Subsumptions of SPO Rules

Thierry Boy de la Tour

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Thierry Boy de la Tour
Univ. Grenoble Alpes, CNRS, Grenoble INP, LIG, 38000 Grenoble, France
thierry dot boy-de-la-tour at imag.fr

In a previous paper we have defined a graph transformation that applies rewrite rules simultaneously to an input graph (or object in a suitable category), called the *Global Coherent Transformation*. The expressiveness of this transformation is enhanced by enabling the use of *subsumption morphisms* between rules and between direct (individual) transformations. Since this transformation is not committed to a particular approach to graph rewriting, it is formalized in a general representation of such approaches, called a *Rewriting Environment*. It was shown that environments exist for the Double Pushout (DPO), the Sesqui-Pushout and the Pullback-Pushout approaches, that each satisfy a certain *Correctness Condition*. In the present paper an environment is exhibited for the Single-Pushout (SPO) approach in categories of presheaves, and it is shown that the Correctness Condition holds. The link between SPO and DPO direct transformations is extended to subsumptions and expressed in diagrammatic form.

1 Introduction

Many "approaches" to graph rewriting have been developed. The most familiar and oldest one is the Double-Pushout (DPO) approach [5], that has a property unknown to term rewriting: a matching of a rule in the input object G is not sufficient to apply the rule. Indeed, this approach imposes a strict semantics of replacement where a matched vertex (say) cannot be removed and replaced unless all adjacent edges are similarly matched and removed. Another semantics exists that tolerates the silent removal of such edges: the Single-Pushout (SPO) approach [9]. It is based on pushouts of partial morphisms and has been defined in a restricted class of categories compared to DPO. Other algebraic approaches, namely the Sesqui-Pushout (SqPO) [4] and the Pullback-Pushout (PBPO) [3], provide the possibility to duplicate matched parts of the input.

In this diversity it is difficult to isolate common features. An obvious one is that they all end with a pushout, either in the category $\mathscr{C}$ whose objects are considered for computing (generally graph-like data structures), or an extension of $\mathscr{C}$ to partial morphisms. A closer look reveals that the result of the transformation is always obtained as a pushout of a $\mathscr{C}$-span $D \xleftarrow{k} K \xrightarrow{r} R$, where D is called the *context* and K the *interface*. The object R may or may not (for SqPO transformations) be the right-hand side of the rule. Besides, all approaches define a morphism $G \xleftarrow{f} D$, though in different ways. Hence all approaches are based on specific rule-based transformations, from which a diagram $G \xleftarrow{f} D \xleftarrow{k} K \xrightarrow{r} R$ can always be extracted (by some mapping). Such diagrams are called *partial transformations* in [2].

Hence in order to develop general methods related to rule-based algebraic transformations, methods that are not committed to a specific approach, one can certainly rely on partial transformations. In [2] a transformation is defined that applies algebraic rewrite rules simultaneously to the input object G. This transformation is not restricted to a particular approach to algebraic rewriting, and can even be applied by mixing rules from different approaches.

One important feature that enhances the expressiveness of this transformation is the use of morphisms between rules. The idea is inspired by [11], where the overlap of two matchings in a graph can be

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represented as a common subgraph of the corresponding left-hand sides, or more generally as morphisms between left-hand sides. But since the transformation in [2] is based on partial transformations, it relies on a notion of morphisms between such diagrams, and hence of a category $\mathscr{C}_{\mathrm{pt}}$ of partial transformations (see Definition 2.2 below). A morphism $s : p \to p'$ can be understood as a subsumption (of p by p') due to the following property: the simultaneous application of partial transformations p and p' yields the same result as p' [2, Proposition 5.12].

Hence we should also be able to find subsumption morphisms between the rule-based transformations (usually called *direct transformations*) of any given approach, hence the map from these to partial transformations should involve morphisms; in other words there should be a functor from a category $\mathscr{D}$ of direct transformations of the given approach, to the category $\mathscr{C}_{\mathrm{pt}}$. Similarly, there should be a functor from $\mathscr{D}$ to a category $\mathscr{R}$ of rules whose morphisms can then be understood as subsumptions between rules. This constitutes a *Rewriting Environment (RE)* $\mathscr{R} \stackrel{\mathsf{R}}{\leftarrow} \mathscr{D} \stackrel{\mathsf{P}}{\to} \mathscr{C}_{\mathrm{pt}}$. The simplicity of this model is very convenient as it encompasses many different situations. However, it does not provide a semantics to rule morphisms. If $\mathscr{D}$ is discrete then any non trivial rule morphism is meaningless; it can never specify a subsumption. We therefore need to enforce a *Correctness Condition* that ensures the existence of a (unique) subsumption corresponding to a rule morphism, under a specific circumstance, namely that that the rules are applied with overlapping matchings.

The difficulty is therefore to strike the right balance between the $\mathscr{R}$-morphisms and the $\mathscr{D}$-morphisms, so that the Correctness Condition holds. This has been done in [2] for the Double-Pushout approach (or DPO, see [5]) in adhesive categories, the SqPO approach and the PBPO approach (with a suitable notion of overlap). The object of the present paper is to do the same for the SPO approach.

In Section 2 we define the category $\mathscr{C}_{\mathrm{pt}}$ and the Correctness Condition suitable for the SPO approach. This approach will be developed in categories of presheaves, which are equivalent to categories of Σ-algebras for monadic many-sorted signatures Σ, as used in [9]. A number of results concerning these categories are developed in Section 3. The SPO direct derivations and their subsumption morphisms are defined in Section 4, where the Correctness Condition is proved. In Section 5 the classical comparison between DPO and SPO direct transformations (the former being a particular case of the latter) is extended to a comparison of their respective subsumption morphisms. Some perspectives are given as conclusion in Section 6.

2 The Correctness Condition

The category-theoretic notions and notations are compatible with [10]. For any category $\mathscr{C}$, we write $G \in \mathscr{C}$ to indicate that G is a $\mathscr{C}$-object, and $|\mathscr{C}|$ is the discrete category on $\mathscr{C}$-objects. Then G also denotes the *embedding* (faithful functor injective on objects, or equivalently, left-cancellable functor) from the terminal category $\mathbf{1}$ to $|\mathscr{C}|$ that maps the object of $\mathbf{1}$ to G. The *slice* category $\mathscr{C} \setminus G$ has as objects $\mathscr{C}$-morphisms of codomain G, and as morphisms $h : f \to g$ $\mathscr{C}$-morphisms such that $g \circ h = f$.

For any functor $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ and embedding $\mathsf{J} : \mathscr{B}' \rightarrowtail \mathscr{B}$, the *inverse image* of J (or simply $\mathscr{B}'$) along F is the embedding $\mathsf{J}' : \mathscr{A}' \rightarrowtail \mathscr{A}$ where $\mathscr{A}'$ is the subcategory of $\mathscr{A}$ of all $\mathscr{A}$-objects and morphisms whose image by F belongs to the subcategory $\mathsf{J}(\mathscr{B}')$ of $\mathscr{B}$, and J' is the corresponding inclusion functor. Together with the unique functor $\mathsf{F}' : \mathscr{A}' \to \mathscr{B}'$ such that $\mathsf{J} \circ \mathsf{F}' = \mathsf{F} \circ \mathsf{J}'$ we get a pullback in the category of categories:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathsf{F}"] & \mathscr{B} \\ \mathscr{A}' \arrow[u, tail, "\mathsf{J}'"] \arrow[r, "\mathsf{F}'"'] \arrow[ur, phantom, "\lrcorner", very near start] & \mathscr{B}' \arrow[u, tail, "\mathsf{J}"'] \end{tikzcd}$$

A close look at the SPO approach reveals that the result of an SPO direct transformation, i.e., a pushout in a category of partial morphisms (with the same objects as $\mathscr{C}$), is actually obtained as a pushout in $\mathscr{C}$ of a $\mathscr{C}$-span whose precise definition will be given in Section 4. For the time being we focus on the fact that most approaches to graph rewriting produce a $\mathscr{C}$-span $D \xleftarrow{k} K \xrightarrow{r} R$, where D is called the *context*, K the *interface* and R the *right-hand side*. The context D is obtained from the input G by deletions/duplications so that, equivalently, we can see G as being obtained from D by additions/mergings (inverse to deletions/duplications) hence that there is a morphism $f : D \to G$.

Definition 2.1 (partial transformation)**.** A *partial transformation* τ in $\mathscr{C}$ is a diagram

$$G \xleftarrow{\ f\ } D \xleftarrow{\ k\ } K \xrightarrow{\ r\ } R$$

The transformation is partial in the sense that its result is not given, though it is only one pushout away, and also because it does not contain a rule or a matching.

Following the informal description of a subsumption, we say that a partial transformation τ is subsumed by τ' if the latter's context D' is be obtained from the former's D (from the same input G) by further deletions/duplications, so that there must be a morphism from D' to D. Similarly, τ' should make further additions/mergings than τ, hence there should ba a morphism from R to R' and also from K to K' (the interface glues the new part to the old, and further additions may require a bigger interface).

Definition 2.2 (category $\mathscr{C}_{\mathrm{pt}}$, functor In, Rewriting Environments)**.** For any category $\mathscr{C}$, let $\mathscr{C}_{\mathrm{pt}}$ be the category whose objects are partial transformations and morphisms $\nu : \tau \to \tau'$ are triples (ν_1, ν_2, ν_3) of $\mathscr{C}$-morphisms such that

$$\begin{tikzcd} G & D \arrow[l, "f"'] & K \arrow[l, "k"'] \arrow[r, "r"] \arrow[d, "\nu_2"'] & R \arrow[d, "\nu_3"] \\ G' \arrow[u, dash, "="] & D' \arrow[l, "f'"] \arrow[u, "\nu_1"] & K' \arrow[l, "k'"] \arrow[r, "r'"'] & R' \end{tikzcd}$$

commutes in $\mathscr{C}$, with the obvious composition $(\nu'_1, \nu'_2, \nu'_3) \circ (\nu_1, \nu_2, \nu_3) := (\nu_1 \circ \nu'_1, \nu'_2 \circ \nu_2, \nu'_3 \circ \nu_3)$.

Let $\mathsf{In} : \mathscr{C}_{\mathrm{pt}} \to |\mathscr{C}|$ be the *input functor* defined as $\mathsf{In}\,\tau := G$.

A *Rewriting Environment* (or *RE*) $\mathscr{R}$ for $\mathscr{C}$ consists of a category $\mathscr{D}$ of *direct transformations*, a category $\mathscr{R}$ of *rules* and two functors

$$\mathscr{R} \xleftarrow{\ \mathsf{R}\ } \mathscr{D} \xrightarrow{\ \mathsf{P}\ } \mathscr{C}_{\mathrm{pt}}$$

In [2] a *rule system* in $\mathscr{R}$ is a category $\mathscr{S}$ with an embedding $\mathsf{J} : \mathscr{S} \rightarrowtail \mathscr{R}$. Equivalently, $\mathscr{S}$ is a subcategory of $\mathscr{R}$ and J is the canonical embedding, so that $\mathscr{S}$ simply picks up rules in $\mathscr{R}$ but also morphisms between them.

Given a rule system and an *input* $\mathscr{C}$-object G, the categories $\mathscr{D}|_G$, $\mathscr{D}|_G^{\mathscr{S}}$ and functors J_G, $\mathsf{J}_{\mathscr{S}}$ are obtained as inverse images of the embeddings G (whose sole image is the object $G \in \mathscr{C}$) and J, in the following way:

$$
\begin{tikzcd}
\mathscr{S} \arrow[r, tail, "\mathsf{J}"] & \mathscr{R} & & \\
& \mathscr{D} \arrow[u, "\mathsf{R}"] \arrow[r, "\mathsf{P}"] & \mathscr{C}_{\mathrm{pt}} \arrow[r, "\mathsf{In}"] & |\mathscr{C}| \\
\mathscr{D}|_G^{\mathscr{S}} \arrow[uu, "\mathsf{R}'"] \arrow[r, tail, "\mathsf{J}_{\mathscr{S}}"'] \arrow[ur, phantom, "\lrcorner", very near start] & \mathscr{D}|_G \arrow[u, tail, "\mathsf{J}_G"] \arrow[rr] \arrow[urr, phantom, "\lrcorner", very near start] & & \mathbf{1} \arrow[u, tail, "G"']
\end{tikzcd}
$$

It is easy to see that J_G is full. However, neither R nor R' are generally full, for the simple reason that if a rule can be applied with two unrelated (say disjoint) matchings in G, then the corresponding direct transformations may not subsume each other. Indeed, their contexts may be obtained by deleting disjoint parts of G, and then none can be obtained by further deletions from the other. Hence, not every morphism in $\mathscr{S}$ is reflected by a morphism in $\mathscr{D}|_G^{\mathscr{S}}$.

As already mentioned, we should still expect morphisms between rules to be reflected by morphisms between direct transformations that are obtained by applying these rules with overlapping matchings. This requires some clarity on what we mean by matchings and overlaps. To keep things simple, we stick to standard matchings as $\mathscr{C}$-morphisms (this is suitable for SPO but not for PBPO rules).

Definition 2.3 (overlap)**.** Given two matchings $m : L \to G$ and $m' : L' \to G$ in the same input object $G \in \mathscr{C}$, m is *overlapped* by m' if there is a $\mathscr{C}$-morphism $f : L \to L'$ such that $m = m' \circ f$ (and f is called an *overlap of m by m'*).

Note that $f : L \to L'$ means that the overlapping L' can be obtained from the overlapped L by additions/mergings, which corresponds to intuition (at least if we admit only monomorphisms). We also need a way of expressing the compatibility between overlaps and morphisms between rules. This can be done by assuming a (generally obvious) functor $\mathsf{L} : \mathscr{R} \to \mathscr{C}$ that yields the *left-hand side* of a rule. The condition for reflecting morphisms between rules can then be expressed as follows:

> for all direct transformations $\delta, \delta' \in \mathscr{D}|_G^{\mathscr{S}}$ and all $\mathscr{S}$-morphism $\sigma : \mathsf{R}'\delta \to \mathsf{R}'\delta'$, if the matchings used in δ and δ' (say m_δ and $m_{\delta'}$) overlap according to $\mathsf{L}\sigma$ (i.e., if $m_\delta = m_{\delta'} \circ \mathsf{L}\sigma$) then σ is reflected by a unique morphism between δ and δ', i.e., there exists a unique $\mathscr{D}|_G^{\mathscr{S}}$-morphism $\mu : \delta \to \delta'$ such that $\mathsf{R}'\mu = \sigma$. (1)

But we expect this condition to hold for any $\mathscr{S}$ and G. We can express this independently of $\mathscr{S}$ and G as follows:

> for all $\delta, \delta' \in \mathscr{D}$ and all $\mathscr{R}$-morphism $\sigma : \mathsf{R}\delta \to \mathsf{R}\delta'$, if $m_\delta = m_{\delta'} \circ \mathsf{L}\sigma$ then there exists a unique $\mathscr{D}$-morphism $\mu : \delta \to \delta'$ such that $\mathsf{R}\mu = \sigma$. (2)

that we call the *Correctness Condition*.

Proposition 2.4. *Condition (1) holds for all $\mathscr{S}$ and G iff the Correctness Condition (2) holds.*

Proof. ($\Rightarrow$) Let $\delta, \delta' \in \mathscr{D}$ and $\sigma : \mathsf{R}\delta \to \mathsf{R}\delta'$ s.t. $m_\delta = (\mathsf{L}\sigma) \circ m_{\delta'}$, since (1) holds for the subcategory $\mathscr{S}$ of $\mathscr{R}$ restricted to σ and for the common codomain G of m_δ and $m_{\delta'}$, and since $\delta, \delta' \in \mathscr{D}|_G^{\mathscr{S}}$, $\mathsf{R}'\delta = \mathsf{R}\delta$ and $\mathsf{R}'\delta' = \mathsf{R}\delta'$ then there exists a unique $\mu : \delta \to \delta'$ such that $\mathsf{R}'\mu = \mathsf{R}\mu = \sigma$, hence (2) holds.

($\Leftarrow$) Let $\delta, \delta' \in \mathscr{D}|_G^{\mathscr{S}}$ and $\sigma : \mathsf{R}'\delta \to \mathsf{R}'\delta'$ s.t. $m_\delta = (\mathsf{L}\sigma) \circ m_{\delta'}$, then $\mathsf{J}_G \mathsf{J}_{\mathscr{S}} \delta, \mathsf{J}_G \mathsf{J}_{\mathscr{S}} \delta' \in \mathscr{D}$ and $\mathsf{J}\,\sigma : \mathsf{R}\,\mathsf{J}_G \mathsf{J}_{\mathscr{S}} \delta \to \mathsf{R}\,\mathsf{J}_G \mathsf{J}_{\mathscr{S}} \delta'$ hence by (2) there exists a unique $\mu : \mathsf{J}_G \mathsf{J}_{\mathscr{S}} \delta \to \mathsf{J}_G \mathsf{J}_{\mathscr{S}} \delta'$ such that $\mathsf{R}\mu = \mathsf{J}\,\sigma$. Since J_G is fully faithful there is a unique $\mu' : \mathsf{J}_{\mathscr{S}} \delta \to \mathsf{J}_{\mathscr{S}} \delta'$ s.t. $\mathsf{J}_G \mu' = \mu$, hence s.t. $\mathsf{R}\,\mathsf{J}_G \mu' = \mathsf{J}\,\sigma$. Since $\mathsf{J}_{\mathscr{S}}$ is the inverse image of J along $\mathsf{R} \circ \mathsf{J}_G$ then μ' is a $\mathscr{D}|_G^{\mathscr{S}}$-morphism and $\mathsf{J}\,\mathsf{R}'\mu' = \mathsf{R}\,\mathsf{J}_G \mu' = \mathsf{J}\,\sigma$, hence $\mathsf{R}'\mu' = \sigma$ since J is faithful. $\square$

Note that there may be no direct transformation δ that corresponds to a rule and a matching (this is the case in the DPO approach when the gluing condition does not hold), and then the Correctness Condition does not require that rule morphisms be reflected. It may also be the case that a rule and a matching defines several (non isomorphic) direct transformations, and then a rule morphism should be reflected by one $\mathscr{D}$-morphism for each pair (δ, δ') corresponding to the two rules and their overlapping matchings.

3 Relevant Properties of Presheaves

In order to prove the Correctness Condition, and even simply to define SPO direct transformations, we need to prove a number of properties of categories of presheaves. A first set concerns the notion of direct image of morphisms between presheaves. A second set is about inverse images and their relations with direct images. A third set exhibits a lattice structure in categories of presheaves, and its connection with direct and inverse images.

We first briefly explain the use of presheaves.

Definition 3.1 (categories $\widehat{\mathscr{C}}$ and $\mathscr{I}$, order $\sqsubseteq$)**.** For any small category $\mathscr{C}$, the category of *presheaves on* $\mathscr{C}$, denoted $\widehat{\mathscr{C}}$, is the functor category $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$. Notations are as follows: for every $A \in \widehat{\mathscr{C}}$ and $f : c \to c'$ in $\mathscr{C}$, Af is a function from Ac' to Ac, and for every $h : A \to B$ in $\widehat{\mathscr{C}}$ the diagram

$$\begin{tikzcd} A\,c' \arrow[r, "A\,f"] \arrow[d, "h\,c'"'] & A\,c \arrow[d, "h\,c"] \\ B\,c' \arrow[r, "B\,f"'] & B\,c \end{tikzcd}$$

commutes in **Set** (h is a natural transformation).

An *inclusion* is a $\widehat{\mathscr{C}}$-morphism $i : A \to B$ such that $(i\,c)(x) = x$ for all $c \in \mathscr{C}$ and all $x \in A\,c$; these morphisms will be indicated by hooked arrows. Since identities are inclusions and the composition of two inclusions is an inclusion, there is a subcategory $\mathscr{I}$ of all inclusions of $\widehat{\mathscr{C}}$, and it is obviously a partial order. We write $A \sqsubseteq B$ if there exists an inclusion (necessarily unique) $i : A \hookrightarrow B$. When B can be deduced from the context, the inclusion i may be written A, so that the slice category $\mathscr{I} \setminus B$ can be identified to a (small) subcategory of $\mathscr{I}$.

The SPO approach in [9] has been developed in *graph structures*, i.e., categories of Σ-algebras for many-sorted signatures Σ with only unary operators. Such signatures can obviously be identified with graphs whose vertices are sorts and arrows are operators (whose source, resp. target, is the domain, resp. codomain, of the operator). Thus a Σ-algebra A is a function that maps sorts s to sets $A\,s$ and arrows $o : s \to s'$ to functions $A\,o : A\,s \to A\,s'$, very much like presheaves. Similarly, Σ-homomorphisms $h : A \to B$ are functions that map every sort s to a function $h\,s : A\,s \to B\,s$ such that $h\,s' \circ A\,o = B\,o \circ h\,s$ for every operator $o : s \to s'$, very much like a morphism of presheaves.

So if we let $\mathscr{C}$ be the (small) category freely generated by the graph Σ (i.e., $\mathscr{C}$ is the category whose morphisms f are the finite paths $o_1 \cdots o_n$ in Σ and composition is concatenation of paths, so that $f = o_n \circ \cdots \circ o_1$), we see that any Σ-algebra A can be trivially extended to a functor from $\mathscr{C}$ to **Set** (by $A\,f := A\,o_n \circ \cdots \circ A\,o_1$) and every such functor extends a Σ-algebra. Hence every Σ-homomorphism $h : A \to B$ is also a natural transformation between the corresponding functors since for every $f = o_n \circ \cdots \circ o_1 : s \to s'$ the diagram

$$\begin{tikzcd}
A\,s \arrow[r, "A\,o_1"'] \arrow[d, "h\,s"'] \arrow[rrrr, bend left, "A\,f"] & A\,s_1 \arrow[r, dash] \arrow[d, "h\,s_1"] & \cdots \arrow[r] & A\,s_{n-1} \arrow[r, "A\,o_n"'] \arrow[d, "h\,s_{n-1}"'] & A\,s' \arrow[d, "h\,s'"] \\
B\,s \arrow[r, "B\,o_1"] \arrow[rrrr, bend right, "B\,f"'] & B\,s_1 \arrow[r, dash] & \cdots \arrow[r] & B\,s_{n-1} \arrow[r, "B\,o_n"] & B\,s'
\end{tikzcd}$$

commutes (and conversely every natural transformation is a Σ-homomorphism). Hence $\mathbf{Set}^{\mathscr{C}}$ is isomorphic to the category of Σ-algebras.

Note that there is no other reason to use $\mathscr{C}^{\mathrm{op}}$ than to stick to the standard definition of presheaves. This way we dispense with many-sorted signatures and algebras and we will simply transpose the relevant definitions from [9] to more standard category theoretic notations.

3.1 Direct images

Definition 3.2 (Im h, functor h^+)**.** For any $\widehat{\mathscr{C}}$-morphism $h : A \to B$, let $\operatorname{Im} h$ be the presheaf defined by $(\operatorname{Im} h)\,c := \{h\,c(x) \mid x \in A\,c\}$ for all $c \in \mathscr{C}$. The functor $h^+ : \mathscr{I} \setminus A \to \mathscr{I} \setminus B$ is defined by $h^+ A' := \operatorname{Im}(h \circ A')$ for all $A' \sqsubseteq A$. Let $h \downharpoonleft A' : A' \to h^+ A'$ be the unique epimorphism such that

$$\begin{tikzcd}
A \arrow[r, "h"] & B \\
A' \arrow[u, hook] \arrow[r, two heads, "h \downharpoonleft A'"'] & h^+ A' \arrow[u, hook]
\end{tikzcd}$$

commutes.

We see that $(k \circ h)^+ = k^+ \circ h^+$ for all $k : B \to C$, and if $h \in \mathscr{I}$ then $h^+ A' = h \circ A'$ for all $A' \sqsubseteq A$.

Lemma 3.3. *For all $h : A \to B$, $A' \sqsubseteq A$ and $B' \sqsubseteq B$, we have $h^+ A' \sqsubseteq B'$ iff there exists h' such that*

$$\begin{tikzcd}
A \arrow[r, "h"] & B \\
A' \arrow[u, hook] \arrow[r, dashed, "h'"'] & B' \arrow[u, hook]
\end{tikzcd}$$

commutes. If h' exists it is unique.

Proof. If $i : h^+ A' \hookrightarrow B'$ then $h' = i \circ (h \downharpoonleft A')$. Conversely, if h' exists but $h^+ A' \not\sqsubseteq B'$ then there exists $c \in \mathscr{C}$ and $x \in A'\,c$ such that $h\,c(x) \notin B'\,c$, though $h\,c(x) = h'\,c(x) \in B'\,c$, a contradiction. $\square$

It is well known that $\widehat{\mathscr{C}}$ is complete and cocomplete, and that its limits and colimits can be computed objectwise, i.e., for any diagram $\mathsf{F} : \mathscr{J} \to \widehat{\mathscr{C}}$ we have $(\varprojlim \mathsf{F})c \simeq \varprojlim \mathsf{F}_c$ and $(\varinjlim \mathsf{F})c \simeq \varinjlim \mathsf{F}_c$ for all $c \in \mathscr{C}$, where $\mathsf{F}_c\,j := (\mathsf{F}\,j)c$ for all $\mathscr{J}$-morphism or object j.

3.2 Inverse images

In particular, we know that for any $h : A \to B$ in **Set** and any $B' \subseteq B$, the inclusion $A' := \{x \in A \mid h(x) \in B'\} \subseteq A$ together with the restriction of h to A' and B' is a pullback of h and the inclusion $B' \subseteq B$, hence the same holds in $\widehat{\mathscr{C}}$: not all pullbacks along an inclusion are inclusions, but at least one is. In fact, since $\mathscr{I}$ is closed under decomposition (if $i \circ h \in \mathscr{I}$ and $i \in \mathscr{I}$ then $h \in \mathscr{I}$) and the only isomorphisms in $\mathscr{I}$ are the identities, there is exactly one such pullback.

Definition 3.4 (functor h^-)**.** For any $\widehat{\mathscr{C}}$-morphism $h : A \to B$, the functor $h^- : \mathscr{I} \setminus B \to \mathscr{I} \setminus A$ is defined by, for all $B' \sqsubseteq B$, let $h^- B'$ be the unique $\widehat{\mathscr{C}}$-object and $h \restriction B'$ the unique $\widehat{\mathscr{C}}$-morphism such that there is a pullback square

$$\begin{tikzcd} A \arrow[r, "h"] & B \\ h^- B' \arrow[u, hook] \arrow[r, "h \restriction B'"'] \arrow[ur, phantom, "\urcorner", very near start] & B' \arrow[u, hook] \end{tikzcd}$$

By pullback composition we easily see that $(k \circ h)^- = h^- \circ k^-$. We also see that h^- is right adjoint to h^+.

Lemma 3.5. *For all $h : A \to B$, we have $h^+ \dashv h^-$.*

Proof. For all $A' \sqsubseteq A$ and $B' \sqsubseteq B$, we consider the diagram

$$\begin{tikzcd} & A \arrow[rr, "h"] & & B \\ & h^- B' \arrow[u, hook] \arrow[rr, "h \restriction B'"'] \arrow[urr, phantom, "\urcorner", very near start] & & B' \arrow[u, hook] \\ A' \arrow[uur, hook] \arrow[ur, dashed, hook, "u"] \arrow[rr, two heads, "h \lfloor A'"'] & & h^+ A' \arrow[uur, hook] \arrow[ur, dashed, hook, "i"'] & \end{tikzcd}$$

If $h^+ A' \sqsubseteq B'$ then i exists and the diagram (without u) commutes, hence by the pullback there exists a unique $u : A' \to h^- B'$ such that the whole diagram commutes, so that $u \in \mathscr{I}$, hence $A' \sqsubseteq h^- B'$. Conversely, if $A' \sqsubseteq h^- B'$ then $u \in \mathscr{I}$ exists hence $h \circ A' = B' \circ h \restriction B' \circ u$ and by Lemma 3.3 we get $h^+ A' \sqsubseteq B'$. □

There obviously follows that $A' \sqsubseteq h^-(h^+ A')$ and $h^+(h^- B') \sqsubseteq B'$.

The standard notion of inverse image (pullback of monomorphism) can be related to h^- in the following way.

Lemma 3.6. *If*

$$\begin{tikzcd} A \arrow[r, "h"] & B \\ A' \arrow[u, tail, "m"] \arrow[r, "h'"'] & B' \arrow[u, tail, "n"'] \end{tikzcd}$$

commutes, where m and n are monomorphisms, then this square is a pullback iff $h^- \operatorname{Im} n = \operatorname{Im} m$.

Proof. Let $i : \mathrm{Im}\, m \hookrightarrow A$ and $j : \mathrm{Im}\, n \hookrightarrow B$, since $i \circ m \lfloor A' = m$ and $j \circ n \lfloor B' = n$ then $m \lfloor A'$ and $n \lfloor B'$ are bimorphisms, and since $\widehat{\mathscr{C}}$ is balanced they are isomorphisms.

If $h^- \, \mathrm{Im}\, n = \mathrm{Im}\, m$ we have two pullback squares

$$\begin{tikzcd}
A \arrow[r, "h"] & B \\
\mathrm{Im}\, m \arrow[u, hook, "i"] \arrow[r, "h \upharpoonright \mathrm{Im}\, n"'] \arrow[ur, phantom, "\urcorner", very near start] & \mathrm{Im}\, n \arrow[u, hook, "j"'] \\
A' \arrow[u, two heads, tail, "m \lfloor A'"] \arrow[r, "h'"'] \arrow[ur, phantom, "\urcorner", very near start] & B' \arrow[u, two heads, tail, "n \lfloor B'"']
\end{tikzcd}$$

and we conclude by pullback composition.

Conversely, if the square is a pullback then

$$\begin{tikzcd}
 & A \arrow[rr, "h"] & & B \\
 & A' \arrow[u, tail, "m"] \arrow[rr, "h'"] \arrow[urr, phantom, "\urcorner", very near start] & & B' \arrow[u, tail, "n"'] \\
h^- \, \mathrm{Im}\, n \arrow[uur, hook] \arrow[ur, dashed, "u"] \arrow[rr, "h \upharpoonright B'"'] & & \mathrm{Im}\, n \arrow[uur, hook, "j"] \arrow[ur, tail, two heads, "(n \lfloor B')^{-1}"'] &
\end{tikzcd}$$

commutes (without u), hence there exists a unique $u : h^- \, \mathrm{Im}\, n \to A'$ such that the whole diagram commutes. Since the diagonal face is also a pullback, then so is the bottom face by pullback decomposition, hence u is an isomorphism. We see that $i \circ (m \lfloor A') \circ u \in \mathscr{I}$ hence $(m \lfloor A') \circ u \in \mathscr{I}$, and since this is an isomorphism it must be an identity, so that $h^- \, \mathrm{Im}\, n = \mathrm{Im}\, m$. □

3.3 The lattice structure of $\mathscr{I}$

Definition 3.7 (meet $\sqcap$, join $\sqcup$)**.** For any $A \in \widehat{\mathscr{C}}$, the partial order $\sqsubseteq$ in $\mathscr{I} \setminus A$ is a complete lattice, where the meet of a set $\mathscr{J}$ of objects is $\sqcap \mathscr{J}$ (with $(\sqcap \mathscr{J})c := \cap \{Jc \mid J \in \mathscr{J}\}$ for all $c \in \mathscr{C}$) and its join is $\sqcup \mathscr{J}$ (with $(\sqcup \mathscr{J})c := \cup \{Jc \mid J \in \mathscr{J}\}$ for all $c \in \mathscr{C}$). This lattice is compatible with direct and inverse images in the sense that for every $h : A \to B$:

- for all $A' \sqsubseteq A$ and $A'' \sqsubseteq A$, $h^+(A' \sqcup A'') = (h^+ A') \sqcup (h^+ A'')$,
- for all $B' \sqsubseteq B$ and $B'' \sqsubseteq B$, $h^-(B' \sqcup B'') = (h^- B') \sqcup (h^- B'')$,
- for all $B' \sqsubseteq B$, $h^+(h^- B') = B' \sqcap \mathrm{Im}\, h$.

These equations are well-known to hold in **Set** and it is therefore a trivial matter to show that they also hold in $\widehat{\mathscr{C}}$ (noting that $(h^- B')\, c = \{x \in A\, c \mid h\, c(x) \in B'\, c\}$ for all $c \in \mathscr{C}$).

We can use the lattice structure of $\mathscr{I}$ to prove that

Lemma 3.8. *If*

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "k'"] \\
C \arrow[r, "h'"'] & D \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout square in $\widehat{\mathscr{C}}$ *then* $D = \mathrm{Im}\, h' \sqcup \mathrm{Im}\, k'$.

Proof. Let $i : \mathrm{Im}\, h' \hookrightarrow D$ and $j : \mathrm{Im}\, k' \hookrightarrow D$, so that $h' = i \circ (h' \lfloor C)$ and $k' = j \circ (k' \lfloor B)$. Let $i' : \mathrm{Im}\, h' \hookrightarrow \mathrm{Im}\, h' \sqcup \mathrm{Im}\, k'$, $j' : \mathrm{Im}\, k' \hookrightarrow \mathrm{Im}\, h' \sqcup \mathrm{Im}\, k'$ and $e : \mathrm{Im}\, h' \sqcup \mathrm{Im}\, k' \hookrightarrow D$, so that $i = e \circ i'$ and $j = e \circ j'$.

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dd, "k"'] & & B \arrow[d, two heads, "k' \lfloor B"] & \\
 & & \mathrm{Im}\, k' \arrow[d, hook, "j"'] \arrow[ddr, hook, "j'"] & \\
C \arrow[r, two heads, "h' \lfloor C"'] & \mathrm{Im}\, h' \arrow[r, hook, "i"] \arrow[drr, hook, "i'"'] & D \arrow[dr, dashed, "u"] & \\
 & & & \mathrm{Im}\, h' \sqcup \mathrm{Im}\, k'
\end{tikzcd}$$

Since

$$e \circ i' \circ (h' \lfloor C) \circ k = h' \circ k = k' \circ h = e \circ j' \circ (k' \lfloor C) \circ h$$

and e is a monomorphism then $i' \circ (h' \lfloor C) \circ k = j' \circ (k' \lfloor B) \circ h$, hence there exists a unique $u : D \to \mathrm{Im}\, h' \sqcup \mathrm{Im}\, k'$ such that $i' \circ (h' \lfloor C) = u \circ h' = u \circ i \circ (h' \lfloor C)$ and $j' \circ (k' \lfloor B) = u \circ k' = u \circ j \circ (k' \lfloor B)$. Since $h' \lfloor C$ and $K' \lfloor B$ are epimorphisms then $i' = u \circ i$ and $j' = u \circ j$, so that $e \circ u \circ i = e \circ i' = i$ and $e \circ u \circ j = e \circ j' = j$, hence

$$\begin{cases} e \circ u \circ h' = e \circ u \circ i \circ (h' \lfloor C) = i \circ (h' \lfloor C) = h' \\ e \circ u \circ k' = e \circ u \circ j \circ (k' \lfloor B) = j \circ (k' \lfloor B) = k'. \end{cases}$$

Since (h', k') is an epi-sink then $e \circ u = 1_D \in \mathscr{I}$ hence $u \in \mathscr{I}$ and we obtain $D = \mathrm{Im}\, h' \sqcup \mathrm{Im}\, k'$. □

We will also need the following key lemma, for which we only give a purely set-theoretic proof.

Lemma 3.9. *If*

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "k'"] \\
C \arrow[r, "h'"'] & D \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout square in $\widehat{\mathscr{C}}$ *and* $J \sqsubseteq B$ *such that* $k'^{-}(k'^{+} J \sqcap \mathrm{Im}\, h) = J \sqcap \mathrm{Im}\, h$ *then* $k'^{-}(k'^{+} J) = J$.

Proof. We need only prove $k'^{-}(k'^{+} J) \sqsubseteq J$. Suppose this is not true, then there exists a $c \in \mathscr{C}$ and an $x \in J c$ such that $k' c^{-}(k' c^{+}(x)) \nsubseteq J c$. By hypothesis we must therefore have $x \notin \mathrm{Im}\, h c$. Besides, this also entails the existence of an $x' \in B c$ such that $k' c(x') = k' c(x)$ and $x' \notin J c$, i.e., $x \neq x'$. But we know that

$$\begin{tikzcd}
A\,c \arrow[r, "h\,c"] \arrow[d, "k\,c"'] & B\,c \arrow[d, "k'\,c"] \\
C\,c \arrow[r, "h'\,c"'] & D\,c \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout in **Set**, hence by the Gluing Condition (see, e.g., [6, Definition 3.9]) the identification points x and x' should be gluing points, i.e., elements of $\operatorname{Im} h\,c$, a contradiction. $\square$

4 SPO Rules and Transformations

4.1 Pushouts of partial morphisms

Definition 4.1 (category $\widetilde{\mathscr{C}}$)**.** The category of *partial presheaf morphisms* on $\mathscr{C}$, denoted $\widetilde{\mathscr{C}}$, has the same objects as $\widehat{\mathscr{C}}$, its morphisms are $\widehat{\mathscr{C}}$-spans $(i,h) : A \to B$ where $i : A' \hookrightarrow A$ and $h : A' \to B$. Their composition is defined by $(j,k) \circ (i,h) := (i \circ h^-\, j, k \circ (h \restriction j))$, and the identities are $(1_A, 1_A)$.

$$\begin{tikzcd}
 & & B & & \\
 & A' \arrow[ur, "h"] \arrow[dl, hook, "i"'] & & B' \arrow[ul, "j"'] \arrow[dr, "k"] & \\
A & & h^-\, j \arrow[ll, hook] \arrow[rr] \arrow[ul, hook] \arrow[ur, "h \restriction j"'] \arrow[u, phantom, "\wedge"] & & C
\end{tikzcd}$$

The pushouts in $\widetilde{\mathscr{C}}$ can be constructed in three steps as defined below.

Definition 4.2 (construction of K, R and D)**.** Let $(S,r) : L \to T$ and $(L',m) : L \to G$ be a span in $\widetilde{\mathscr{C}}$, and let

1. $K := \sqcup\{J \sqsubseteq S \sqcap L' \mid r^-(r^+ J) = J \text{ and } m^-(m^+ J) = J\}$,
2. $R := \sqcup\{J \sqsubseteq T \mid r^- J \sqsubseteq K\}$ and $D := \sqcup\{J \sqsubseteq G \mid m^- J \sqsubseteq K\}$,
3. $n : R \to H$, $g : D \to H$ a pushout of the span $r \restriction R : K \to R$, $m \restriction D : K \to D$ in $\widehat{\mathscr{C}}$.

$$\begin{tikzcd}
L & S \arrow[l, hook'] \arrow[r, "r"] & T \\
L' \arrow[u, hook] \arrow[d, "m"'] & K \arrow[l, hook'] \arrow[u, hook] \arrow[r, "r \restriction R"] \arrow[d, "m \restriction D"'] \arrow[ur, phantom, "\urcorner", very near start] \arrow[dl, phantom, "\llcorner", very near start] & R \arrow[u, hook] \arrow[d, "n"] \\
G & D \arrow[l, hook'] \arrow[r, "g"'] & H \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Note that $r^+ K \in \{J \sqsubseteq T \mid r^- J \sqsubseteq K\}$ by definition of K, hence $r^+ K \sqsubseteq R$ by definition of R, so that $K = r^-(r^+ K) \sqsubseteq r^- R \sqsubseteq K$ and hence $K = r^- R$, which explains why the upper right square is a pullback. Similarly $K = m^- D$, the lower left square is a pullback and therefore $(r \restriction R, m \restriction D)$ is a $\widehat{\mathscr{C}}$-span.

It is proven in [9, Section 2] (see also the proof in [1, Section 7]) that the cospan $(R,n) : T \to H$, $(D,g) : G \to H$ given in Definition 4.2 is a pushout of $(S,r) : L \to T$, $(L',m) : L \to G$ in $\widetilde{\mathscr{C}}$. This justifies the definition of direct SPO-transformations in Definition 4.3 below, though in the sequel we do not need to know that these are pushouts in $\widetilde{\mathscr{C}}$.

4.2 SPO transformations and subsumptions

In the SPO approach to algebraic rewriting, a rule is any $\widetilde{\mathscr{C}}$-morphism $(S,r) : L \to T$, a matching of this rule in a $\widetilde{\mathscr{C}}$-object G is a $\widetilde{\mathscr{C}}$-morphism $(1_L, m) : L \to G$ and a direct transformation is a pushout of the two. However, in order to define in a simple and intuitive way subsumption morphisms between SPO rules and transformations, they will be treated as diagrams in $\widehat{\mathscr{C}}$.

Definition 4.3 (SPO Rewriting Environment $\mathscr{R}_1 \xleftarrow{\mathsf{R}_1} \mathscr{D}_1 \xrightarrow{\mathsf{P}_1} (\widehat{\mathscr{C}})_{\mathrm{pt}}$)**.** An *SPO-rule* ρ in $\widehat{\mathscr{C}}$ is a span diagram

$$L \xhookleftarrow{\;i\;} S \xrightarrow{\;r\;} T$$

in $\widehat{\mathscr{C}}$, where $i \in \mathscr{I}$. (Diagrams are functors from an index category to $\widehat{\mathscr{C}}$, and it will sometimes be convenient to refer to the objects and morphisms of this index category; they will be denoted by the corresponding roman letters, here $\rho\mathrm{L} = L$, $\rho\mathrm{r} = r$, etc.)

A *subsumption morphism* $\sigma : \rho \to \rho'$ is a triple $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of $\widehat{\mathscr{C}}$-monomorphisms such that

$$\begin{tikzcd}
L \arrow[d, tail, "\sigma_1"'] & S \arrow[l, hook', "i"'] \arrow[r, "r"] \arrow[d, tail, "\sigma_2"] & T \arrow[d, tail, "\sigma_3"] \\
L' & S' \arrow[l, hook', "i'"] \arrow[r, "r'"'] & T'
\end{tikzcd}$$

(where $L' = \rho'\mathrm{L}$ etc.) commutes and both squares are pushouts. Composition is componentwise $\sigma' \circ \sigma := (\sigma'_1 \circ \sigma_1,\, \sigma'_2 \circ \sigma_2,\, \sigma'_3 \circ \sigma_3)$ and the obvious identities are $1_\rho := (1_L,\, 1_S,\, 1_T)$ (this is a subcategory of $\widehat{\mathscr{C}}^{\cdot\leftarrow\cdot\rightarrow\cdot}$). The category of SPO-rules and subsumption morphisms is denoted $\mathscr{R}_1$. The functor $\mathsf{L} : \mathscr{R}_1 \to \mathscr{C}$ is defined by $\mathsf{L}\rho := \rho\mathrm{L}$ and $\mathsf{L}\sigma := \sigma_1$.

A *direct SPO-transformation* δ is a diagram

$$\begin{tikzcd}
 & S \arrow[dl, hook', "i"'] \arrow[r, "r"] & T \\
L \arrow[d, "m"'] & K \arrow[l, hook'] \arrow[u, hook] \arrow[r, "r\restriction R"] \arrow[d, "m\restriction D"'] & R \arrow[u, hook] \arrow[d, "n"] \\
G & D \arrow[l, hook'] \arrow[r, "g"'] & H
\end{tikzcd}$$

in $\widehat{\mathscr{C}}$ such that K, R, D, n and g are defined as in Definition 4.2 (with $L' = L$).

A *subsumption morphism* $(\sigma, \nu) : \delta \to \delta'$ is a pair of triples of $\widehat{\mathscr{C}}$-monomorphisms such that

$$\begin{tikzcd}[column sep=small]
 & G \arrow[dl, dash, "="'] & & L \arrow[ll, "m"'] \arrow[dl, tail, "\sigma_1"'] & & S \arrow[ll, hook', "i"'] \arrow[rr, "r"] \arrow[dl, tail, "\sigma_2"'] & & T \arrow[dl, tail, "\sigma_3"'] \\
G' & & L' \arrow[ll, "m'"'] & & S' \arrow[ll, hook', "i'"] \arrow[rr, "r'"] & & T' & \\
 & D \arrow[uu] & & K \arrow[ll, "m\restriction D"] \arrow[uu] \arrow[rr, dash, "="] & & K \arrow[uu] \arrow[rr, "r\restriction R"] \arrow[dl, "\nu_2"] & & R \arrow[uu] \arrow[dl, "\nu_3"] \\
D' \arrow[uu] \arrow[ur, "\nu_1"] & & K' \arrow[ll, "m'\restriction D'"] \arrow[uu] \arrow[rr, dash, "="'] & & K' \arrow[uu] \arrow[rr, "r'\restriction R'"'] & & R' \arrow[uu] &
\end{tikzcd} \tag{3}$$

commutes, $\nu_1 \in \mathscr{I}$ and the two rightmost top squares are pushouts. Composition is componentwise $(\sigma', \nu') \circ (\sigma, \nu) := (\sigma' \circ \sigma,\, \nu' \circ \nu)$ where $\nu' \circ \nu := (\nu_1 \circ \nu_1',\, \nu_2' \circ \nu_2,\, \nu_3' \circ \nu_3)$ and the identities are obvious. The category of direct SPO-transformations and their subsumption morphisms is denoted $\mathscr{D}_1$.

Let R_1 be the obvious functor from $\mathscr{D}_1$ to $\mathscr{R}_1$, i.e., such that $(\mathsf{R}_1\,\delta)\mathsf{L} := \delta\mathsf{L}$ etc. and $\mathsf{R}_1(\sigma, \nu) := \sigma$. Let P_1 be the obvious functor from $\mathscr{D}_1$ to $(\widehat{\mathscr{C}})_{\mathrm{pt}}$, i.e., such that $(\mathsf{P}_1\,\delta)\mathsf{G} := \delta\mathsf{G}$ etc., and $\mathsf{P}_1(\sigma, \nu) := \nu$. The span of functors $\mathscr{R}_1 \xleftarrow{\mathsf{R}_1} \mathscr{D}_1 \xrightarrow{\mathsf{P}_1} (\widehat{\mathscr{C}})_{\mathrm{pt}}$ is the *SPO Rewriting Environment*.

We now prove the main result of this paper, namely that the SPO Rewriting Environment satisfies the Correctness Condition.

Theorem 4.4. *For all $\delta, \delta' \in \mathscr{D}_1$, all $\sigma : \mathsf{R}_1\,\delta \to \mathsf{R}_1\,\delta'$ such that $m = m' \circ \mathsf{L}\sigma$ (with m and m' from the diagrams δ and δ'), there exists a unique ν such that $(\sigma, \nu) : \delta \to \delta'$.*

Proof. To prove the existence and unicity of ν_2, by Lemma 3.3 we need only prove that $\sigma_2^+\,K \sqsubseteq K'$.

Let $J' := m'^-(m^+\,K)$, we first see that

$$J' \sqsubseteq m'^- \circ m'^+\,J' = m'^- \circ m'^+ \circ m'^- \circ m^+\,K \sqsubseteq m'^- \circ m^+\,K = J'$$

hence $m'^-(m'^+\,J') = J'$. Next, we have $J' \sqsubseteq L' = S' \sqcup \operatorname{Im}\sigma_1$ by Lemma 3.8. Since $m = m' \circ \sigma_1$ then $\sigma_1^-\,J' = m^-(m^+\,K) = K$ by construction of K (Definition 4.2), hence $\sigma_1^+\,K = \sigma_1^+(\sigma_1^-\,J') = J' \sqcap \operatorname{Im}\sigma_1$ and therefore

$$J' = J' \sqcap L' = J' \sqcap (S' \sqcup \operatorname{Im}\sigma_1) = (J' \sqcap S') \sqcup \sigma_1^+\,K.$$

But $\sigma_1^+\,K = i' \circ \sigma_2^+\,K \sqsubseteq i'$, hence $J' \sqsubseteq S'$. We also have $\operatorname{Im}\sigma_1 = i' \circ \operatorname{Im}\sigma_2$, hence by identifying now J' to its inclusion in S' we get $\sigma_2^+\,K = J' \sqcap \operatorname{Im}\sigma_2$. We now consider the rightmost back and top squares, i.e., the diagram

$$\begin{tikzcd}
K \arrow[r, hook] \arrow[d, "r\restriction R"'] & S \arrow[r, tail, "\sigma_2"] \arrow[d, "r"] & S' \arrow[d, "r'"] \\
R \arrow[r, hook] & T \arrow[r, tail, "\sigma_3"'] & T'
\end{tikzcd}$$

The left square is a pullback by construction of K and R. The right square is a pushout along a monomorphism, and since $\widehat{\mathscr{C}}$ is adhesive by [8, Corollary 3.6] then it is also a pullback by [8, Lemma 4.3]. By pullback composition we deduce that the outer square is a pullback, and by Lemma 3.6 that $r'^-(\sigma_3^+\,R) = \sigma_2^+\,K$. Hence

$$\sigma_2^+\,K \sqsubseteq r'^- \circ r'^+ \circ \sigma_2^+\,K = r'^- \circ \sigma_3^+ \circ r'^+\,K \sqsubseteq r'^- \circ \sigma_3^+\,R = \sigma_2^+\,K$$

since $r'^+\,K \sqsubseteq R$, hence $r'^-(r'^+\,J' \sqcap \operatorname{Im}\sigma_2) = J' \sqcap \operatorname{Im}\sigma_2$. We can therefore apply Lemma 3.9 to the right square, yielding $r'^-(r'^+\,J') = J'$.

By the properties of J' w.r.t. r' and m' and by the construction of K' we get $J' \sqsubseteq K'$, and since $\sigma_2^+\,K \sqsubseteq J'$ we are done with ν_2.

To prove the existence and unicity of ν_3 we need only prove that $\sigma_3^+\,R \sqsubseteq R'$. We have proved that $r'^-(\sigma_3^+\,R) = \sigma_2^+\,K \sqsubseteq K'$, hence by construction of R' we get $\sigma_3^+\,R \sqsubseteq R'$ and we are done with ν_3.

We now prove the existence of $\nu_1 \in \mathscr{I}$, i.e., that $D' \sqsubseteq D$. Let $J := \sigma_2^-\,K'$. It is obvious that $J \sqsubseteq S$. We next see that

$$\sigma_3^+ \circ r^+\,J = r'^+ \circ \sigma_2^+\,J = r'^+(\sigma_2^+ \circ \sigma_2^-\,K') \sqsubseteq r'^+\,K'$$

and hence by Lemma 3.5 we have $r^+ J \sqsubseteq \sigma_3^- \circ r'^+ K'$, so that

$$J \sqsubseteq r^-(r^+ J) \sqsubseteq r^- \circ \sigma_3^- \circ r'^+ K' = \sigma_2^- \circ r'^- \circ r'^+ K' = \sigma_2^- K' = J$$

since $r'^- \circ r'^+ K' = K'$ by construction of K'. Hence we have $r^-(r^+ J) = J$.

Let us consider the diagram

$$\begin{tikzcd}
J \arrow[r, hook] \arrow[d, tail, "\sigma_2 \upharpoonright K'"'] & S \arrow[d, tail, "\sigma_2"] & L \arrow[l, hook', "i"'] \arrow[d, tail, "\sigma_1"] \\
K' \arrow[r, hook] & S' & L' \arrow[l, hook', "i'"]
\end{tikzcd}$$

where the left square is a pullback by definition of J and the right one is a pulation square as above. Hence the outer square is a pullback, and by Lemma 3.6 we have $\sigma_1^- K' = J$. We deduce that $m^+ J = m'^+ \circ \sigma_1^+ \circ \sigma_1^- K' \sqsubseteq m'^+ K'$, hence

$$J \sqsubseteq m^- \circ m^+ J \sqsubseteq \sigma_1^- \circ m'^- \circ m'^+ K' = \sigma_1^- K' = J$$

since $m'^- \circ m'^+ K' = K'$ by construction of K', so that $m^-(m^+ J) = J$.

By the properties of J w.r.t. r and m and by the construction of K we therefore have $J \sqsubseteq K$. But then we have

$$m^- D' = \sigma_1^- \circ m'^- D' = \sigma_1^- K' = J \sqsubseteq K$$

since $m'^- D' = K'$ by construction of D'. We thus obtain $D' \sqsubseteq D$ by the construction of D.

There remains to establish the commuting properties. Let $j : D \hookrightarrow G$ and $j' : R' \hookrightarrow T'$, one easily get $j \circ \nu_1 \circ m' \upharpoonright D' \circ \nu_2 = j \circ m \upharpoonright D$ and $j' \circ r' \upharpoonright R' \nu_2 = j' \circ \nu_3 \circ r \upharpoonright R$ by diagram chasing, and since j and j' are monomorphisms then the bottom faces of (3) commute, so that $(\sigma, \nu) : \delta \to \delta'$. □

It is easy to find examples showing that this property fails if in Definition 4.3 of subsumption morphisms between SPO-rules, either square is not a pushout, or either of σ_1, σ_3 is not a monomorphism.

5 Comparison with DPO subsumptions

Definition 5.1. A *DPO-rule* ρ in $\widehat{\mathscr{C}}$ is a span diagram

$$L \xleftarrow{\;l\;} K \xrightarrow{\;r\;} R$$

in $\widehat{\mathscr{C}}$, where l is a monomorphism.

A *subsumption morphism* $\sigma : \rho \to \rho'$ is a triple $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of $\widehat{\mathscr{C}}$-monomorphisms such that

$$\begin{tikzcd}
L \arrow[d, "\sigma_1"'] & K \arrow[l, tail, "l"'] \arrow[r, "r"] \arrow[d, "\sigma_2"] & R \arrow[d, "\sigma_3"] \\
L' & K' \arrow[l, tail, "l'"] \arrow[r, "r'"'] & R'
\end{tikzcd}$$

(where $L' = \rho'$L etc.) commutes and the left square is a pullback. Composition is componentwise. The category of DPO-rules and subsumption morphisms is denoted $\mathscr{R}_2$.

A *direct DPO-transformation* δ in $\widehat{\mathscr{C}}$ is a diagram

$$\begin{tikzcd}
L \arrow[d, "m"'] & K \arrow[l, tail, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "n"] \\
G & D \arrow[l, hook, "f"] \arrow[r, "g"'] & H
\end{tikzcd}$$

in $\widehat{\mathscr{C}}$ such that l is a monomorphism and the two squares are pushouts.

A *subsumption morphism* $\mu : \delta \to \delta'$ is a 4-tuple $(\mu_1, \mu_2, \mu_3, \mu_4)$ of $\widehat{\mathscr{C}}$-morphisms such that the following diagram

$$\begin{tikzcd}
 & L \arrow[dl, "\mu_1"'] \arrow[dd, "m" near start] & & K \arrow[ll, tail, "l"'] \arrow[rr, "r"] \arrow[dl, "\mu_2"'] \arrow[dd, "k"] & & R \arrow[dl, "\mu_3"] \\
L' \arrow[dd, "m'"'] & & K' \arrow[ll, tail, "l'"'] \arrow[rr, "r'"] \arrow[dd, "k'"' near start] & & R' & \\
 & G \arrow[dl, dash, "="'] & & D \arrow[ll, hook] & & \\
G' & & D' \arrow[ll, hook, "f'"] \arrow[ur, "f"', "\mu_4"] & & &
\end{tikzcd} \tag{4}$$

commutes and the top left square is a pullback, with componentwise composition. Let R_2 be the obvious functor from $\mathscr{D}_2$ to $\mathscr{R}_2$ (such that $(\mathsf{R}_2\ \delta)\,\mathrm{L} = \delta\,\mathrm{L}$, etc.) and P_2 be the obvious functor from $\mathscr{D}_2$ to $\mathscr{C}_{\mathrm{pt}}$, the *DPO Rewriting Environment* is $\mathscr{R}_2 \xleftarrow{\mathsf{R}_2} \mathscr{D}_2 \xrightarrow{\mathsf{P}_2} \mathscr{C}_{\mathrm{pt}}$.

Compared to $\mathscr{D}_{\mathrm{DPO}}$ in [2, Definition 3.8] we assume in $\mathscr{D}_2$ that $f \in \mathscr{I}$ (and since $\mathscr{I}$ is stable by decomposition this entails that $\mu_4 \in \mathscr{I}$ too). The reason is that we work in adhesive categories $\widehat{\mathscr{C}}$ where we know that f is a monomorphism (since these are stable by pushouts [9, Lemma 4.2]), and among the isomorphic copies of D we can always pick one such that $D \sqsubseteq G$. Thus we only miss isomorphic copies of direct DPO-transformations, so that $\mathscr{D}_2$ is equivalent to $\mathscr{D}_{\mathrm{DPO}}$.

The DPO Rewriting Environment has been shown to satisfy the Correctness Condition in all adhesive categories, see [2, Proposition 6.4], hence it also holds in $\widehat{\mathscr{C}}$.

Since every inclusion is a monomorphism then SPO-rules can be seen as DPO-rules. Besides, in $\widehat{\mathscr{C}}$ every pushout along a monomorphism is a pullback (by [8]), hence $\mathscr{R}_1$-morphisms can also be seen as $\mathscr{R}_2$-morphisms.

Definition 5.2. Let J_1 be the obvious embedding of $\mathscr{R}_1$ in $\mathscr{R}_2$ (with $(\mathsf{J}_1\ \rho)\,\mathrm{K} = \rho\,\mathrm{S}$, $(\mathsf{J}_1\ \rho)\,\mathrm{R} = \rho\,\mathrm{T}$ etc.) and let $\mathsf{J}_2 : \mathscr{D}_2^{\hookrightarrow} \to \mathscr{D}_2$ be the inverse image of J_1 along R_2.

$\mathscr{D}_2^{\hookrightarrow}$ is the subcategory of $\mathscr{D}_2$ that contains all direct DPO-transformations of SPO rules, with morphisms corresponding to morphisms of SPO rules, i.e., in Diagram 4 we have $l, l' \in \mathscr{I}$, μ_1, μ_2, μ_3 are monomorphisms and the two top squares are pushouts.

We also know that $\mathscr{D}_2$-objects can be seen as $\mathscr{D}_1$-objects. We establish this correspondence solely with the results of Section 3, and extend it to subsumption morphisms through a full embedding.

Theorem 5.3. *There exists a unique full embedding* $\mathsf{J}_2^1 : \mathscr{D}_2^{\hookrightarrow} \rightarrowtail \mathscr{D}_1$ *such that*

$$
\begin{tikzcd}
\mathscr{R}_2 & \mathscr{D}_2 \arrow[l, "\mathsf{R}_2"'] \arrow[r, "\mathsf{P}_2"] & \mathscr{C}_{\mathrm{pt}} \arrow[dd, no head, "="] \\
\mathscr{R}_1 \arrow[u, tail, "\mathsf{J}_1"] \arrow[d, no head, "="'] & \mathscr{D}_2^{\hookrightarrow} \arrow[l, "\mathsf{R}_2{}'"] \arrow[u, tail, "\mathsf{J}_2"'] \arrow[d, tail, dashed, "\mathsf{J}_2^1"] & \\
\mathscr{R}_1 & \mathscr{D}_1 \arrow[l, "\mathsf{R}_1"] \arrow[r, "\mathsf{P}_1"'] & \mathscr{C}_{\mathrm{pt}}
\end{tikzcd}
\tag{5}
$$

commutes.

Proof. Let δ be a direct DPO-transformation as in Definition 5.1, the only possibility to define a direct SPO-transformation $\mathsf{J}_2^1\,\delta$ so that (5) commutes is as the diagram

$$
\begin{tikzcd}
 & K \arrow[dl, hook, "l"'] \arrow[r, "r"] \arrow[d, no head, "="] & R \arrow[d, no head, "="] \\
L \arrow[d, "m"'] \arrow[r, phantom, "\text{(a)}"] & K \arrow[l, hook'] \arrow[r, "r"] \arrow[d, "k"'] & R \arrow[d, "n"] \\
G & D \arrow[l, hook', "f"] \arrow[r, "g"'] & H
\end{tikzcd}
$$

This proves the unicity of J_2^1 on objects, if indeed it exists. We first prove that $\mathsf{J}_2^1\,\delta$ is a direct SPO-transformation. Let $K' := \bigsqcup\{J \sqsubseteq K \mid r^-(r^+ J) = J \text{ and } m^-(m^+ J) = J\}$, $R' := \bigsqcup\{J \sqsubseteq R \mid r^- J \sqsubseteq K'\}$ and $D' := \bigsqcup\{J \sqsubseteq G \mid m^- J \sqsubseteq K'\}$, we must prove that $K' = K$, $R' = R$, $D' = D$ and $k = m \restriction D$ (that $r = r \restriction R$ is obvious).

We see that $K' \sqsubseteq K$. Conversely, we have $r^-(r^+ K) = K$ since the domain of r is K. By Lemma 3.3 we have $m^+ K \sqsubseteq D$. Square (a) is a pushout along a monomorphism hence it is a pullback, hence by Lemma 3.6 we have $m^- D = m^- \operatorname{Im} f = \operatorname{Im} l = K$, so that $K \sqsubseteq m^-(m^+ K) \sqsubseteq m^- D = K$. This proves that $K \sqsubseteq K'$ and therefore that $K' = K$. Then we have $R' = \bigsqcup\{J \sqsubseteq R \mid r^- J \sqsubseteq K\} \sqsubseteq R$, and obviously $r^- R \sqsubseteq K$, hence $R' = R$.

We also have $D' = \bigsqcup\{J \sqsubseteq G \mid m^- J \sqsubseteq K\}$, hence we get $D \sqsubseteq D'$. Conversely, for any $J \sqsubseteq G$ such that $m^- J \sqsubseteq K$, we have $J \sqcap \operatorname{Im} m = m^+(m^- J) \sqsubseteq m^+ K \sqsubseteq D$ and $G = D \sqcup \operatorname{Im} m$ by Lemma 3.8, so that

$$J = J \sqcap G = J \sqcap (D \sqcup \operatorname{Im} m) = (J \sqcap D) \sqcup (J \sqcap \operatorname{Im} m) \sqsubseteq D,$$

hence $D' \sqsubseteq D$ and therefore $D = D'$. Finally, since (a) is the unique bullback of m and $f \in \mathscr{I}$ with $l \in \mathscr{I}$, then $K = m^- D$ and $k = m \restriction D$, and we have proved that $\mathsf{J}_2^1\,\delta$ is a direct SPO-transformation. Besides, it is obvious that J_2^1 is injective on objects.

Similarly, given a $\mathscr{D}_2^{\hookrightarrow}$-morphism $\mu : \delta \to \delta'$, by the required commutation of (5) we must define $\mathsf{J}_2^1\,\mu$ as $(\sigma, \nu) : \mathsf{J}_2^1\,\delta \to \mathsf{J}_2^1\,\delta'$ with $\sigma := (\mu_1, \mu_2, \mu_3)$ and $\nu := (\mu_4, \mu_2, \mu_3)$, and it is obvious that each of μ and $\mathsf{J}_2^1\,\mu$ is uniquely defined by the other, hence J_2^1 thus defined is a unique full embedding such that (5) commutes, if it is indeed a functor. But it is again obvious that $\mathsf{J}_2^1 1_\delta = 1_{\mathsf{J}_2^1\delta}$ and that, for any $\mu' : \delta' \to \delta''$,

$$\begin{aligned}
\mathsf{J}_2^1(\mu' \circ \mu) &= \mathsf{J}_2^1(\mu_1' \circ \mu_1, \mu_2' \circ \mu_2, \mu_3' \circ \mu_3, \mu_4 \circ \mu_4') \\
&= ((\mu_1' \circ \mu_1, \mu_2' \circ \mu_2, \mu_3' \circ \mu_3), (\mu_4 \circ \mu_4', \mu_2' \circ \mu_2, \mu_3' \circ \mu_3)) \\
&= ((\mu_1', \mu_2', \mu_3') \circ (\mu_1, \mu_2, \mu_3), (\mu_4', \mu_2', \mu_3') \circ (\mu_4, \mu_2, \mu_3)) \\
&= (\mathsf{J}_2^1 \mu') \circ (\mathsf{J}_2^1 \mu).
\end{aligned}$$

□

6 Conclusion

The Correctness Condition is much more difficult to obtain in the SPO approach than in the other algebraic approaches, but this is related to the elaborate construction of pushouts in the category $\widetilde{\mathscr{C}}$ of partial morphisms of presheaves. The definition of SPO-subsumptions is remarkably simple and requires only very basic algebraic constructions (monomorphisms and pushouts in $\widehat{\mathscr{C}}$), as is the case in the other approaches. Theorem 5.3 also yields a simple diagrammatic expression of the relationship between DPO and SPO direct transformations and the corresponding subsumptions.

The proof of Theorem 4.4 uses only well-known facts in Category Theory, some results on adhesive categories and the properties obtained in Section 3. It it were to be attempted in a different class of categories where the SPO approach can be developed, as in [7], the most challenging problem would probably be to prove Lemma 3.9, that relies heavily on the fact that morphisms in $\widetilde{\mathscr{C}}$ can be decomposed as functions.

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