and d' are similarly unrelated (hence R is generally not full).

For this reason we need to enhance REs with a notion of matchings of left-hand sides of rules, and of morphisms between matchings. This notion should be general enough to encompass the standard situation, where matchings are $\mathcal{C}$-morphisms, but also the non standard matchings used in PBPO direct transformations (that involve two consecutive $\mathcal{C}$-morphisms, a match and a co-match), and possibly many others. This is the subject of Section 3, where it is also shown how different notions of matchings can be combined, and how the standard notions of monic matches and identity matching may be generalized.

In Section 4 this general notion of matching is connected to Rewriting Environments, and it is shown how it can contribute to analyze their properties. The analysis involves the definition of a category of redexes, or pairs of rules and of matchings of their left-hand side. This allows to define a so-called X-functor whose properties are shown to characterize some key properties of REs. It is also shown how different REs can be combined and how they can be restricted to strict matchings.

But first we need to compile some definitions and results, mostly concerning functors.

2 Background

The standard notions of Category Theory are assumed, see [10]. For any category $\mathcal{C}$, we write $G \in \mathcal{C}$ to indicate that G is a $\mathcal{C}$-object, and $|\mathcal{C}|$ is the discrete category on $\mathcal{C}$-objects. Then G may also denote the functor from the terminal category $\mathbf{1}$ to $\mathcal{C}$ or to $|\mathcal{C}|$ (as specified in the context) that maps the object of $\mathbf{1}$ to G.

2.1 Functors

We will use the following notion from [2].

Definition 1 (right-full). *A functor* $\mathsf{F} : \mathcal{A} \to \mathcal{B}$ *is* right-full *if for all* $a' \in \mathcal{A}$, *all* $b \in \mathcal{B}$ *and all* $\mathcal{B}$*-morphism* $g : b \to \mathsf{F}\,a'$, *there exist* $a \in \mathcal{A}$ *and an* $\mathcal{A}$*-morphism* $f : a \to a'$ *such that* $\mathsf{F}\,f = g$.

$$
\begin{tikzcd}
a \arrow[r, dotted, "\mathsf{F}"] \arrow[d, dashed, "f"'] \arrow[d, phantom, ""{name=L}] & b \arrow[d, "g"] \arrow[d, phantom, ""{name=R}] \\
a' \arrow[r] & \mathsf{F}\,a'
\arrow[from=L, to=R, dotted]
\end{tikzcd}
$$

Note that a full functor may not be right-full (since b may have no preimage) and a right-full functor may not be full (since a depends on g). It is obvious that right-fullness is closed by composition and that all isomorphisms are right-full.

Lemma 1. *For any functors* $\mathsf{F} : \mathcal{A} \to \mathcal{B}$ *and* $\mathsf{G} : \mathcal{B} \to \mathcal{C}$ *such that* G *is faithful then* F *is faithful iff* $\mathsf{G} \circ \mathsf{F}$ *is faithful. If* G *is faithful and right-full then* F *is right-full iff* $\mathsf{G} \circ \mathsf{F}$ *is right-full.*

Proof. Suppose $\mathsf{G} \circ \mathsf{F}$ is right-full and let $a' \in \mathcal{A}$, $b \in \mathcal{B}$ and $g : b \to \mathsf{F}\,a'$ in $\mathcal{B}$, then $\mathsf{G}\,g : \mathsf{G}\,b \to \mathsf{G}\,\mathsf{F}\,a'$ is a $\mathcal{C}$-morphism, hence there exists $a \in \mathcal{A}$ and $f : a \to a'$ in $\mathcal{A}$ such that $\mathsf{G}\,\mathsf{F}\,f = \mathsf{G}\,g$, hence such that $\mathsf{F}\,f = g$, which shows that F is right-full. The other claims are proven similarly (or well known).

Definition 2 (embedding, meet, inverse image). *We call* embedding *a functor that is both faithful and injective on objects (equivalently, that is left-cancellable). If* $\mathcal{A}$ *is a subcategory of* $\mathcal{B}$ *then the* canonical embedding of $\mathcal{A}$ into

$\mathcal{B}$ is the functor $\mathsf{J} : \mathcal{A} \to \mathcal{B}$ defined by $\mathsf{J}\,a \coloneqq a$ for all $a \in \mathcal{A}$ and $\mathsf{J}\,f \coloneqq f$ for all $\mathcal{A}$-morphisms f.

A meet *of two functors $\mathsf{F} : \mathcal{A} \to \mathcal{C}$ and $\mathsf{G} : \mathcal{B} \to \mathcal{C}$ consists in a category $\mathcal{P}$ and two functors $\mathsf{G}' : \mathcal{P} \to \mathcal{A}$ and $\mathsf{F}' : \mathcal{P} \to \mathcal{B}$ such that $\mathsf{F} \circ \mathsf{G}' = \mathsf{G} \circ \mathsf{F}'$ and for all functors F'', G'' such that $\mathsf{F} \circ \mathsf{G}'' = \mathsf{G} \circ \mathsf{F}''$ there is a unique functor H such that $\mathsf{F}'' = \mathsf{F}' \circ \mathsf{H}$ and $\mathsf{G}'' = \mathsf{G}' \circ \mathsf{H}$.*

If F is an embedding then F' is called an inverse image of F along G.

Since meets have the universal property of pullbacks (they are pullbacks in the "category" of categories) they will be pictured as are standard pullbacks:

$$\begin{tikzcd}
\mathcal{P}' \arrow[dr, dashed, "\mathsf{H}"] \arrow[drr, bend left, "\mathsf{G}''"] \arrow[ddr, bend right, "\mathsf{F}''"'] & & \\
& \mathcal{P} \arrow[r, "\mathsf{G}'"] \arrow[d, "\mathsf{F}'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "\mathsf{F}"] \\
& \mathcal{B} \arrow[r, "\mathsf{G}"'] & \mathcal{C}
\end{tikzcd}$$

It is well-known (see [8]) that meets always exist: take for $\mathcal{P}$ the subcategory of $\mathcal{A} \times \mathcal{B}$ with objects (a, b) such that $\mathsf{F}\,a = \mathsf{G}\,b$ and with morphisms (f, g) such that $\mathsf{F}\,f = \mathsf{G}\,g$, and take for G' the projection $\pi_{\mathcal{A}} : \mathcal{P} \to \mathcal{A}$ on the first coordinate and for F' the projection $\pi_{\mathcal{B}} : \mathcal{P} \to \mathcal{B}$ on the second coordinate. All meets are isomorphic to this one, that we may therefore call *the* meet of F and G.

If F is an embedding of $\mathcal{A}$ into $\mathcal{C}$, then $\mathsf{F}(\mathcal{A})$ is a subcategory of $\mathcal{C}$ and (a restriction of) F is an isomorphism from $\mathcal{A}$ to its image $\mathsf{F}(\mathcal{A})$, with inverse, say, $\mathsf{F}^{-1} : \mathsf{F}(\mathcal{A}) \to \mathcal{A}$. Let $\mathcal{B}'$ be the subcategory of $\mathcal{B}$ with all $b \in \mathcal{B}$ such that $\mathsf{G}\,b \in \mathsf{F}(\mathcal{A})$ and all $\mathcal{B}$-morphisms g such that $\mathsf{G}\,g$ is a $\mathsf{F}(\mathcal{A})$-morphism. It is easy to see that the second projection $\pi_2 : \mathcal{P} \to \mathcal{B}'$ is an isomorphism, that $\pi_{\mathcal{B}} \circ {\pi_2}^{-1}$ is the canonical embedding of $\mathcal{B}'$ into $\mathcal{B}$ and that $\pi_{\mathcal{A}} \circ {\pi_2}^{-1} = \mathsf{F}^{-1} \circ \mathsf{G}'$ where $\mathsf{G}' : \mathcal{B}' \to \mathsf{F}(\mathcal{A})$ is a restriction of G. Hence all inverse images of F along G are isomorphic to $\mathcal{B}'$, that we may therefore call *the* inverse image of F along G.

Besides, the following properties of functors are preserved by meets:

Lemma 2. *If F is right-full (resp. full, resp. faithful, resp. an embedding, resp. surjective on objects) then so is F'.*

Proof. Suppose F is right-full and let (a', b') be an object in *the* meet of F and G, let $b \in \mathcal{B}$ and $g : b \to b'$ in $\mathcal{B}$ (since $\pi_{\mathcal{B}}(a', b') = b'$), then $\mathsf{G}\,g : \mathsf{G}\,b \to \mathsf{F}\,a'$ is a $\mathcal{C}$-morphism (since $\mathsf{F}\,a' = \mathsf{G}\,b'$), hence there exists $a \in \mathcal{A}$ and an $\mathcal{A}$-morphism $f : a \to a'$ such that $\mathsf{F}\,f = \mathsf{G}\,g$ (and hence $\mathsf{F}\,a = \mathsf{G}\,b$), so that $(f, g) : (a, b) \to (a', b')$ is a morphism such that $\pi_{\mathcal{B}}(f, g) = g$, hence $\pi_{\mathcal{B}}$ is right-full. This holds for F' since it is obtained by composing $\pi_{\mathcal{B}}$ with an isomorphism. The other claims are proven similarly.

We will also use an instance of the well-known pullback composition and decomposition lemma (see [1, Proposition 11.10]), applied to meets of functors:

Lemma 3. *If the following diagram of functors*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{C} \arrow[dd] \\
& \mathcal{B} \arrow[ur] \arrow[dd] & \\
\mathcal{A}' \arrow[rr] \arrow[dr] & & \mathcal{C}' \\
& \mathcal{B}' \arrow[ur] &
\end{tikzcd}$$

commutes and the right face is a meet, then the left face is a meet iff the back face is a meet.

Similarly we will use the fact that meets are mono-sources (see [1, Proposition 11.6]):

Lemma 4. *If $\mathcal{P}$ with F' and G' is a meet of F and G, and $\mathsf{H}, \mathsf{H}' : \mathcal{P}' \to \mathcal{P}$ are such that $\mathsf{F}' \circ \mathsf{H} = \mathsf{F}' \circ \mathsf{H}'$ and $\mathsf{G}' \circ \mathsf{H} = \mathsf{G}' \circ \mathsf{H}'$ then $\mathsf{H} = \mathsf{H}'$.*

Definition 3 (sum $\mathcal{A}_1 + \mathcal{A}_2$, injections $\mathsf{I}_{\mathcal{A}_i}$, functors $[\mathsf{F}_1, \mathsf{F}_2]$, $\mathsf{F}_1 + \mathsf{F}_2$). *Given two categories $\mathcal{A}_1$ and $\mathcal{A}_2$, their* sum *is the category $\mathcal{A}_1 + \mathcal{A}_2$ whose objects are pairs (i, a) where $i \in \{1, 2\}$ and $a \in \mathcal{A}_i$, and morphisms $f : (i, a) \to (i, a')$ are the $\mathcal{A}_i$-morphisms $f : a \to a'$, with the obvious composition (the union of the compositions in $\mathcal{A}_1$ and $\mathcal{A}_2$). The* injections *are the two embeddings $\mathsf{I}_{\mathcal{A}_i} : \mathcal{A}_i \to \mathcal{A}_1 + \mathcal{A}_2$ defined by $\mathsf{I}_{\mathcal{A}_i}\, a \coloneqq (i, a)$ for all $a \in \mathcal{A}_i$ and $\mathsf{I}_{\mathcal{A}_i}\, f \coloneqq f$ for all $\mathcal{A}_i$-morphisms f.*

For any functors $\mathsf{F}_i : \mathcal{A}_i \to \mathcal{C}$ let $[\mathsf{F}_1, \mathsf{F}_2] : \mathcal{A}_1 + \mathcal{A}_2 \to \mathcal{C}$ be the functor defined by $[\mathsf{F}_1, \mathsf{F}_2](i, a) \coloneqq \mathsf{F}_i\, a$ for all $(i, a) \in \mathcal{A}_1 + \mathcal{A}_2$ and $[\mathsf{F}_1, \mathsf{F}_2]f \coloneqq \mathsf{F}_i\, f$ for all $\mathcal{A}_i$-morphisms f.

For any functors $\mathsf{G}_i : \mathcal{A}_i \to \mathcal{B}_i$ let $\mathsf{G}_1 + \mathsf{G}_2 : \mathcal{A}_1 + \mathcal{A}_2 \to \mathcal{B}_1 + \mathcal{B}_2$ be the functor defined by $\mathsf{G}_1 + \mathsf{G}_2 \coloneqq [\mathsf{I}_{\mathcal{B}_1} \circ \mathsf{G}_1, \mathsf{I}_{\mathcal{B}_2} \circ \mathsf{G}_2]$.

Injections have the universal property of coproducts, i.e., that $[\mathsf{F}_1, \mathsf{F}_2]$ is the unique functor such that $[\mathsf{F}_1, \mathsf{F}_2] \circ \mathsf{I}_{\mathcal{A}_i} = \mathsf{F}_i$ for $i = 1, 2$. From this it is easy to deduce that $[\mathsf{I}_{\mathcal{A}_1}, \mathsf{I}_{\mathcal{A}_2}] = 1_{\mathcal{A}_1 + \mathcal{A}_2}$, $[\mathsf{F} \circ \mathsf{F}_1, \mathsf{H} \circ \mathsf{F}_2] = \mathsf{F} \circ [\mathsf{F}_1, \mathsf{F}_2]$, $[\mathsf{F}_1 \circ \mathsf{G}_1, \mathsf{F}_2 \circ \mathsf{G}_2] = [\mathsf{F}_1, \mathsf{F}_2] \circ (\mathsf{G}_1 + \mathsf{G}_2)$ and $(\mathsf{H}_1 + \mathsf{H}_2) \circ (\mathsf{G}_1 + \mathsf{G}_2) = (\mathsf{H}_1 \circ \mathsf{G}_1) + (\mathsf{H}_2 \circ \mathsf{G}_2)$.

As above the following properties of functors are preserved by sums.

Lemma 5. *If $\mathsf{F}_i : \mathcal{A}_i \to \mathcal{B}_i$ are right-full (resp. full, resp. faithful, resp. embeddings) for $i = 1, 2$ then so is $\mathsf{F}_1 + \mathsf{F}_2$.*

Proof. Suppose F_1 and F_2 are right-full and let $(i', a') \in \mathcal{A}_1 + \mathcal{A}_2$, $(i, b) \in \mathcal{B}_1 + \mathcal{B}_2$ and $g : (i, b) \to (\mathsf{F}_1 + \mathsf{F}_2)(i', a')$ a $\mathcal{B}_1 + \mathcal{B}_2$-morphism, since $(\mathsf{F}_1 + \mathsf{F}_2)(i', a') = (i', \mathsf{F}_{i'}\, a')$ then $i = i'$ and $g : b \to \mathsf{F}_i\, a'$ is a $\mathcal{B}_i$-morphism, hence there exist $a \in \mathcal{A}_i$ and an $\mathcal{A}_i$-morphism $f : a \to a'$ such that $\mathsf{F}_i\, f = g$, and therefore $(\mathsf{F}_1 + \mathsf{F}_2)f = (\mathsf{I}_{\mathcal{B}_i} \circ \mathsf{F}_i)f = \mathsf{F}_i\, f = g$, hence $\mathsf{F}_1 + \mathsf{F}_2$ is right-full. The other claims are proven similarly.

Finally, we see that meets are also preserved by sums.

Lemma 6. *If*
$$\begin{tikzcd} \mathcal{P}_i \arrow[r, "\mathsf{G}'_i"] \arrow[d, "\mathsf{F}'_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}_i \arrow[d, "\mathsf{F}_i"] \\ \mathcal{B}_i \arrow[r, "\mathsf{G}_i"'] & \mathcal{C}_i \end{tikzcd}$$
is a meet for $i = 1, 2$ then so is

$$\begin{tikzcd} \mathcal{P}_1 + \mathcal{P}_2 \arrow[r, "\mathsf{G}'_1 + \mathsf{G}'_2"] \arrow[d, "\mathsf{F}'_1 + \mathsf{F}'_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}_1 + \mathcal{A}_2 \arrow[d, "\mathsf{F}_1 + \mathsf{F}_2"] \\ \mathcal{B}_1 + \mathcal{B}_2 \arrow[r, "\mathsf{G}_1 + \mathsf{G}_2"'] & \mathcal{C}_1 + \mathcal{C}_2 \end{tikzcd}$$

The proof is left to the reader.

2.2 Rewriting Environments

We now introduce some definitions from [2], starting with subsumption morphisms. The idea of a subsumption $s : d \to d'$ between direct transformations is that d performs a part of the transformation d', and hence that d can be extended to d'. This means that d removes less than d' and also glues less. But this also means that the context D' of d' should be smaller than the context D of d. On the other hand, D may also be obtained by duplicating a part of the input G and then D' should be obtained by making at least as many duplications than d, so that D' may be bigger than D. But in both cases there should be a morphism from D' to D, which explains the contravariance of s_1 below.

Definition 4 (category $\mathcal{C}_{\text{pt}}$, functor In, Rewriting Environments). *A* partial transformation *p in $\mathcal{C}$ is a diagram*

$$G \xleftarrow{f} D \xleftarrow{k} K \xrightarrow{r} R$$

For any category $\mathcal{C}$, let $\mathcal{C}_{\text{pt}}$ be the category whose objects are partial transformations and morphisms $s : p \to p'$ are triples (s_1, s_2, s_3) of $\mathcal{C}$-morphisms such that

$$\begin{tikzcd} G & D \arrow[l, "f"'] & K \arrow[l, "k"'] \arrow[r, "r"] \arrow[d, "s_2"'] & R \arrow[d, "s_3"] \\ G' \arrow[u, dash, "="] & D' \arrow[l, "f'"] \arrow[u, "s_1"] & K' \arrow[l, "k'"] \arrow[r, "r'"'] & R' \end{tikzcd}$$

commutes in $\mathcal{C}$, with the obvious composition $(s'_1, s'_2, s'_3) \circ (s_1, s_2, s_3) \coloneqq (s_1 \circ s'_1, s'_2 \circ s_2, s'_3 \circ s_3)$.

Let $\mathsf{In} : \mathcal{C}_{\text{pt}} \to |\mathcal{C}|$ be the input functor *defined as $\mathsf{In}p = G$.*

A Rewriting Environment *(or* RE*)* $\mathscr{R}$ *for* $\mathcal{C}$ *consists of a category* $\mathcal{D}$ *of* direct transformations*, a category* $\mathcal{R}$ *of* rules *and two functors*

$$\mathcal{R} \xleftarrow{\mathsf{R}} \mathcal{D} \xrightarrow{\mathsf{P}} \mathcal{C}_{\mathrm{pt}}$$

A rule system *in* $\mathscr{R}$ *is a category* $\mathcal{S}$ *with an embedding* $\mathsf{J} : \mathcal{S} \to \mathcal{R}$ *(alternately,* $\mathcal{S}$ *is a subcategory of* $\mathcal{R}$ *and* J *is the canonical embedding).*

Given a rule system and an input *$\mathcal{C}$-object G, we build the categories* $\mathcal{D}|_G$, $\mathcal{D}|_G^{\mathcal{S}}$ *and functors* J_G, $\mathsf{J}_{\mathcal{S}}$ *as inverse images of the embeddings G and* J.

$$\begin{tikzcd}
\mathcal{S} \arrow[r, tail, "\mathsf{J}"] & \mathcal{R} & & \\
& \mathcal{D} \arrow[u, "\mathsf{R}"] \arrow[r, "\mathsf{P}"] & \mathcal{C}_{\mathrm{pt}} \arrow[r, "\mathsf{In}"] & |\mathcal{C}| \\
\mathcal{D}|_G^{\mathcal{S}} \arrow[uu, "\mathsf{R}'"] \arrow[r, tail, "\mathsf{J}_{\mathcal{S}}"'] \arrow[ur, phantom, "\lrcorner", very near start] & \mathcal{D}|_G \arrow[u, tail, "\mathsf{J}_G"] \arrow[rr] \arrow[ur, phantom, "\lrcorner", very near start] & & \mathbf{1} \arrow[u, tail, "G"']
\end{tikzcd}$$

Note that by our construction of inverse images and by Lemma 2 (since $G : \mathbf{1} \to |\mathcal{C}|$ is full), $\mathcal{D}|_G$ is a full subcategory of $\mathcal{D}$. Similarly, $\mathcal{D}|_G^{\mathcal{S}}$ is a subcategory of $\mathcal{D}$ that may not be full if J is not full. Hence $\mathcal{D}|_G^{\mathcal{S}}$ contains all direct transformations of G by the rules in $\mathcal{S}$, and all the subsumptions between these whose image by R also belongs to $\mathcal{S}$ (or its image by J). In this way rule systems are used to specify subcategories of direct transformations.

It seems that in most cases the functor R is faithful. This is a consequence of a property that seems ubiquitous in all approaches to algebraic rewriting (see [2, Propositions 6.4, 6.6, 6.9] and [12, Theorem 3.5]), and we will see in Section 4 that this property (generalized as condition (1) in Theorem 1 below) has other consequences. Understanding this property is one motivation of the present work.

Another important property is whether R' is right-full, for this means that for any direct transformation $d' \in \mathcal{D}|_G^{\mathcal{S}}$ and any subsumption $s : r \to \mathsf{R}'\, d'$ in the rule system $\mathcal{S}$, there exists a transformation $d \in \mathcal{D}|_G^{\mathcal{S}}$ subsumed by d' (with subsumption s' s.t. $\mathsf{R}'\, s' = s$), i.e., when a rule applies its subsumed rules necessarily apply. This is a property that seems all too natural but it does not always hold, e.g., in the DPO approach with unrestricted matchings. It is shown in [2, Proposition 6.3] that R' is right-full whenever R is right-full, hence we will focus on R.

One important feature of REs is that they can be combined:

Definition 5. *Given two REs* $\mathscr{R}_i = \mathcal{R}_i \xleftarrow{\mathsf{R}_i} \mathcal{D}_i \xrightarrow{\mathsf{P}_i} |\mathcal{C}|$ *for* $\mathcal{C}$ *for* $i = 1, 2$*, their* sum $\mathscr{R}_1 + \mathscr{R}_2$ *is the RE*

$$\mathcal{R}_1 + \mathcal{R}_2 \xleftarrow{\mathsf{R}_1 + \mathsf{R}_2} \mathcal{D}_1 + \mathcal{D}_2 \xrightarrow{[\mathsf{P}_1, \mathsf{P}_2]} \mathcal{C}_{\mathrm{pt}}$$

By Lemma 5 we obviously have:

Proposition 1. *If* R_1 *and* R_2 *are right-full (resp. faithful) then so is* $\mathsf{R}_1 + \mathsf{R}_2$.

3 Matching Environments

A matching is generally understood as a kind of relation between a source object, that we call a *pattern* p, and a target object, in our case the input object $G \in \mathcal{C}$. In most approaches this is simply a $\mathcal{C}$-morphism, so that matchings may be composed. In Term Rewriting a matching is a substitution together with a position in G, but once again matchings can be composed since G can be understood also as a pattern (a term with variables). However, in the PBPO approach a matching in G is a commuting diagram

$$\begin{tikzcd} L \arrow[d, "m"] \arrow[dd, bend right, "t_L"'] \\ G \arrow[d, "c"] \\ T_L \end{tikzcd}$$

where t_L belongs to the PBPO rule. More precisely, the pattern appears to be the $\mathcal{C}$-morphism t_L, and the matching that relates it to G is the pair (m, c) that factors t_L through G. Anyway, the pattern and the target have different natures and it therefore seems difficult to consider a target as a pattern. Thus we have to drop the possibility to compose matchings.

For this reason we treat matchings and patterns as objects each in their respective category. A matching designates the location in G of its pattern, and such locations can be connected by morphisms only if their patterns are likewise connected (and they are located in the same G). The converse is false; connected patterns can be matched at unrelated locations. This yields a definition similar to REs.

Definition 6 (Matching Environment $\mathscr{M}$, categories $\mathcal{M}|^G$, $\mathcal{M}|_p$, $\mathcal{M}|_p^G$). *A* Matching Environment *(or* ME*)* $\mathscr{M}$ *for* $\mathcal{C}$ *consists of a category* $\mathcal{M}$ *whose objects are called* matchings*, a category* $\mathcal{P}$ *whose objects are called* patterns*, and two* source *and* target *functors*

$$\mathcal{P} \xleftarrow{\mathsf{S}} \mathcal{M} \xrightarrow{\mathsf{T}} |\mathcal{C}|$$

An object $m \in \mathcal{M}$ *is a matching* of $\mathsf{S}\,m$ into $\mathsf{T}\,m$.

As in REs we extract from $\mathcal{M}$ *the category* $\mathcal{M}|_p$ *of matchings* of *a pattern* $p \in \mathcal{P}$*, the category* $\mathcal{M}|^G$ *of matchings* into *an object* $G \in \mathcal{C}$*, and the category* $\mathcal{M}|_p^G$ *of matchings* of p into G*, by taking the inverse image of the corresponding embeddings* $p : \mathbf{1} \to \mathcal{P}$ *and* $G : \mathbf{1} \to |\mathcal{C}|$ *along* S *and* T *as pictured below (so that we obtain subcategories of* $\mathcal{M}$*).*

$$\begin{tikzcd}
\mathbf{1} \arrow[r, tail, "p"] & \mathcal{P} & \\
\mathcal{M}|_p \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\urcorner", very near start] & \mathcal{M} \arrow[u, "\mathsf{S}"'] \arrow[r, "\mathsf{T}"] & |\mathcal{C}| \\
\mathcal{M}|_p^G \arrow[u, tail] \arrow[r, tail] \arrow[ur, phantom, "\urcorner", very near start] & \mathcal{M}|^G \arrow[u, tail] \arrow[r] \arrow[ur, phantom, "\urcorner", very near start] & \mathbf{1} \arrow[u, tail, "G"']
\end{tikzcd}$$

This general definition is easily illustrated by the standard notion of matchings, where patterns are $\mathcal{C}$-objects and matchings are $\mathcal{C}$-morphisms to G and are connected by pattern morphisms restricted by a trivial commuting condition. For PBPO rules the notions of matchings and patterns should be clear, but their morphisms may not. The notion of subsumption morphism between PBPO rules given in [2, Definition 6.8] spills the beans: a morphism from t_L to $t_{L'}$ is a factorization of t_L through $t_{L'}$.

Definition 7 (Matching Environments $\mathscr{M}_{\mathrm{std}}$, $\mathscr{M}_{\mathrm{fct}}$). *Let $\mathcal{M}_{\mathrm{std}}$ be the category whose objects are the $\mathcal{C}$-morphisms and whose morphisms $f : m \to m'$ are $\mathcal{C}$-morphisms such that*

$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "m"'] & L' \arrow[d, "m'"] \\
G \arrow[r, dash, "="] & G'
\end{tikzcd}$$

commutes, with the same composition as $\mathcal{C}$. The functor $\mathsf{S}_{\mathrm{std}} : \mathcal{M}_{\mathrm{std}} \to \mathcal{C}$ is defined by $\mathsf{S}_{\mathrm{std}}\, m \coloneqq L$ where L is the domain of m, and $\mathsf{S}_{\mathrm{std}}\, f \coloneqq f$. The functor $\mathsf{T}_{\mathrm{std}} : \mathcal{M}_{\mathrm{std}} \to |\mathcal{C}|$ is defined by $\mathsf{T}_{\mathrm{std}}\, m \coloneqq G$ where G is the codomain of m, and $\mathsf{T}_{\mathrm{std}}\, f \coloneqq 1_G$ for all $f : m \to m'$ in $\mathcal{M}_{\mathrm{std}}$ and G is the common codomain of m and m'. Let $\mathscr{M}_{\mathrm{std}}$ be the Matching Environment $\mathcal{C} \xleftarrow{\mathsf{S}_{\mathrm{std}}} \mathcal{M}_{\mathrm{std}} \xrightarrow{\mathsf{T}_{\mathrm{std}}} |\mathcal{C}|$.

Let $\mathcal{P}_{\mathrm{fct}}$ be the category whose objects are $\mathcal{C}$-morphisms and whose morphisms are pairs of $\mathcal{C}$-morphisms $(f, g) : p \to p'$ such that

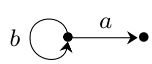
$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "p"'] & L' \arrow[d, "p'"] \\
T & T' \arrow[l, "g"]
\end{tikzcd}$$

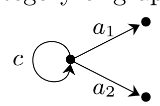
commutes, with obvious composition $(f', g') \circ (f, g) \coloneqq (f' \circ f, g \circ g')$. The category of matchings $\mathcal{M}_{\mathrm{fct}}$ has as objects pairs (m, c) of consecutive $\mathcal{C}$-morphisms, i.e., such that $c \circ m$ exists, and as morphisms pairs of $\mathcal{C}$-morphisms $(f, g) : (m, c) \to (m', c')$ such that

$$
\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "m"'] & L' \arrow[d, "m'"] \\
G \arrow[r, dash, "="] \arrow[d, "c"'] & G' \arrow[d, "c'"] \\
T & T' \arrow[l, "g"]
\end{tikzcd}
$$

commutes, with composition as in $\mathcal{P}_{\text{fct}}$. The functor $\mathsf{S}_{\text{fct}} : \mathcal{M}_{\text{fct}} \to \mathcal{P}_{\text{fct}}$ is defined by $\mathsf{S}_{\text{fct}}\,(m, c) \coloneqq c \circ m$ and $\mathsf{S}_{\text{fct}}\,(f, g) \coloneqq (f, g)$. The functor $\mathsf{T}_{\text{fct}} : \mathcal{M}_{\text{fct}} \to |\mathcal{C}|$ is defined by $\mathsf{S}_{\text{fct}}\,(m, c) \coloneqq G$ and $\mathsf{S}_{\text{fct}}\,(f, g) \coloneqq 1_G$ for all $(f, g) : (m, c) \to (m', c')$, where G is the common codomain of m, m' and domain of c, c'. Let $\mathscr{M}_{\text{fct}}$ be the Matching Environment $\mathcal{P}_{\text{fct}} \xleftarrow{\mathsf{S}_{\text{fct}}} \mathcal{M}_{\text{fct}} \xrightarrow{\mathsf{T}_{\text{fct}}} |\mathcal{C}|$.

Example 1. In $\mathscr{M}_{\text{std}}$ where $\mathcal{C}$ is the category of graphs, we consider the pattern

$p =$ 

and the graph $G =$ 

There are two monic matchings $m_i \coloneqq \{b \mapsto c,\ a \mapsto a_i\}$ of p into G for $i = 1, 2$, and one non monic matching $m_0 \coloneqq \{a, b \mapsto c\}$. The $\mathcal{C}$-morphism $f \coloneqq \{a, b \mapsto b\} : p \to p$ is an $\mathcal{M}_{\text{std}}$-morphism from m_0 to m_1 since $m_1 \circ f = m_0$, and also a $\mathcal{M}_{\text{std}}$-morphism from m_0 to m_2 since $m_2 \circ f = m_0$. Hence the category $\mathcal{M}_{\text{std}}|_p^G$ is a span $m_1 \leftarrow m_0 \to m_2$.

We have the same problem with MEs as with REs since the notion allows $\mathcal{M}$ to be discrete even when $\mathcal{P}$ is not. One may hope to constrain $\mathcal{M}$-morphisms by requiring S to be full, but neither S_{std} nor S_{fct} are full. Fortunately they share the following properties.

Lemma 7. *The functors S_{std} and S_{fct} are right-full and faithful.*

Proof. Faithfulness is obvious. To show that S_{std} is right-full, take $m' : L' \to G'$ an object of $\mathcal{M}_{\text{std}}$ and $f : L \to \mathsf{S}_{\text{std}}\, m'$ a morphism in $\mathcal{C}$, since $\mathsf{S}_{\text{std}}\, m' = L'$ then $f : m' \circ f \to m'$ is a morphism in $\mathcal{M}_{\text{std}}$ such that $\mathsf{S}_{\text{std}}\, f = f$.

Let $(m', c') \in \mathcal{M}_{\text{fct}}$ and $(f, g) : p \to c' \circ m'$ in $\mathcal{P}_{\text{fct}}$, then $(f, g) : (m' \circ f, g \circ c') \to (m', c')$ is a morphism in $\mathcal{M}_{\text{fct}}$, hence S_{fct} is also right-full.

It therefore seems reasonable to require, at least, that S be right-full, which obviously imposes a constraint on $\mathcal{M}$-morphisms w.r.t. $\mathcal{P}$-morphisms. This also entails that for any $\mathcal{P}$-morphisms $g : p \to p'$ and matching m' of p' in G there is a matching m of p in G and a $\mathcal{M}$-morphism $f : m \to m'$; for this reason we may say that p' is an *instance* of p and that g is an instance morphism. The morphism f can be viewed as an inclusion of m in m'.

Next we see that, as in Definition 5 for REs, MEs can easily be combined while preserving the properties of the source functors.

Definition 8 (sum of ME). *The* sum $\mathscr{M}_1+\mathscr{M}_2$ *of two Matching Environments* $\mathscr{M}_i = \mathcal{P}_i \xleftarrow{\mathsf{S}_i} \mathcal{M}_i \xrightarrow{\mathsf{T}_i} |\mathcal{C}|$ *for* $i = 1, 2$ *is the Matching Environment*

$$\mathcal{P}_1 + \mathcal{P}_2 \xleftarrow{\mathsf{S}_1 + \mathsf{S}_2} \mathcal{M}_1 + \mathcal{M}_2 \xrightarrow{[\mathsf{T}_1, \mathsf{T}_2]} |\mathcal{C}|$$

By Lemma 5 we have:

Proposition 2. *If* S_1 *and* S_2 *are right-full (resp. faithful) then so is* $\mathsf{S}_1 + \mathsf{S}_2$*.*

3.1 Strict and trivial matchings

We now try to generalize the notion of monic matchings in $\mathscr{M}_{\text{std}}$. Since matchings do not compose, we cannot rely on the standard notion of regularity (or cancellability) on the left. Intuitively, we wish to avoid matchings that relate distinct items (vertices, edges or whatever) of a pattern to the same item in G, and hence to avoid that locations that exist in patterns should be confused by matchings. If we see a matching as a map, we want every part of the map to have a unique location. The location of a part of a map within the whole map being given by a morphism, we are asking for a 1-1 correspondence between such morphisms and the parts of the map. This idea leads to a simple definition.

Definition 9 (strict matchings). *A matching* $s \in \mathcal{M}$ *is* strict *if for all morphisms* $f : m \to s$ *and* $f' : m' \to s$ *in* $\mathcal{M}$*,* $m = m'$ *entails* $f = f'$*.*

We first check that this notion corresponds to monic standard matchings.

Proposition 3. *A matching in* $\mathcal{M}_{\text{std}}$ *is strict iff it is monic.*

Proof. Obvious since for all $f : m \to s$ and $f' : m' \to s$ in $\mathcal{M}_{\text{std}}$ we have $m = m'$ iff $s \circ f = s \circ f'$.

We then see that strict factor matchings appear as a quite natural extension of the standard case.

Proposition 4. *A matching* $(m, c) \in \mathcal{M}_{\text{fct}}$ *is strict iff* m *is monic and* c *is epic.*

Proof. Suppose (m, c) is strict, for all $\mathcal{C}$-morphisms f_1, f_2, g_1, g_2 such that $m \circ f_1 = m \circ f_2$ and $g_1 \circ c = g_2 \circ c$, since $(f_1, g_1) : (m \circ f_1, g_1 \circ c) \to (m, c)$ and $(f_2, g_2) : (m \circ f_2, g_2 \circ c) \to (m, c)$ are $\mathcal{M}_{\text{fct}}$-morphisms with identical domains, then $f_1 = f_2$ and $g_1 = g_2$, hence m is monic and c is epic. The converse is similar.

Besides, in a category of strict matchings we only accept $\mathcal{M}$-monomorphisms, and similarly we restrict the category of patterns to $\mathcal{P}$-monomorphisms. We therefore need source functors that preserve monomorphisms, but we only know that monomorphisms are reflected by faithful functors [6, Proposition 12.8]. Fortunately we can use right-fullness.

Proposition 5. *If* S *is right-full and faithful then* S *preserves monomorphisms.*

Proof. For any $\mathcal{M}$-monomorphism $f : m \to m'$ and $\mathcal{P}$-morphisms h and h' such that $(\mathsf{S}\, f) \circ h = (\mathsf{S}\, f) \circ h'$, since S is right-full there exist $\mathcal{M}$-morphisms g and g' such that $\mathsf{S}\, g = h$ and $\mathsf{S}\, g' = h'$, hence $\mathsf{S}(f \circ g) = \mathsf{S}(f \circ g')$, so that $f \circ g = f \circ g'$, hence $g = g'$ and $h = h'$.

Thanks to this property we can restrict MEs to strict matchings and monomorphisms, provided we assume that the source functor is right-full and faithful.

Definition 10 (strict restriction $\mathscr{M}^\star$ of $\mathscr{M}$). *Given a ME $\mathscr{M}$ as in Definition 6 and such that S is right-full and faithful, let $\mathcal{M}^\star$ be the subcategory of strict matchings of $\mathcal{M}$ and $\mathcal{M}$-monomorphisms between them, and $\mathcal{P}^\star$ be the subcategory of $\mathcal{P}$ with all $\mathcal{P}$-objects and all $\mathcal{P}$-monomorphisms. Let $\mathsf{T}^\star : \mathcal{M}^\star \to |\mathcal{C}|$ be the restriction of T to $\mathcal{M}^\star$ and $\mathsf{S}^\star : \mathcal{M}^\star \to \mathcal{P}^\star$ be the restriction of S to $\mathcal{M}^\star$ (that exists according to Proposition 5). The* strict restriction *$\mathscr{M}^\star$ of $\mathscr{M}$ is the Matching Environment*

$$\mathcal{P}^\star \xleftarrow{\mathsf{S}^\star} \mathcal{M}^\star \xrightarrow{\mathsf{T}^\star} |\mathcal{C}|$$

Example 2. The pattern category of the standard ME $\mathscr{M}_{\text{std}}$ is $\mathcal{C}$, hence its restriction $\mathcal{C}^\star$ is the category with all $\mathcal{C}$-objects and whose morphisms are the $\mathcal{C}$-monomorphisms. Similarly, the $\mathcal{M}^\star_{\text{std}}$-objects are the $\mathcal{C}$-monomorphisms, and the $\mathcal{M}^\star_{\text{std}}$-morphisms $f : m \to m'$ are the $\mathcal{C}$-monomorphisms such that $m = m' \circ f$. The functors $\mathsf{S}^\star_{\text{std}}$ and $\mathsf{T}^\star_{\text{std}}$ are the obvious restrictions of S_{std} and T_{std} to monic matchings and monomorphisms. Note that $\mathcal{M}^\star_{\text{std}}$ is a full subcategory of $\mathcal{M}_{\text{std}}$ (since $m' \circ f$ monic entails f monic).

We can then prove that the relevant property of the source functor is preserved by this restriction. We first need an easy lemma.

Lemma 8. *For any $\mathcal{M}$-monomorphism $f : m \to m'$, if m' is strict then so is m.*

Proof. Let $g : n \to m$ and $g' : n' \to m$, if $n = n'$ then $f \circ g = f \circ g' : n \to m'$, hence $g = g'$.

Proposition 6. *The functor $\mathsf{S}^\star$ in Definition 10 is right-full and faithful.*

Proof. That $\mathsf{S}^\star$ is faithful is obvious. Let m' be a strict matching, p be a pattern and $g : p \to \mathsf{S}\, m'$ be a $\mathcal{P}^\star$-morphism, i.e., a $\mathcal{P}$-monomorphism, since S is right-full there exists $m \in \mathcal{M}$ and a $\mathcal{M}$-morphism $f : m \to m'$ such that $\mathsf{S}\, f = g$. Since S is faithful then f is an $\mathcal{M}$-monomorphism, hence by Lemma 8 m is strict and therefore f is a $\mathcal{M}^\star$-morphism such that $\mathsf{S}^\star\, f = g$.

It is easy to see that $(\mathscr{M}^\star)^\star = \mathscr{M}^\star$, hence this restriction is one-shot. Similarly we see that restrictions and sums commute, i.e., $(\mathscr{M}_1 + \mathscr{M}_2)^\star = \mathscr{M}_1^\star + \mathscr{M}_2^\star$.

Another notion that seems difficult to generalize in absence of composable matchings is that of an identity matching, that first requires an identity between a pattern and the input object G, which is out of reach. Obviously the notion can only be generalized up to a point, and only up to $\mathcal{M}$-isomorphisms. The idea is to say that a matching into G is trivial if it is wider than all other matchings into G, in the sense that they are all a part of it.

Definition 11 (trivial matchings). *For any $g \in \mathcal{C}$, a matching into G is* trivial *if it is terminal in $\mathcal{M}|^G$.*

Hence a matching m into G is trivial if for all matchings m' into G there is a unique $f : m' \to m$. This unicity obviously entails that m is strict.

We see that trivial standard matchings are only an approximation of identity matchings, but the best we can expect.

Proposition 7. *A matching in $\mathcal{M}_{\text{std}}$ is trivial iff it is an isomorphism.*

Proof. If $m \in \mathcal{M}_{\text{std}}$ is a trivial matching into G, since 1_G is a matching into G then there exists $f : 1_G \to m$, so that $m \circ f = 1_G$, i.e., m is a retraction, and since m is monic by Proposition 3, then m is an isomorphism (by [6, Proposition 6.7]). The converse is obvious.

Proposition 8. *A matching $(m, c) \in \mathcal{M}_{\text{fct}}$ is trivial iff m and c are isomorphisms.*

Proof. If (m, c) is a strict matching into G then there exists $(f, g) : (1_G, 1_G) \to (m, c)$, hence $m \circ f = 1_G$ and $g \circ c = 1_G$, i.e., m is a retraction and c is a section, and since m is monic and c is epic by Proposition 4 then m and c are isomorphisms. The converse is obvious.

This obviously means that a pattern $p \in \mathcal{P}_{\text{fct}}$ admits trivial matchings (in $\mathcal{M}|_p$) iff p is a $\mathcal{C}$-isomorphism (and then only into $\mathcal{C}$-objects isomorphic to p's domain and codomain).

4 Rewriting Environments with Matchings

We can now enhance Rewriting Environments by stating that every direct transformation has a matching of the left-hand side of its rule into its input object. This matching and the left-hand side are accessed by means of two new functors, and their relations are expressed as commuting conditions.

Definition 12 (REM). *A* Rewriting Environment with Matchings *(or* REM*) $\mathscr{E}$ consists of a Rewriting Environment $\mathscr{R}$ as in Definition 4, a Matching Environment $\mathscr{M}$ as in Definition 6 and two functors L and M such that*

$$\begin{tikzcd}
\mathcal{R} \arrow[d, "\mathsf{L}"'] & \mathcal{D} \arrow[l, "\mathsf{R}"'] \arrow[r, "\mathsf{P}"] \arrow[d, "\mathsf{M}"] & \mathcal{C}_{\text{pt}} \arrow[d, "\mathsf{In}"] \\
\mathcal{P} & \mathcal{M} \arrow[l, "\mathsf{S}"] \arrow[r, "\mathsf{T}"'] & |\mathcal{C}|
\end{tikzcd}$$

commutes. For every rule $r \in \mathcal{R}$, the pattern $\mathsf{L}r$ is the left-hand side *of r.*

Example 3. In [2] the category $\mathcal{R}_{\text{DPO}}$ of DPO rules (in $\mathcal{C}$) is defined with morphisms as triples (s_1, s_2, s_3) of $\mathcal{C}$-morphisms such that

$$\begin{tikzcd}
L & K \arrow[l, tail] \arrow[r] \arrow[d, "s_2"] \arrow[dl, phantom, very near start, "\llcorner"] & R \arrow[d, "s_3"] \\
L' & K' \arrow[l, tail] \arrow[r] & R'
\arrow[from=1-1, to=2-1, "s_1"']
\end{tikzcd}$$

commutes and the left square is a pullback. Hence there is an obvious functor $\mathsf{L}_{\mathrm{DPO}} : \mathcal{R}_{\mathrm{DPO}} \to \mathcal{C}$ where $\mathsf{L}_{\mathrm{DPO}}(s_1, s_2, s_3) = s_1$. Similarly, the category $\mathcal{D}_{\mathrm{DPO}}$ has DPO diagrams as objects, and morphisms (s_1, s_2, s_3, s_4) such that

$$\begin{tikzcd}
& L \arrow[dl, "s_1"'] \arrow[dd, "m"] & & K \arrow[ll, tail] \arrow[rr] \arrow[dl, "s_2"] \arrow[dd] & & R \arrow[dl, "s_3"] \\
L' \arrow[dd, "m'"'] & & K' \arrow[ll, tail] \arrow[rr] \arrow[dd] & & R' & \\
& G & & D \arrow[ll, tail] & & \\
G' \arrow[ur, dash, "="] & & D' \arrow[ll] \arrow[ur, "s_4"'] & & &
\end{tikzcd}$$

commutes and the top left square is a pullback. Again there is an obvious functor $\mathsf{M}_{\mathrm{DPO}} : \mathcal{D}_{\mathrm{DPO}} \to \mathcal{M}_{\mathrm{std}}$ where $\mathsf{M}_{\mathrm{DPO}}(s_1, s_2, s_3, s_4) = s_1$. The DPO RE together with $\mathscr{M}_{\mathrm{std}}$ and the two functors $\mathsf{L}_{\mathrm{DPO}}$ and $\mathsf{M}_{\mathrm{DPO}}$ constitute a REM. These functors are neither full nor faithful. If $\mathcal{C}$ does not have pullbacks they are not right-full either.

In a REM the category $\mathcal{D}$ is still free to be discrete and it does not seem that we can prevent this by requiring some standard property on the functor M. In order to formulate a realistic constraint we need further tools.

Definition 13 (redex category $\mathcal{X}$, X-functor). *In a REM $\mathscr{E}$ as in Definition 12, the* redex category $\mathcal{X}$ *of $\mathscr{E}$ and the projection functors $\pi_{\mathcal{R}} : \mathcal{X} \to \mathcal{R}$, $\pi_{\mathcal{M}} : \mathcal{X} \to \mathcal{M}$ are obtained as the meet of L and S. We call* X-functor *of $\mathscr{E}$ the unique functor $\mathsf{X} : \mathcal{D} \to \mathcal{X}$ such that $\mathsf{R} = \pi_{\mathcal{R}} \circ \mathsf{X}$ and $\mathsf{M} = \pi_{\mathcal{M}} \circ \mathsf{X}$.*

$$\begin{tikzcd}
& & \mathcal{D} \arrow[dll, bend right, "\mathsf{R}"'] \arrow[dl, dashed, "\mathsf{X}"'] \arrow[ddl, bend left, "\mathsf{M}"] \\
\mathcal{R} \arrow[d, "\mathsf{L}"'] & \mathcal{X} \arrow[l, "\pi_{\mathcal{R}}"'] \arrow[d, "\pi_{\mathcal{M}}"] \arrow[dl, phantom, very near start, "\llcorner"] & \\
\mathcal{P} & \mathcal{M} \arrow[l, "\mathsf{S}"] &
\end{tikzcd}$$

As explained in Section 2.2 we are interested in whether R is right-full and faithful. We easily see that these are equivalent to properties of the X-functor.

Proposition 9. *If S is faithful then R is faithful iff X is faithful. If S is right-full and faithful then R is right-full iff X is right-full.*

Proof. By Lemma 2 $\pi_{\mathcal{R}}$ is faithful (resp. right-full and faithful), hence by Lemma 1, X is faithful (resp. right-full) iff so is $\pi_{\mathcal{R}} \circ \mathsf{X} = \mathsf{R}$.

Theorem 1. *If S is faithful, then the X-functor is fully faithful iff*

$$\begin{array}{l}\text{for all } d, d' \in \mathcal{D},\ f : \mathsf{R}\,d \to \mathsf{R}\,d' \text{ in } \mathcal{R} \text{ and } g : \mathsf{M}\,d \to \mathsf{M}\,d' \text{ in } \mathcal{M} \text{ such that} \\ \mathsf{L}\,f = \mathsf{S}\,g, \text{ there exists a unique } s : d \to d' \text{ in } \mathcal{D} \text{ such that } \mathsf{R}\,s = f.\end{array} \tag{1}$$

Proof. Suppose X is fully faithfull and let d, d', f and g as in (1), then $(f, g) : (\mathsf{R}\,d, \mathsf{M}\,d) \to (\mathsf{R}\,d', \mathsf{M}\,d')$ is a $\mathcal{X}$-morphism, hence there exists a unique $s : d \to d'$ in $\mathcal{D}$ such that $\mathsf{X}\,s = (f, g)$, hence $\mathsf{R}\,s = f$. Since R is faithful by Proposition 9, then s is indeed unique such that $\mathsf{R}\,s = f$, hence property (1) holds.

Suppose now that (1) holds. We first see that R is faithful: for all $s, s' : d \to d'$ in $\mathcal{D}$ such that $\mathsf{R}\,s = \mathsf{R}\,s'$, then $\mathsf{R}\,s : \mathsf{R}\,d \to \mathsf{R}\,d'$ is a $\mathcal{R}$-morphism and $\mathsf{M}\,s : \mathsf{M}\,d \to \mathsf{M}\,d'$ is an $\mathcal{M}$-morphism such that $\mathsf{L}(\mathsf{R}\,s) = \mathsf{S}(\mathsf{M}\,s)$, hence there is a unique $s'' : d \to d'$ such that $\mathsf{R}\,s'' = \mathsf{R}\,s = \mathsf{R}\,s'$, hence $s = s'$. By Proposition 9 we get that X is faithful.

To prove that X is full, let $d, d' \in \mathcal{D}$ and $(f, g) : \mathsf{X}\,d \to \mathsf{X}\,d'$ be a $\mathcal{X}$-morphism, then $f = \pi_{\mathcal{R}}(f, g) : \mathsf{R}\,d \to \mathsf{R}\,d'$ is a $\mathcal{R}$-morphism and $g = \pi_{\mathcal{M}}(f, g) : \mathsf{M}\,d \to \mathsf{M}\,d'$ is a $\mathcal{M}$-morphism such that $\mathsf{L}\,f = \mathsf{S}\,g$, hence by (1) there exists $s : d \to d'$ in $\mathcal{D}$ such that $\mathsf{R}\,s = f$. But then $\mathsf{S}\,\mathsf{M}\,s = \mathsf{L}\,\mathsf{R}\,s = \mathsf{L}\,f = \mathsf{S}\,g$, and since S is faithful then $\mathsf{M}\,s = g$, hence $\mathsf{X}\,s = (\mathsf{R}\,s, \mathsf{M}\,s) = (f, g)$.

Example 4. It is shown in [2, Proposition 6.4] that if $\mathcal{C}$ is adhesive [7] then for all $d, d' \in \mathcal{D}_{\mathrm{DPO}}$, all $(s_1, s_2, s_3) : \mathsf{R}_{\mathrm{DPO}}\,d \to \mathsf{R}_{\mathrm{DPO}}\,d'$ in $\mathcal{R}_{\mathrm{DPO}}$ such that $m = m' \circ s_1$ there exists a unique $s : d \to d'$ in $\mathcal{D}_{\mathrm{DPO}}$ such that $\mathsf{R}_{\mathrm{DPO}}\,s = (s_1, s_2, s_3)$. Since m and m' refer to the corresponding morphisms in the DPO diagrams d and d', the equation $m = m' \circ s_1$ means that $s_1 : \mathsf{M}_{\mathrm{DPO}}\,d \to \mathsf{M}_{\mathrm{DPO}}\,d'$ is a morphism in $\mathcal{M}_{\mathrm{std}}$, and since $\mathsf{L}_{\mathrm{DPO}}(s_1, s_2, s_3) = s_1 = \mathsf{S}_{\mathrm{std}}\,s_1$, it is equivalent to $\mathsf{L}_{\mathrm{DPO}}(s_1, s_2, s_3) = \mathsf{S}_{\mathrm{std}}\,g$ for any $g : \mathsf{M}_{\mathrm{DPO}}\,d \to \mathsf{M}_{\mathrm{DPO}}\,d'$. Hence property (1) holds for the DPO REM, and its X-functor is therefore fully faithful (in adhesive categories).

We leave it to the reader to check that this also holds for the SPO REM (in categories of presheaves) by [12, Theorem 3.5]), for the SqPO REM by [2, Proposition 6.6] and for the PBPO REM by [2, Proposition 6.9].

4.1 Combining REMs

We can easily combine REMs as we did with REs and MEs. This can be done by preserving the properties of the X-functors.

Definition 14 (sum of REMs). *The* sum $\mathscr{E}_1 + \mathscr{E}_2$ *of two REMs* $\mathscr{E}_i$ *given by*

$$\begin{tikzcd} \mathcal{R}_i \arrow[d, "\mathsf{L}_i"'] & \mathcal{D}_i \arrow[l, "\mathsf{R}_i"'] \arrow[r, "\mathsf{P}_i"] \arrow[d, "\mathsf{M}_i"] & \mathcal{C}_{\mathrm{pt}} \arrow[d, "\mathsf{In}"] \\ \mathcal{P}_i & \mathcal{M}_i \arrow[l, "\mathsf{S}_i"] \arrow[r, "\mathsf{T}_i"'] & |\mathcal{C}| \end{tikzcd}$$

for $i = 1, 2$ is the REM

$$\begin{tikzcd}
\mathcal{R}_1 + \mathcal{R}_2 \arrow[d, "\mathsf{L}_1 + \mathsf{L}_2"'] & \mathcal{D}_1 + \mathcal{D}_2 \arrow[l, "\mathsf{R}_1 + \mathsf{R}_2"'] \arrow[r, "{[\mathsf{P}_1, \mathsf{P}_2]}"] \arrow[d, "\mathsf{M}_1 + \mathsf{M}_2"] & \mathcal{C}_{\mathrm{pt}} \arrow[d, "\mathsf{In}"] \\
\mathcal{P}_1 + \mathcal{P}_2 & \mathcal{M}_1 + \mathcal{M}_2 \arrow[l, "\mathsf{S}_1 + \mathsf{S}_2"] \arrow[r, "{[\mathsf{T}_1, \mathsf{T}_2]}"'] & |\mathcal{C}|
\end{tikzcd}$$

Note that this diagram is easily seen to commute.

Proposition 10. *If the X_1-functor and the X_2-functor are right-full (resp. full, resp. faithful, resp. embeddings) then so is the X-functor of the sum.*

Proof. Let $\mathcal{X}$ be the redex category of the sum, i.e., the meet of $\mathsf{L}_1 + \mathsf{L}_2$ and $\mathsf{S}_1 + \mathsf{S}_2$, and $\mathsf{X} : \mathcal{D}_1 + \mathcal{D}_2 \to \mathcal{X}$ be the unique functor such that $\mathsf{R}_1 + \mathsf{R}_2 = \pi_{\mathcal{R}_1 + \mathcal{R}_2} \circ \mathsf{X}$ and $\mathsf{M}_1 + \mathsf{M}_2 = \pi_{\mathcal{M}_1 + \mathcal{M}_2} \circ \mathsf{X}$. By Lemma 6 $\mathcal{X}_1 + \mathcal{X}_2$ with $\pi_{\mathcal{R}_1} + \pi_{\mathcal{R}_2}$ and $\pi_{\mathcal{M}_1} + \pi_{\mathcal{M}_2}$ is also a meet of $\mathsf{L}_1 + \mathsf{L}_2$ and $\mathsf{S}_1 + \mathsf{S}_2$, hence there exists a unique isomorphism $\mathsf{H} : \mathcal{X}_1 + \mathcal{X}_2 \to \mathcal{X}$ such that $\pi_{\mathcal{R}_1} + \pi_{\mathcal{R}_2} = \pi_{\mathcal{R}_1 + \mathcal{R}_2} \circ \mathsf{H}$ and $\pi_{\mathcal{M}_1} + \pi_{\mathcal{M}_2} = \pi_{\mathcal{M}_1 + \mathcal{M}_2} \circ \mathsf{H}$.

$$\begin{tikzcd}
 & & & \mathcal{D}_1 + \mathcal{D}_2 \arrow[dl, "\mathsf{X}_1 + \mathsf{X}_2"'] \arrow[dddll, bend right=60, "\mathsf{R}_1 + \mathsf{R}_2"'] \arrow[ddll, bend left=40, "\mathsf{M}_1 + \mathsf{M}_2"] \\
 & & \mathcal{X}_1 + \mathcal{X}_2 \arrow[dll, bend right, "\pi_{\mathcal{R}_1} + \pi_{\mathcal{R}_2}"'] \arrow[dl, dashed, "\mathsf{H}"] \arrow[ddl, "\pi_{\mathcal{M}_1} + \pi_{\mathcal{M}_2}"] & \\
\mathcal{R}_1 + \mathcal{R}_2 \arrow[d, "\mathsf{L}_1 + \mathsf{L}_2"'] & \mathcal{X} \arrow[l, "\pi_{\mathcal{R}_1 + \mathcal{R}_2}"'] \arrow[d, "\pi_{\mathcal{M}_1 + \mathcal{M}_2}"'] \arrow[dl, phantom, "\lrcorner"] & & \\
\mathcal{P}_1 + \mathcal{P}_2 & \mathcal{M}_1 + \mathcal{M}_2 \arrow[l, "\mathsf{S}_1 + \mathsf{S}_2"] & &
\end{tikzcd}$$

But then we see that

$$\begin{aligned}
\pi_{\mathcal{R}_1 + \mathcal{R}_2} \circ \mathsf{H} \circ (\mathsf{X}_1 + \mathsf{X}_2) &= (\pi_{\mathcal{R}_1} + \pi_{\mathcal{R}_2}) \circ (\mathsf{X}_1 + \mathsf{X}_2) \\
&= (\pi_{\mathcal{R}_1} \circ \mathsf{X}_1) + (\pi_{\mathcal{R}_2} \circ \mathsf{X}_2) = \mathsf{R}_1 + \mathsf{R}_2
\end{aligned}$$

and similarly that $\pi_{\mathcal{M}_1 + \mathcal{M}_2} \circ \mathsf{H} \circ (\mathsf{X}_1 + \mathsf{X}_2) = \mathsf{M}_1 + \mathsf{M}_2$. By Lemma 4 we therefore have $\mathsf{X} = \mathsf{H} \circ (\mathsf{X}_1 + \mathsf{X}_2)$ and we conclude with Lemma 5.

4.2 The strict restriction of a REM

We finally see that it is possible, under suitable premise, to construct a REM corresponding to the restriction to strict matchings, while preserving the properties of the X-functor.

Definition 15 (strict restriction $\mathscr{E}^\star$ of REM $\mathscr{E}$). *Given a REM $\mathscr{E}$ as in Definition 12 such that S is right-full and faithful, so that there is a strict restriction $\mathscr{M}^\star$ of its ME $\mathscr{M}$ as in Definition 10, let $\mathsf{J}_\mathcal{R} : \mathcal{R}^\star \to \mathcal{R}$ be the inverse image of the canonical embedding $\mathsf{J}_\mathcal{P} : \mathcal{P}^\star \to \mathcal{P}$ along L, and $\mathsf{L}^\star : \mathcal{R}^\star \to \mathcal{P}^\star$ be the corresponding restriction of L.*

Similarly, let $\mathsf{J}_\mathcal{D} : \mathcal{D}^\star \to \mathcal{D}$ be the inverse image of the canonical embedding $\mathsf{J}_\mathcal{M} : \mathcal{M}^\star \to \mathcal{M}$ along M, and $\mathsf{M}^\star : \mathcal{D}^\star \to \mathcal{M}^\star$ be the corresponding restriction of M.

Let $\mathsf{P}^\star = \mathsf{P} \circ \mathsf{J}_\mathcal{D}$.

$$
\begin{tikzcd}
 & \mathcal{R}^\star \arrow[dl, "\mathsf{L}^\star"'] \arrow[dd, tail] \arrow[dl, phantom, shift left=3, "\llcorner"] & & \mathcal{D}^\star \arrow[ll, dashed, "\mathsf{R}^\star"'] \arrow[dl, "\mathsf{M}^\star"'] \arrow[dd, tail] \arrow[ddr, "\mathsf{P}^\star"] \arrow[dl, phantom, shift left=3, "\llcorner"] & \\
\mathcal{P}^\star \arrow[dd, tail, "\mathsf{J}_\mathcal{P}"'] & & \mathcal{M}^\star \arrow[ll, "\mathsf{S}^\star"] \arrow[dd, tail] \arrow[ddr, "\mathsf{T}^\star"] & & \\
 & \mathcal{R} \arrow[dl, "\mathsf{L}"'] & & \mathcal{D} \arrow[ll, "\mathsf{R}"'] \arrow[r, "\mathsf{P}"] \arrow[dl, "\mathsf{M}"] & \mathcal{C}_{\mathrm{pt}} \arrow[dl, "\mathsf{In}"] \\
\mathcal{P} & & \mathcal{M} \arrow[ll, "\mathsf{S}"] \arrow[r, "\mathsf{T}"'] & |\mathcal{C}| &
\end{tikzcd}
$$

It is easy to see that $\mathsf{J}_\mathcal{P} \circ \mathsf{S}^\star \circ \mathsf{M}^\star = \mathsf{L} \circ \mathsf{R} \circ \mathsf{J}_\mathcal{D}$, and since $\mathcal{R}^\star$ with $\mathsf{J}_\mathcal{R}, \mathsf{L}^\star$ is a meet of L and $\mathsf{J}_\mathcal{P}$, then there exists a unique $\mathsf{R}^\star : \mathcal{D}^\star \to \mathcal{R}^\star$ such that $\mathsf{L}^\star \circ \mathsf{R}^\star = \mathsf{S}^\star \circ \mathsf{M}^\star$ and $\mathsf{J}_\mathcal{R} \circ \mathsf{R}^\star = \mathsf{R} \circ \mathsf{J}_\mathcal{D}$. Hence we get a REM $\mathscr{E}^\star$, called the strict restriction *of $\mathscr{E}$.*

Note that by our special construction of inverse images, the functors $\mathsf{J}_\mathcal{R}$ and $\mathsf{J}_\mathcal{D}$ are canonical embeddings, so that $\mathsf{L}^\star$, $\mathsf{M}^\star$, $\mathsf{R}^\star$ and $\mathsf{P}^\star$ are restrictions of L, M, R and P respectively. Besides, since $\mathcal{P}^\star$ and $\mathcal{P}$ have the same objects ($\mathsf{J}_\mathcal{P}$ is surjective on objects) then so do $\mathcal{R}^\star$ and $\mathcal{R}$ ($\mathsf{J}_\mathcal{R}$ is surjective on objects by Lemma 2), so that all rules are preserved by the restriction.

Example 5. We have build $\mathscr{M}^\star_{\mathrm{std}}$ in Example 2. We now build the strict restriction of the DPO REM. We first see that $\mathcal{R}^\star_{\mathrm{DPO}}$ is obtained as the inverse image of the canonical embedding of $\mathcal{C}^\star$ in $\mathcal{C}$ along $\mathsf{L}_{\mathrm{DPO}}$, hence the $\mathcal{R}^\star_{\mathrm{DPO}}$-objects are all the $\mathcal{R}_{\mathrm{DPO}}$-objects (all DPO-rules), and the $\mathcal{R}^\star_{\mathrm{DPO}}$-morphisms are all the $(s_1, s_2, s_3) : r \to r'$ such that $\mathsf{L}_{\mathrm{DPO}}(s_1, s_2, s_3)$ is in $\mathcal{C}^\star$, i.e., such that s_1 is monic. This is the category $\mathcal{R}_{\mathrm{mDPO}}$ in [2, Definition 3.3].

Similarly, the $\mathcal{D}^\star_{\mathrm{DPO}}$-objects are the direct DPO-transformations $d \in \mathcal{D}_{\mathrm{DPO}}$ such that $\mathsf{M}_{\mathrm{DPO}}\, d$ is monic, and the $\mathcal{D}^\star_{\mathrm{DPO}}$-morphisms are the $\mathcal{D}_{\mathrm{DPO}}$-morphisms $(s_1, s_2, s_3, s_4) : d \to d'$ such that s_1 is monic. This is a full subcategory of $\mathcal{D}_{\mathrm{DPO}}$, denoted $\mathcal{D}_{\mathrm{mDPO}}$ in [2, Definition 3.8].

Proposition 11. *If X is right-full (resp. full, resp. faithful, resp. an embedding) then so is $\mathsf{X}^\star$.*

Proof. We consider the following diagram where both X-functors are depicted, together with their construction.

$$\begin{tikzcd}
 & \mathcal{R}^\star \arrow[dl, "\mathsf{L}^\star"'] \arrow[dd, tail] & & \mathcal{X}^\star \arrow[ll, "\pi_{\mathcal{R}^\star}"'] \arrow[dl, "\pi_{\mathcal{M}^\star}"'] \arrow[dd, tail, dashed] & & \mathcal{D}^\star \arrow[ll, "\mathsf{X}^\star"'] \arrow[llll, bend right, "\mathsf{R}^\star"'] \arrow[dlll, "\mathsf{M}^\star"] \arrow[dd, tail] \\
\mathcal{P}^\star \arrow[dd, tail, "\mathsf{J}_{\mathcal{P}}"'] & & \mathcal{M}^\star \arrow[ll, "\mathsf{S}^\star"] \arrow[dd, tail] & & & \\
 & \mathcal{R} \arrow[dl, "\mathsf{L}"'] & & \mathcal{X} \arrow[ll, "\pi_{\mathcal{R}}"'] \arrow[dl, "\pi_{\mathcal{M}}"] & & \mathcal{D} \arrow[ll, "\mathsf{X}"'] \arrow[llll, bend right, "\mathsf{R}"'] \arrow[dlll, "\mathsf{M}"] \\
\mathcal{P} & & \mathcal{M} \arrow[ll, "\mathsf{S}"] & & &
\end{tikzcd}$$

It is easy to see that $\mathsf{S} \circ \mathsf{J}_{\mathcal{M}} \circ \pi_{\mathcal{M}^\star} = \mathsf{L} \circ \mathsf{J}_{\mathcal{R}} \circ \pi_{\mathcal{R}^\star}$, and since $\mathcal{X}$ with $\pi_{\mathcal{R}}$, $\pi_{\mathcal{M}}$ is a meet of L and S then there exists a unique $\mathsf{J}_{\mathcal{X}} : \mathcal{X}^\star \to \mathcal{X}$ such that $\pi_{\mathcal{M}} \circ \mathsf{J}_{\mathcal{X}} = \mathsf{J}_{\mathcal{M}} \circ \pi_{\mathcal{M}^\star}$ and $\pi_{\mathcal{R}} \circ \mathsf{J}_{\mathcal{X}} = \mathsf{J}_{\mathcal{R}} \circ \pi_{\mathcal{R}^\star}$.

We therefore have a commuting cube from $\mathcal{X}^\star$ to $\mathcal{P}$. Since its left and top faces are meets, then by Lemma 3 the diagonal square $(\pi_{\mathcal{M}^\star}, \mathsf{J}_{\mathcal{P}} \circ \mathsf{S}^\star, \mathsf{J}_{\mathcal{R}} \circ \pi_{\mathcal{R}^\star}, \mathsf{L})$ is a meet, and since its bottom face is also a meet then again by Lemma 3 its right face is a meet. Hence $\mathsf{J}_{\mathcal{X}}$ is an embedding by Lemma 2.

We also have a square of X-functors, and we now show that it commutes. Indeed, it is easy to see that $\pi_{\mathcal{M}} \circ \mathsf{X} \circ \mathsf{J}_{\mathcal{D}} = \pi_{\mathcal{M}} \circ \mathsf{J}_{\mathcal{X}} \circ \mathsf{X}^\star$ and that $\pi_{\mathcal{R}} \circ \mathsf{X} \circ \mathsf{J}_{\mathcal{D}} = \pi_{\mathcal{R}} \circ \mathsf{J}_{\mathcal{X}} \circ \mathsf{X}^\star$, hence by Lemma 4 we get $\mathsf{X} \circ \mathsf{J}_{\mathcal{D}} = \mathsf{J}_{\mathcal{X}} \circ \mathsf{X}^\star$.

We can thus apply again Lemma 3 to get that the square of X-functors is a meet, and we conclude with Lemma 2.

5 Conclusion

We conclude that any combination of the REMs for DPO, SPO, SqPO and PBPO approaches and their strict restrictions yields a REM with a fully faithful X-functor.

At the abstract level it seems indispensable to require that the X-functor be fully faithful and the source functor S be faithful, at the very least. For then we see that for all $d, d' \in \mathcal{D}$ with a morphism $g : \mathsf{M}\, d \to \mathsf{M}\, d'$ and a subsumption morphism $f : \mathsf{R}\, d \to \mathsf{R}\, d'$ that is not an identity, there is a subsumption morphism $s : d \to d'$ that is not an identity either.

Requiring further that S be right-full enables the strict restriction. Other constructions that seem universal (independent of rule semantics) could be investigated.

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