Subsumptions of SPO Rules

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Abstract

In [2] a *Global Coherent Transformation* is defined, that applies simultaneously algebraic rewriting rules to an input object. The expressiveness of this transformation is enhanced by enabling the use of *subsumption morphisms* between rules, as in Global Transformations [8]. However, it is not committed to a particular approach to algebraic rewriting, and is therefore formalized in a general representation of such approaches, called a *Rewriting Environment*. It was shown that suitable environments exist for the Double Pushout, the Sesqui-Pushout and the Pullback-Pushout approaches. In the present paper an environment is exhibited for the Single-Pushout (SPO) approach in categories of presheaves, and it is shown that it enjoys the same property linking subsumptions between SPO rules and subsumptions between direct SPO transformations.

1 Introduction

An "approach" to algebraic rewriting is characterized by a particular notion of *rules* in a category $\mathcal{C}$, whose objects are those we wish to rewrite (generally graph-like data structures), together with a notion of *direct transformations* that defines how a rule transforms an input object G.

It is generally expected that such transformations proceed by local replacement of a part of G by something else, as in term rewriting. However, the Pullback-Pushout (PBPO) approach enables non local modifications of G. This replacement is usually decomposed as the deletion of a part of G followed by the addition of a new part. However, the deletion step may involve some duplications in the Sesqui-Pushout (SqPO) and the PBPO approaches.

All in all there is no general agreement on what is an algebraic rewrite rule and how it should be applied. A closer look reveals that the disagreement concerns the deletion/duplication step, while all approaches agree that the addition step is a pushout, though in the SPO approach both steps are performed by a pushout in a category of partial morphisms, see [6].

This diversity of approaches makes it difficult to design a general definition of algebraic rewriting, let alone of rule subsumption. Informally, a rule is subsumed by another rule if it performs fewer modifications to G; that is both fewer deletions/duplications and fewer additions. It is not based on a comparison of the *results* of applying both rules to G, and it cannot be: earning less and spending less can make one either poorer or richer in the end. More importantly, we will view subsumptions as (composable) morphisms between rules, and also between direct transformations. This will result in categories $\mathscr{R}$ of rules and $\mathscr{D}$ of direct transformations.

But for now $\mathscr{R}$ and $\mathscr{D}$ are only abstract categories, which leaves us free to understand, say, graph morphisms as subsumtions. We need more than this. One thing we know is that to every direct transformation corresponds a rule, and that to subsuming transformations must correspond subsuming rules, i.e., that there is a functor $\mathsf{R} : \mathscr{D} \to \mathscr{R}$. It is important to notice that, since subsuming rules can be applied at unrelated locations in G, the corresponding transformations may not be related by subsumption, hence that R is generally not full.

But the notion of subsumption can only be made precise by separating the deletion/duplication step from the pushout step. In [2] this is done by defining a *partial transformation* τ as a diagram

$$G \xleftarrow{\ f\ } D \xleftarrow{\ k\ } K \xrightarrow{\ r\ } R$$

in $\mathcal{C}$, where G is the *input* object, D is generally known as the *context*, K as the *interface* and R as the *right-hand side*. The transformation is partial in the sense that D is supposed to be obtained from G by applying the deletion/duplication step, and it is assumed that the result of the transformation is the pushout of the $\mathcal{C}$-span $D \overset{k}{\leftarrow} K \overset{r}{\rightarrow} R$, not yet performed.

We understand easily that if τ deletes less than a partial transformation τ', then the context D' of τ' should be smaller than D. Hence a subsumption morphism $\nu : \tau \to \tau'$ is a triple (ν_1, ν_2, ν_3) such that

$$\begin{array}{ccccccc}
G & \xleftarrow{\ f\ } & D & \xleftarrow{\ k\ } & K & \xrightarrow{\ r\ } & R \\
{\scriptstyle =}\Big\| & & \Big\uparrow{\scriptstyle \nu_1} & & {\scriptstyle \nu_2}\Big\downarrow & & \Big\downarrow{\scriptstyle \nu_3} \\
G' & \xleftarrow[\ f'\ ]{} & D' & \xleftarrow[\ k'\ ]{} & K' & \xrightarrow[\ r'\ ]{} & R'
\end{array}$$

commutes in $\mathcal{C}$. The obvious composition $(\nu'_1, \nu'_2, \nu'_3) \circ (\nu_1, \nu_2, \nu_3) := (\nu_1 \circ \nu'_1, \nu'_2 \circ \nu_2, \nu'_3 \circ \nu_3)$ yields the category $\mathcal{C}_{\mathrm{pt}}$.

Now we can admit that a direct transformation (an object of $\mathscr{D}$) is anything from which a partial transformation can be obtained, and similarly for their subsumptions. In other words, there should be a functor $\mathsf{P} : \mathscr{D} \to \mathcal{C}_{\mathrm{pt}}$. The span of functors $\mathscr{R} \overset{\mathsf{R}}{\leftarrow} \mathscr{D} \overset{\mathsf{P}}{\rightarrow} \mathcal{C}_{\mathrm{pt}}$ is what have been called a *Rewriting Environment* (RE) in [2]. This is an abstract representation of an approach to algebraic rewriting in $\mathcal{C}$, with subsumption morphisms between rules and between direct transformations.

However, this representation is too weak in one respect: it allows the category $\mathscr{D}$ to be discrete even if $\mathscr{R}$ is not. More precisely, it does not impose any constraints on the morphisms in $\mathscr{D}$ w.r.t. the morphisms in $\mathscr{R}$, so that subsumptions between rules may not be reflected as subsumptions between (suitable) direct transformations using these rules. This is why in [2] the REs defined for the Double-Pushout (DPO), the SqPO and the PBPO approaches all come with a property[1] that fills the gap between the corresponding $\mathscr{R}$ and $\mathscr{D}$, namely Propositions 6.4, 6.6 and 6.9.

The purpose of the present paper is therefore to design a RE for the SPO approach, with subsumption morphisms that support a property similar to those obtained in [2]. We slightly generalize the constructions of [6] from graph structures to categories of presheaves, which avoids any reference to Σ-algebras. The necessary tools are developed in Section 2.

One difficulty is that SPO transformations are designed in [6] as pushouts in a category of partial morphisms, and not as pushouts in $\mathcal{C}$ as we have assumed. However, these transformations can be understood as diagrams in $\mathcal{C}$, where the result is indeed obtained as a pushout of a $\mathcal{C}$-span. The difficulty lies in the construction of this span, given explicitly in Section 3 where it is used to obtain the required property on SPO subsumptions as Theorem 3.5.

[1] In a forthcoming paper it will be shown that this property is equivalent to a specific functor being fully faithful.

2 Relevant Properties of Presheaves

The category-theoretic notions and notations are compatible with [7]. For any category $\mathcal{C}$, we write $G \in \mathcal{C}$ to indicate that G is a $\mathcal{C}$-object, and $|\mathcal{C}|$ is the discrete category on $\mathcal{C}$-objects. Then G also denotes the functor from the terminal category $\mathbf{1}$ to $|\mathcal{C}|$ that maps the object of $\mathbf{1}$ to G. The *slice* category $\mathcal{C} \setminus G$ has as objects $\mathcal{C}$-morphisms of codomain G, and as morphisms $h : f \to g$ $\mathcal{C}$-morphisms such that $g \circ h = f$.

2.1 Direct images

Definition 2.1 (categories $\widehat{\mathcal{C}}$, $\mathcal{I}$, order $\sqsubseteq$, $\operatorname{Im} h$, functor $h^{\uparrow}$)**.** For any small category $\mathcal{C}$, the category of *presheaves on* $\mathcal{C}$, denoted $\widehat{\mathcal{C}}$, is the functor category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$. Notations are as follows: for every $A \in \widehat{\mathcal{C}}$ and $f : c \to c'$ in $\mathcal{C}$, Af is a function from Ac' to Ac, and for every $h : A \to B$ in $\widehat{\mathcal{C}}$ the diagram

$$\begin{CD} Ac' @>{hc'}>> Bc' \\ @V{Af}VV @VV{Bf}V \\ Ac @>>{hc}> Bc \end{CD}$$

commutes in $\mathbf{Set}$ (h is a natural transformation).

An *inclusion* is a $\widehat{\mathcal{C}}$-morphism $i : A \to B$ such that $(ic)(x) = x$ for all $c \in \mathcal{C}$ and all $x \in Ac$; these morphisms will be indicated by hooked arrows. Since identities are inclusions and the composition of two inclusions is an inclusion, there is a subcategory $\mathcal{I}$ of all inclusions of $\widehat{\mathcal{C}}$, and it is obviously a partial order. We write $A \sqsubseteq B$ if there exists an inclusion (necessarily unique) $i : A \hookrightarrow B$. When B can be deduced from the context, the inclusion i may be written A, so that the slice category $\mathcal{I} \setminus B$ can be identified to a (small) subcategory of $\mathcal{I}$.

For any $\widehat{\mathcal{C}}$-morphism $h : A \to B$, let $\operatorname{Im} h$ be the presheaf defined by $(\operatorname{Im} h)c := \{h(x) \mid x \in Ac\}$ for all $c \in \mathcal{C}$. The functor $h^{\uparrow} : \mathcal{I} \setminus A \to \mathcal{I} \setminus B$ is defined by $h^{\uparrow} A' := \operatorname{Im}(h \circ A')$ for all $A' \sqsubseteq A$. Let $h \downharpoonleft A' : A' \to h^{\uparrow} A'$ be the unique epimorphism such that

$$\begin{CD} A @>{h}>> B \\ @AAA @AAA \\ A' @>>{h \downharpoonleft A'}> h^{\uparrow} A' \end{CD}$$

commutes.

We see that, if $h \in \mathcal{I}$ then $h^{\uparrow} A' = h \circ A'$ for all $A' \sqsubseteq A$. Besides, for all $k : B \to C$ we have $(k \circ h)^{\uparrow} = k^{\uparrow} \circ h^{\uparrow}$.

Lemma 2.2. *For all $h : A \to B$, $A' \sqsubseteq A$ and $B' \sqsubseteq B$, we have $h^{\uparrow} A' \sqsubseteq B'$ iff there is a unique h' such that*

$$\begin{CD} A @>{h}>> B \\ @AAA @AAA \\ A' @>>{h'}> B' \end{CD}$$

commutes.

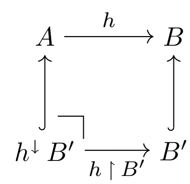
Proof. If $i : h^{\uparrow} A' \hookrightarrow B'$ then $h' = i \circ (h \downharpoonleft A')$. Conversely, if h' exists but $h^{\uparrow} A' \not\sqsubseteq B'$ then there exists $c \in \mathcal{C}$ and $x \in A'c$ such that $hc(x) \notin B'c$, though $hc(c) = h'c(x) \in B'c$, a contradiction. □

It is well known that $\widehat{\mathcal{C}}$ is complete and cocomplete, and that its limits and colimits can be computed objectwise, i.e., for any diagram $\mathsf{F} : \mathcal{J} \to \widehat{\mathcal{C}}$ we have $(\varprojlim \mathsf{F})c \simeq \varprojlim \mathsf{F}_c$ and $(\varinjlim \mathsf{F})c \simeq \varinjlim \mathsf{F}_c$ for all $c \in \mathcal{C}$, where $\mathsf{F}_c j := (\mathsf{F}j)c$ for all $\mathcal{J}$-morphism or object j.

2.2 Inverse images

In particular, we know that for any $h : A \to B$ in **Set** and any $B' \subseteq B$, the inclusion $A' := \{x \in A \mid h(x) \in B'\} \subseteq A$ together with the restriction of h to A' and B' is a pullback of h and the inclusion $B' \subseteq B$, hence the same holds in $\widehat{\mathcal{C}}$: not all pullbacks along an inclusion are inclusions, but at least one is. In fact, since $\mathcal{I}$ is closed under decomposition (if $i \circ h \in \mathcal{I}$ and $i \in \mathcal{I}$ then $h \in \mathcal{I}$) and the only isomorphisms in $\mathcal{I}$ are the identities, there is exactly one such pullback.

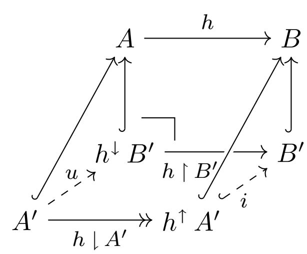
Definition 2.3 (functor $h^{\downarrow}$)**.** For any $\widehat{\mathcal{C}}$-morphism $h : A \to B$, the functor $h^{\downarrow} : \mathcal{I} \setminus B \to \mathcal{I} \setminus A$ is defined by, for all $B' \sqsubseteq B$, let $h^{\downarrow} B'$ be the unique $\widehat{\mathcal{C}}$-object and $h \upharpoonright B'$ the unique $\widehat{\mathcal{C}}$-morphism such that there is a pullback square



By pullback composition we easily see that $(k \circ h)^{\downarrow} = h^{\downarrow} \circ k^{\downarrow}$. We also see that $h^{\downarrow}$ is right adjoint to $h^{\uparrow}$.

Lemma 2.4. *For all $h : a \to B$, we have $h^{\uparrow} \dashv h^{\downarrow}$.*

Proof. For all $A' \sqsubseteq A$ and $B' \sqsubseteq B$, we consider the diagram

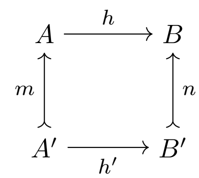


If $h^{\uparrow} A' \sqsubseteq B'$ then i exists and the diagram (without u) commutes, hence by the pullback there exists a unique $u : A' \to h^{\downarrow} B'$ such that the whole diagram commutes, so that $u \in \mathcal{I}$, hence $A' \sqsubseteq h^{\downarrow} B'$. Conversely, if $A' \sqsubseteq h^{\downarrow} B'$ then $u \in \mathcal{I}$ exists and the diagram (without i) commutes, hence by Lemma 2.2 we have $h^{\uparrow} A' \sqsubseteq B'$. □

There obviously follows that $A' \sqsubseteq h^{\downarrow}(h^{\uparrow} A')$ and $h^{\uparrow}(h^{\downarrow} B') \sqsubseteq B'$.

The standard notion of inverse image (pullback of monomorphism) can be related to $h^{\downarrow}$ in the following way.

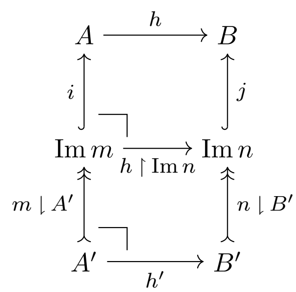
Lemma 2.5. *If*



commutes, where m *and* n *are monomorphisms, then this square is a pullback iff* $h^{\downarrow}\operatorname{Im} n = \operatorname{Im} m$.

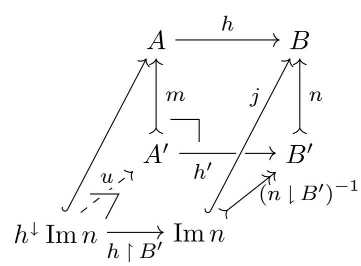
Proof. Let $i : \operatorname{Im} m \hookrightarrow A$ and $j : \operatorname{Im} n \hookrightarrow B$, since $i \circ m \downharpoonleft A' = m$ and $j \circ n \downharpoonleft B' = n$ then $m \downharpoonleft A'$ and $n \downharpoonleft B'$ are bimorphisms, and since $\widehat{\mathcal{C}}$ is balanced they are isomorphisms.

If $h^{\downarrow}\operatorname{Im} n = \operatorname{Im} m$ we have two pullback squares



and we conclude by pullback composition.

Conversely, if the square is a pullback then



commutes (without u), hence there exists a unique $u : h^{\downarrow}\operatorname{Im} n \to A'$ such that the whole diagram commutes. Since the diagonal face is also a pullback, then so is the bottom face by pullback decomposition, hence u is an isomorphism. We see that $i \circ (m \downharpoonleft A') \circ u \in \mathcal{I}$ hence $(m \downharpoonleft A') \circ u \in \mathcal{I}$, and since this is an isomorphism it must be an identity, so that $h^{\downarrow}\operatorname{Im} n = \operatorname{Im} m$. □

2.3 The lattice structure of $\mathcal{I}$

Definition 2.6 (meet $\sqcap$, join $\sqcup$)**.** For any $A \in \widehat{\mathcal{C}}$, the partial order $\sqsubseteq$ in $\mathcal{I} \setminus A$ is a complete lattice, where the meet of a set $\mathcal{J}$ of objects is $\sqcap\mathcal{J}$ (with $(\sqcap\mathcal{J})c := \cap\{Jc \mid J \in \mathcal{J}\}$ for all $c \in \mathcal{C}$) and its join is $\sqcup\mathcal{J}$ (with $(\sqcup\mathcal{J})c := \cup\{Jc \mid J \in \mathcal{J}\}$ for all $c \in \mathcal{C}$).

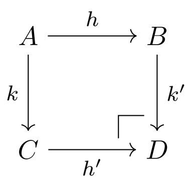
We leave to the reader the proof of the following properties for every $h : A \to B$:

- for all $A' \sqsubseteq A$ and $A'' \sqsubseteq A$, $h^{\uparrow}(A' \sqcup A'') = (h^{\uparrow} A') \sqcup (h^{\uparrow} A'')$.

- for all $B' \sqsubseteq B$ and $B'' \sqsubseteq B$, $h^{\downarrow}(B' \sqcup B'') = (h^{\downarrow} B') \sqcup (h^{\downarrow} B'')$ and $h^{\uparrow}(h^{\downarrow} B') = B' \sqcap \operatorname{Im} h$.

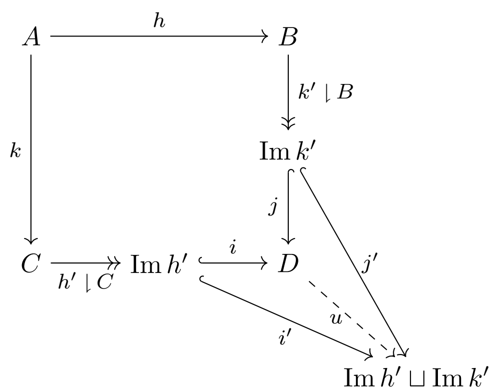
We can use the lattice structure of $\mathcal{I}$ to prove that

Lemma 2.7. *If*



is a pushout square in $\widehat{\mathcal{C}}$ *then* $D = \operatorname{Im} h' \sqcup \operatorname{Im} k'$.

Proof. Let $i : \operatorname{Im} h' \hookrightarrow D$ and $j : \operatorname{Im} k' \hookrightarrow D$, so that $h' = i \circ (h' \downharpoonleft C)$ and $k' = j \circ (k' \downharpoonleft B)$. Let $i' : \operatorname{Im} h' \hookrightarrow \operatorname{Im} h' \sqcup \operatorname{Im} k'$, $j' : \operatorname{Im} k' \hookrightarrow \operatorname{Im} h' \sqcup \operatorname{Im} k'$ and $e : \operatorname{Im} h' \sqcup \operatorname{Im} k' \hookrightarrow D$, so that $i = e \circ i'$ and $j = e \circ j'$.



Since

$$e \circ i' \circ (h' \downharpoonleft C) \circ k = h' \circ k = k' \circ h = e \circ j' \circ (k' \downharpoonleft C) \circ h$$

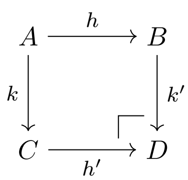
and e is a monomorphism then $i' \circ (h' \downharpoonleft C) \circ k = j' \circ (k' \downharpoonleft B) \circ h$, hence there exists a unique $u : D \to \operatorname{Im} h' \sqcup \operatorname{Im} k'$ such that $i' \circ (h' \downharpoonleft C) = u \circ h' = u \circ i \circ (h' \downharpoonleft C)$ and $j' \circ (k' \downharpoonleft B) = u \circ k' = u \circ j \circ (k' \downharpoonleft B)$. Since $h' \downharpoonleft C$ and $K' \downharpoonleft B$ are epimorphisms then $i' = u \circ i$ and $j' = u \circ j$, so that $e \circ u \circ i = e \circ i' = i$ and $e \circ u \circ j = e \circ j' = j$, hence

$$\begin{cases} e \circ u \circ h' = e \circ u \circ i \circ (h' \downharpoonleft C) = i \circ (h' \downharpoonleft C) = h' \\ e \circ u \circ k' = e \circ u \circ j \circ (k' \downharpoonleft B) = j \circ (k' \downharpoonleft B) = k'. \end{cases}$$

Since (h', k') is an epi-sink then $e \circ u = 1_D \in \mathcal{I}$ hence $u \in \mathcal{I}$ and we obtain $D = \operatorname{Im} h' \sqcup \operatorname{Im} k'$. ☐

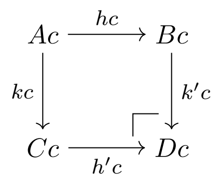
We will also need the following key lemma, for which we only give a purely set-theoretic proof.

Lemma 2.8. *If*



is a pushout square in $\widehat{\mathcal{C}}$ *and* $J \sqsubseteq B$ *such that* $k'^{\downarrow}(k'^{\uparrow} J \sqcap \operatorname{Im} h) = J \sqcap \operatorname{Im} h$ *then* $k'^{\downarrow}(k'^{\uparrow} J) = J$.

Proof. We need only prove $k'^{\downarrow}(k'^{\uparrow} J) \sqsubseteq J$. Suppose this is not true, then there exists a $c \in \mathcal{C}$ and an $x \in Jc$ such that $k'c^{\downarrow}(k'c^{\uparrow}(x)) \not\subseteq Jc$. By hypothesis we must therefore have $x \notin \operatorname{Im} hc$. Besides, this also entails the existence of an $x' \in Bc$ such that $k'c(x') = k'c(x)$ and $x' \notin Jc$, i.e., $x \neq x'$. But we know that

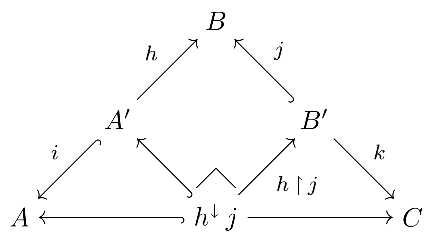


is a pushout in **Set**, hence by the Gluing Condition (see, e.g., [3, Definition 3.9]) the identification points x and x' should be gluing points, i.e., elements of $\operatorname{Im} hc$, a contradiction. □

3 SPO Rules and Transformations

3.1 Pushouts of partial morphisms

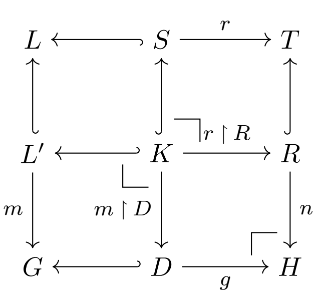
Definition 3.1 (category $\widetilde{\mathcal{C}}$)**.** The category of *partial presheaf morphisms* on $\mathcal{C}$, denoted $\widetilde{\mathcal{C}}$, has the same objects as $\widehat{\mathcal{C}}$, its morphisms are $\widehat{\mathcal{C}}$-spans $(i, h) : A \to B$ where $i : A' \hookrightarrow A$ and $h : A' \to B$. Their composition is defined by $(j, k) \circ (i, h) := (i \circ h^{\downarrow} j, k \circ (h \upharpoonright j))$, and the identities are $(1_A, 1_A)$.



The pushouts in $\widetilde{\mathcal{C}}$ can be constructed in three steps as defined below.

Definition 3.2 (construction of K, R and D)**.** Let $(S, r) : L \to T$ and $(L', m) : L \to G$ be a span in $\widetilde{\mathcal{C}}$, and let

1. $K := \sqcup\{J \sqsubseteq S \sqcap L' \mid r^{\downarrow}(r^{\uparrow} J) = J \text{ and } m^{\downarrow}(m^{\uparrow} J) = J\}$,
2. $R := \sqcup\{J \sqsubseteq T \mid r^{\downarrow} J \sqsubseteq K\}$ and $D := \sqcup\{J \sqsubseteq G \mid m^{\downarrow} J \sqsubseteq K\}$,
3. $n : R \to H$, $g : D \to H$ a pushout of the span $r \upharpoonright R : K \to R$, $m \upharpoonright D : K \to D$ in $\widehat{\mathcal{C}}$.



Note that $r^{\uparrow} K \in \{J \sqsubseteq T \mid r^{\downarrow} J \sqsubseteq K\}$ by definition of K, hence $r^{\uparrow} K \sqsubseteq R$ by definition of R, so that $K = r^{\downarrow}(r^{\uparrow} K) \sqsubseteq r^{\downarrow} R \sqsubseteq K$ and hence $K = r^{\downarrow} R$, which explains why the upper right square is a pullback. Similarly $K = m^{\downarrow} D$, the lower left square is a pullback and therefore $(r \restriction R,\, m \restriction D)$ is a $\widehat{\mathcal{C}}$-span.

Theorem 3.3. *In Definition 3.2 the cospan $(R, n) : T \to H$, $(D, g) : G \to H$ is a pushout of $(S, r) : L \to T$, $(L', m) : L \to G$ in $\widetilde{\mathcal{C}}$.*

The proof is similar to the proof in [6, Section 2] or in [1, Section 7]. Note that the categories in [6] are isomorphic to categories of presheaves. Indeed, a many-sorted signature Σ with unary operator symbols only (a *graph structure*) can be identified to a graph, where the sorts are vertices and the unary operators are edges. If we let $\mathcal{C}$ be the category freely generated by this graph (the category whose morphisms are the paths in this graph), then is is easy to see that $\widehat{\mathcal{C}^{\mathrm{op}}}$ is isomorphic to the category of Σ-algebras.

3.2 SPO transformations and subsumptions

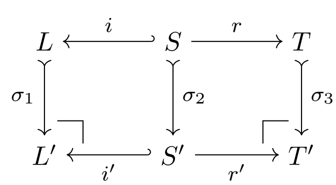
In the SPO approach to algebraic rewriting, a rule is any $\widetilde{\mathcal{C}}$-morphism $(S, r) : L \to T$, a matching of this rule in a $\widetilde{\mathcal{C}}$-object G is a $\widetilde{\mathcal{C}}$-morphism $(1_L, m) : L \to G$ and a direct transformation is a pushout of the two. However, in order to define in a simple and intuitive way subsumption morphisms between SPO rules and transformations, they will be treated as diagrams in $\widehat{\mathcal{C}}$.

Definition 3.4 (RE $\mathscr{R}_{\mathrm{SPO}} \xleftarrow{\mathsf{R}_{\mathrm{SPO}}} \mathscr{D}_{\mathrm{SPO}} \xrightarrow{\mathsf{P}_{\mathrm{SPO}}} (\widehat{\mathcal{C}})_{\mathrm{pt}}$)**.** An *SPO-rule* ρ in $\widehat{\mathcal{C}}$ is a span diagram

$$L \xhookleftarrow{\;i\;} S \xrightarrow{\;r\;} T$$

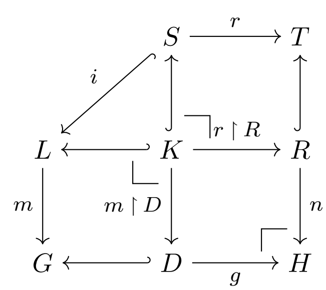
in $\widehat{\mathcal{C}}$, where $i \in \mathcal{I}$. (Diagrams are functors from an index category to $\widehat{\mathcal{C}}$, and it will sometimes be convenient to refer to the objects and morphisms of this index category; they will be denoted by the corresponding roman letters, here $\rho\mathrm{L} = L$, $\rho\mathrm{r} = r$, etc.)

A *subsumption morphism* $\sigma : \rho \to \rho'$ is a triple $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of $\widehat{\mathcal{C}}$-monomorphisms such that



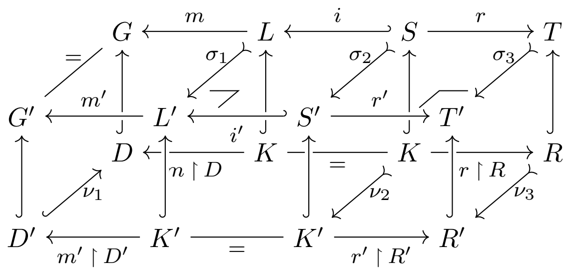
(where $L' = \rho'\mathrm{L}$ etc.) commutes and both squares are pushouts. Composition is componentwise $\sigma' \circ \sigma := (\sigma'_1 \circ \sigma_1,\, \sigma'_2 \circ \sigma_2,\, \sigma'_3 \circ \sigma_3)$ and the obvious identities are $1_\rho := (1_L,\, 1_S,\, 1_T)$ (this is a subcategory of $\widehat{\mathcal{C}}^{\cdot\leftarrow\cdot\rightarrow\cdot}$). The category of SPO-rules and subsumption morphisms is denoted $\mathscr{R}_{\mathrm{SPO}}$.

A *direct SPO-transformation* δ is a diagram



in $\widehat{\mathcal{C}}$ such that K, R, D, n and g are defined as in Definition 3.2 (with $L' = L$).

A *subsumption morphism* $(\sigma, \nu) : \delta \to \delta'$ is a pair of triples of $\widehat{\mathcal{C}}$-monomorphisms such that



(1)

commutes, $\nu_1 \in \mathcal{I}$ and the two rightmost top squares are pushouts. Composition is componentwise $(\sigma', \nu') \circ (\sigma, \nu) := (\sigma' \circ \sigma,\ \nu' \circ \nu)$ where $\nu' \circ \nu := (\nu_1 \circ \nu_1',\ \nu_2' \circ \nu_2,\ \nu_3' \circ \nu_3)$ and the identities are obvious. The category of direct SPO-transformations and their subsumption morphisms is denoted $\mathscr{D}_{\mathrm{SPO}}$.

Let $\mathsf{R}_{\mathrm{SPO}}$ be the obvious functor from $\mathscr{D}_{\mathrm{SPO}}$ to $\mathscr{R}_{\mathrm{SPO}}$, i.e., such that $(\mathsf{R}_{\mathrm{SPO}}\,\delta)\mathrm{L} := \delta\mathrm{L}$ etc. and $\mathsf{R}_{\mathrm{SPO}}(\sigma, \nu) := \sigma$. Let $\mathsf{P}_{\mathrm{SPO}}$ be the obvious functor from $\mathscr{D}_{\mathrm{SPO}}$ to $(\widehat{\mathcal{C}})_{\mathrm{pt}}$, i.e., such that $(\mathsf{P}_{\mathrm{SPO}}\,\delta)\mathrm{G} := \delta\mathrm{G}$ etc., and $\mathsf{P}_{\mathrm{SPO}}(\sigma, \nu) := \nu$. The span of functors $\mathscr{R}_{\mathrm{SPO}} \xleftarrow{\mathsf{R}_{\mathrm{SPO}}} \mathscr{D}_{\mathrm{SPO}} \xrightarrow{\mathsf{P}_{\mathrm{SPO}}} (\widehat{\mathcal{C}})_{\mathrm{pt}}$ is the *SPO Rewriting Environment*.

We now prove the main result of this paper, a property that is a simple translation to the SPO RE, and restriction to categories of presheaves, of [2, Proposition 6.4].

Theorem 3.5. *For all $\delta, \delta' \in \mathscr{D}_{\mathrm{SPO}}$, all $\sigma : \mathsf{R}_{\mathrm{SPO}}\,\delta \to \mathsf{R}_{\mathrm{SPO}}\,\delta'$ such that $m = \sigma_1 \circ m'$ (with m and m' from the diagrams δ and δ'), there exists a unique ν such that $(\sigma, \nu) : \delta \to \delta'$.*

Proof. To prove the existence and unicity of ν_2, by Lemma 2.2 we need only prove that $\sigma_2^{\uparrow} K \sqsubseteq K'$.

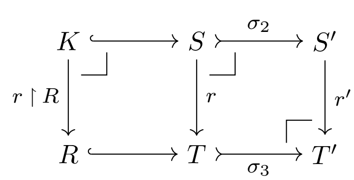
Let $J' := m'^{\downarrow}(m^{\uparrow} K)$, we first see that

$$J' \sqsubseteq m'^{\downarrow} \circ m'^{\uparrow} J' = m'^{\downarrow} \circ m'^{\uparrow} \circ m'^{\downarrow} \circ m^{\uparrow} K \sqsubseteq m'^{\downarrow} \circ m^{\uparrow} K = J'$$

hence $m'^{\downarrow}(m'^{\uparrow} J') = J'$. Next, we have $J' \sqsubseteq L' = S' \sqcup \operatorname{Im} \sigma_1$ by Lemma 2.7. Since $m = m' \circ \sigma_1$ then $\sigma_1^{\downarrow} J' = m^{\downarrow}(m^{\uparrow} K) = K$ by construction of K (Definition 3.2), hence $\sigma_1^{\uparrow} K = \sigma_1^{\uparrow}(\sigma_1^{\downarrow} J') = J' \sqcap \operatorname{Im} \sigma_1$ and therefore

$$J' = J' \sqcap L' = J' \sqcap (S' \sqcup \operatorname{Im} \sigma_1) = (J' \sqcap S') \sqcup \sigma_1^{\uparrow} K.$$

But $\sigma_1^{\uparrow} K = i' \circ \sigma_2^{\uparrow} K \sqsubseteq i'$, hence $J' \sqsubseteq S'$. We also have $\operatorname{Im} \sigma_1 = i' \circ \operatorname{Im} \sigma_2$, hence by identifying now J' to its inclusion in S' we get $\sigma_2^{\uparrow} K = J' \sqcap \operatorname{Im} \sigma_2$. We now consider the rightmost back and top squares, i.e., the diagram



The left square is a pullback by construction of K and R. The right square is a pushout along a monomorphism, and since $\widehat{\mathcal{C}}$ is adhesive by [5, Corollary 3.6] then it is also a pullback by [5, Lemma

4.3]. By pullback composition we deduce that the outer square is a pullback, and by Lemma 2.5 that $r'^{\downarrow}(\sigma_3^{\uparrow} R) = \sigma_2^{\uparrow} K$. Hence

$$\sigma_2^{\uparrow} K \sqsubseteq r'^{\downarrow} \circ r'^{\uparrow} \circ \sigma_2^{\uparrow} K = r'^{\downarrow} \circ \sigma_3^{\uparrow} \circ r'^{\uparrow} K \sqsubseteq r'^{\downarrow} \circ \sigma_3^{\uparrow} R = \sigma_2^{\uparrow} K$$

since $r'^{\uparrow} K \sqsubseteq R$, hence $r'^{\downarrow}(r'^{\uparrow} J' \sqcap \operatorname{Im} \sigma_2) = J' \sqcap \operatorname{Im} \sigma_2$. We can therefore apply Lemma 2.8 to the right square, yielding $r'^{\downarrow}(r'^{\uparrow} J') = J'$.

By the properties of J' w.r.t. r' and m' and by the construction of K' we get $J' \sqsubseteq K'$, and since $\sigma_2^{\uparrow} K \sqsubseteq J'$ we are done with ν_2.

To prove the existence and unicity of ν_3 we need only prove that $\sigma_3^{\uparrow} R \sqsubseteq R'$. We have proved that $r'^{\downarrow}(\sigma_3^{\uparrow} R) = \sigma_2^{\uparrow} K \sqsubseteq K'$, hence by construction of R' we get $\sigma_3^{\uparrow} R \sqsubseteq R'$ and we are done with ν_3.

We now prove the existence of $\nu_1 \in \mathcal{I}$, i.e., that $D' \sqsubseteq D$. Let $J := \sigma_2^{\downarrow} K'$. It is obvious that $J \sqsubseteq S$. We next see that

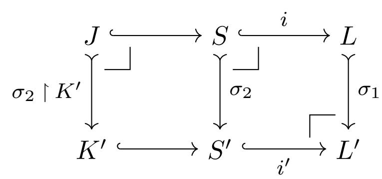
$$\sigma_3^{\uparrow} \circ r^{\uparrow} J = r'^{\uparrow} \circ \sigma_2^{\uparrow} J = r'^{\uparrow}(\sigma_2^{\uparrow} \circ \sigma_2^{\downarrow} K') \sqsubseteq r'^{\uparrow} K'$$

and hence by Lemma 2.4 we have $r^{\uparrow} J \sqsubseteq \sigma_3^{\downarrow} \circ r'^{\uparrow} K'$, so that

$$J \sqsubseteq r^{\downarrow}(r^{\uparrow} J) \sqsubseteq r^{\downarrow} \circ \sigma_3^{\downarrow} \circ r'^{\uparrow} K' = \sigma_2^{\downarrow} \circ r'^{\downarrow} \circ r'^{\uparrow} K' = \sigma_2^{\downarrow} K' = J$$

since $r'^{\downarrow} \circ r'^{\uparrow} K' = K'$ by construction of K'. Hence we have $r^{\downarrow}(r^{\uparrow} J) = J$.

Let us consider the diagram



where the left square is a pullback by definition of J and the right one is a pulation square as above. Hence the outer square is a pullback, and by Lemma 2.5 we have $\sigma_1^{\downarrow} K' = J$. We deduce that $m^{\uparrow} J = m'^{\uparrow} \circ \sigma_1^{\uparrow} \circ \sigma_1^{\downarrow} K' \sqsubseteq m'^{\uparrow} K'$, hence

$$J \sqsubseteq m^{\downarrow} \circ m^{\uparrow} J \sqsubseteq \sigma_1^{\downarrow} \circ m'^{\downarrow} \circ m'^{\uparrow} K' = \sigma_1^{\downarrow} K' = J$$

since $m'^{\downarrow} \circ m'^{\uparrow} K' = K'$ by construction of K', so that $m^{\downarrow}(m^{\uparrow} J) = J$.

By the properties of J w.r.t. r and m and by the construction of K we therefore have $J \sqsubseteq K$. But then we have

$$m^{\downarrow} D' = \sigma_1^{\downarrow} \circ m'^{\downarrow} D' = \sigma_1^{\downarrow} K' = J \sqsubseteq K$$

since $m'^{\downarrow} D' = K'$ by construction of D'. We thus obtain $D' \sqsubseteq D$ by the construction of D.

There remains to establish the commuting properties. Let $j : D \hookrightarrow G$ and $j' : R' \hookrightarrow T'$, one easily get $j \circ \nu_1 \circ m' \restriction D' \circ \nu_2 = j \circ m \restriction D$ and $j' \circ r' \restriction R' \nu_2 = j' \circ \nu_3 \circ r \restriction R$ by diagram chasing, and since j and j' are monomorphisms then the bottom faces of (1) commute, so that $(\sigma, \nu) : \delta \to \delta'$. □

It is easy to find examples showing that this property fails if in Definition 3.4 of subsumption morphisms between SPO-rules, either square is not a pushout, or either of σ_1, σ_3 is not a monomorphism.

4 Conclusion

The proof of Theorem 3.5 uses only well-known facts in Category Theory, some results on adhesive categories and the properties obtained in Section 2. It it were to be attempted in a different class of categories where the SPO approach can be developed, as in [4], the most challenging problem would probably be to prove Lemma 2.8, that relies heavily on the fact that morphisms in $\widehat{\mathcal{C}}$ can be decomposed as functions.

An interesting fact is that subsumption morphisms between SPO-rules are also subsumption morphisms between DPO-rules, as defined in [2, Definition 3.3], so that there is an obvious embedding of $\mathscr{R}_{\mathrm{SPO}}$ into $\mathscr{R}_{\mathrm{DPO}}$. This extends the well-known fact that DPO-transformations are nothing else than SPO-transformations where the gluing condition holds. Hence there should be an equivalence from $\mathscr{D}_{\mathrm{SPO}}^{\mathrm{GC}}$, the full subcategory of $\mathscr{D}_{\mathrm{SPO}}$ objects that satisfy the gluing condition, to the category of DPO-transformations, and it should commute with the previous embedding through the corresponding R functors.

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