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Abstract. We consider several systems of algebras of real- and complex-valued functions, which appear in o-minimal geometry and related geometrically tame contexts. For each such system, we prove its stability under parametric integration and we study the asymptotics of the functions as well as the nature of their parametric Mellin transforms.

Contents

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1. Introduction

The study of parametric integrals of functions belonging to a given tame class arises from the question of the nature of the volume of the fibres X_y of a tame family $(X_y)_{y\in Y}$. More

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precisely, describing the locus of integrability is a counterpart to establishing the nature of the set of points y of Y for which X_y has finite volume. The volumes of globally subanalytic sets have been studied in [LR98, CLR00], where it is proven that, for a globally subanalytic set $X \subseteq \mathbb{R}^{n+m}$ such that the fibers $X_y = X \cap \{\{y\} \times \mathbb{R}^m\}$ have dimension at most k, the set $Y_0 \subseteq \mathbb{R}^n$ of points y such that the k-dimensional volume $v(y)$ of X_y is finite is again globally subanalytic. However, it is necessary to introduce a function which is not globally subanalytic in order to express the volume: the restriction of v to Y_0 has the form $v = P(A_1, \dots, A_r, \log A_1, \dots, \log A_r)$, where P is a polynomial and the A_i are positive globally subanalytic functions.

The class of all functions definable in an o-minimal structure is closed under many natural operations, but is not in general stable under parametric integration. For instance, it follows from the above results that the family $\mathcal{S}$ of all *globally subanalytic functions* is not stable under parametric integration. However, the family $\mathcal{C}$ of *constructible functions* (see Definition 2.1) is, and indeed it is the smallest such collection containing $\mathcal{S}$ (see [CM12]). Moreover, the locus of integrability of a constructible function is the zero-set of a function which is again constructible.

The expansion $\mathbb{R}_{\mathrm{an,exp}}$ of the real field by all restricted analytic functions and the unrestricted exponential is not stable under parametric integration [DMM97], although some of these integrals are definable in larger o-minimal structures. For example, all antiderivatives of functions definable in an o-minimal structure $\mathcal{R}$ are definable in a larger o-minimal structure, called the Pfaffian closure of $\mathcal{R}$ [Spe99]. Other parametric integrals and definable integral transforms of functions definable in $\mathbb{R}_{\mathrm{an,exp}}$ (for example, the restrictions to the real half-line $(1, +\infty)$ of the Gamma function, seen as a Mellin transform, and of the Riemann Zeta function, seen as a quotient of two Mellin transforms) are known to be definable in suitable larger o-minimal structures [DS00, DS98, RSS22]. However, there is no known general o-minimal universe in which all such parametric integrals are definable (and indeed incompatibility results in [RSW03, RSS07, LG10] suggest that such a universe might not exist).

We therefore turn our attention to subcollections of functions definable in a given o-minimal structure (here, $\mathbb{R}_{\mathrm{an,exp}}$) which are stable under taking parametric integrals. There aren't many known such collections. For example, the collection of all functions definable in $\mathbb{R}_{\mathrm{an}}^{\mathrm{pow}}$ (the expansion of $\mathbb{R}_{\mathrm{an}}$ by all real power functions, seen as a reduct of $\mathbb{R}_{\mathrm{an,exp}}$) is not stable under parametric integration and indeed some such integrals are not even definable in $\mathbb{R}_{\mathrm{an,exp}}$ (see [Sou02, Prop. 2.1]). Our first aim is to define a collection $\mathcal{C}^{\mathbb{R}}$ of $\mathbb{R}$-algebras of functions definable in $\mathbb{R}_{\mathrm{an,exp}}$, extending $\mathcal{C}$ and stable under parametric integration (see Definition 2.2 and Theorem 2.4 below, for the case $\mathbb{K} = \mathbb{R}$). The elements of $\mathcal{C}^{\mathbb{R}}$ are called *real power-constructible* functions and they are constructed from real powers and logarithms of globally subanalytic functions.

Parametric integrals of tame functions also appear in the study of functional and geometric analogues of *period conjectures*. Recent breakthroughs in functional transcendence around o-minimality and periods have been made, concerning the transcendence of the coordinates of the Hodge filtration, which are ratios of certain period functions. For instance, Bakker, Klingler and Tsimerman [BKT20] proved that period maps are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$, yielding a new proof of the algebraicity of the Hodge loci. This provides an example of an integration process whose resulting functions remain in the original tame framework. Analogously, our Theorem 2.4 states that parametric integration preserves the class $\mathcal{C}^{\mathbb{R}}$. In the same spirit, we consider (see Definition 2.13 and Theorems 2.16, 2.19) larger classes which we prove to be stable under parametric integration.

Another motivation for considering the collection $\mathcal{C}^{\mathbb{R}}$ lies beyond o-minimality: most integral transforms (Fourier, Mellin...) are usually applied to rapidly decaying or compactly supported unary functions, but they can be extended to classes of functions having an asymptotic expansion (at 0 and/or at ∞) in the scale of real power-log monomials (for example, for such functions it can be shown that the Mellin transform extends to a meromorphic function on the whole complex plane, outside the domain of convergence of the integral, see [Zei06, Section 6.7 (by D. Zagier)]. In order to consider parametric versions of such transforms, one needs some control over the behaviour of the multi-variable functions in the collection to which we want to apply the transform. This is clear for example in the study of oscillatory integrals of the first kind, when the phase and the amplitude are analytic: resolution of singularities in the class of analytic germs is used to recover information about the asymptotic expansion of such parametric integrals. When applying parametric integral transforms to a class $\mathcal{F}$ of functions in several variables, it is hence important to have information about the geometry of the domain of the functions in $\mathcal{F}$ and to have some well-behaved theory of resolution of singularities adapted to the class $\mathcal{F}$. This is where o-minimality plays a central role: the key result here is a version of local resolution of singularities called the *subanalytic preparation theorem* [LR97], [Par94], together with cell-decomposition and piecewise analyticity arguments to patch together the local results into a global stability statement.

More specifically, in this paper we study *parametric* Mellin transforms of functions in $\mathcal{C}^{\mathbb{R}}$, exploiting both the o-minimal (subanalytic) nature of the domain of the functions and a preparation theorem available for the functions in $\mathcal{C}^{\mathbb{R}}$. We define a collection of functions which contains the parametric Mellin transforms of the functions in $\mathcal{C}^{\mathbb{R}}(X)$, for $X \subseteq \mathbb{R}^m$ a globally subanalytic set, and stable under integration with respect to the variable $x \in X$: our starting point is $\mathcal{C}^{\mathbb{R}}$, a collection of functions defined on subanalytic sets. We then apply an integral transform which depends on a complex parameter s, which we want to keep separate from the subanalytic variables, in the sense that we will not integrate with respect to s. For this, we construct a collection $\mathcal{C}^{\mathcal{M}}$ of $\mathbb{C}$-algebras of functions of the variables (s, x) (where s is a single complex variable and x is a tuple of variables ranging in a subanalytic set) which contain the parametric Mellin transforms of power-constructible functions, and stable under parametric integration.

The functions in $\mathcal{C}^{\mathcal{M}}$ will be shown to depend meromorphically on the variable s. This, together with Theorem 2.16, will be used to provide a *meromorphic extension* of the parametric Mellin transform to the whole complex plane. A classical result in this spirit is proven in [Ati70] (see also [Gre10, Th. 1.4] for a more recent and simplified proof): given a real analytic function f defined in a open neighborhood $U$of $0 \in \mathbb{R}^n$, for every $\mathcal{C}^{\infty}$ function φ whose support is compact and contained in U, the integral of $f^s\varphi$, initially defined as a holomorphic function on $\Re(s) > 0$, extends to a meromorphic function on $\mathbb{C}$.

As the Mellin transform is usually considered as a function of a complex parameter, we leave the realm of real-valued functions and of o-minimality. There is hence no reason to restrict ourselves to *real* powers of subanalytic functions. Therefore, we define *complex power-constructible* functions, prove that they form a collection $\mathcal{C}^{\mathbb{C}}$ which is stable under parametric integration (see Definition 2.2 and Theorem 2.4 below, case $\mathbb{K}=\mathbb{C}$) and study their parametric Mellin transforms. The purely imaginary powers of subanalytic functions introduce now some nontrivial oscillatory phenomena, which lead us to invoke results from the theory of continuously distributed functions mod 1 (see Section 3.1).

The paper is organized as follows. In Section 2, we introduce several classes of functions, for which we prove stability under parametric integration: power-constructible functions (Definition 2.2), parametric power-constructible functions (Definition 2.13) and some variants (Section 2.3.1). The main results about these classes are stated in Theorems 2.4, 2.16 and 2.19. In Section 3 we introduce the three basic tools that will be used in the proofs of the main results: a non-compensation argument about finite sums of purely imaginary powers, the properties of parametric strong functions (which are the building blocks in the construction of the class of parametric power-constructible functions) and the previously mentioned subanalytic preparation theorem, from which we derive the consequences needed in our setting. Section 4 is devoted to preparing the functions in the classes under consideration in a particularly simple way with respect to a given subanalytic variable. This will allow in Section 5 to provide a first result about integrating a generator of a class with respect to a single variable. The proofs of the general stability statements are carried out in Section 6.

2. Notation, definitions and main results

A subset X of $\mathbb{R}^m$ is globally subanalytic if it is the image under the canonical projection from $\mathbb{R}^{m+n}$ to $\mathbb{R}^m$ of a globally semianalytic subset of $\mathbb{R}^{m+n}$ (i.e. a subset $Y\subseteq\mathbb{R}^{m+n}$ such that, in a neighborhood of every point of $\mathbb{P}^1(\mathbb{R})^{m+n}$, Y is described by finitely many analytic equations and inequalities). Equivalently, X is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$ (see for example [DD88]). Thus, the logarithm $\log:(0,+\infty)\longrightarrow\mathbb{R}$ and the power map $x^y:(0,+\infty)\times\mathbb{R}\longrightarrow\mathbb{R}$ are functions whose graph is not subanalytic, but they are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ (see for example [DMM94]).

Throughout this paper $X\subseteq\mathbb{R}^m$ will be a globally subanalytic set (from now on, just "*subanalytic set*", for short). Denote by $\mathcal{S}(X)$ the collection of all subanalytic functions on X, i.e. all the functions of domain X whose graph is a subanalytic set, and let $\mathcal{S}_+(X)=\{f\in\mathcal{S}(X):\ f(X)\subseteq(0,+\infty)\}$.

Definition 2.1 (Constructible functions)**.** Let $\mathcal{C}(X)$ be the $\mathbb{R}$-algebra of *constructible functions on X,* generated by all subanalytic functions and their logarithms:

$$\mathcal{C}(X)=\left\{\sum_{i=1}^{N}f_i\prod_{j=1}^{M}\log g_{i,j}:\ M,N\in\mathbb{N}^{\times},\ f_i\in\mathcal{S}(X),g_{i,j}\in\mathcal{S}_+(X)\right\}.$$

Define $\mathcal{C}=\{\mathcal{C}(X):\ \ X\subseteq\mathbb{R}^m \text{ subanalytic},\ m\in\mathbb{N}\}$.

By [LR97, CLR00, CM11], $\mathcal{C}$ is the smallest collection of $\mathbb{R}$-algebras containing $\mathcal{S}$ and stable under parametric integration. Notice that constructible functions are definable in $\mathbb{R}_{\mathrm{an,\,exp}}$.

A function defined on X and taking its values in $\mathbb{C}$ is called a *complex-valued subanalytic (constructible, resp.) function* if its real and imaginary parts are in $\mathcal{S}(X)$ (in $\mathcal{C}(X)$, resp.).

2.1. **Power-constructible functions.** For $\mathbb{K} \subseteq \mathbb{C}$ a subfield, write

$$\mathcal{S}_{+}^{\mathbb{K}}(X) = \{f^{\alpha} : \ f \in \mathcal{S}_{+}(X), \ \alpha \in \mathbb{K}\}.$$

Let $\mathbb{F}_{\mathbb{K}}$ be $\mathbb{R}$ if $\mathbb{K} \subseteq \mathbb{R}$ and $\mathbb{C}$ otherwise.

Definition 2.2 (Power-constructible functions)**.** Let $\mathcal{C}^{\mathbb{K}}(X)$ be the $\mathbb{F}_{\mathbb{K}}$-algebra generated by the logarithms and the $\mathbb{K}$-powers of the subanalytic functions on X, i.e.

$$\mathcal{C}^{\mathbb{K}}(X) = \left\{ \sum_{i=1}^{N} c_i \prod_{j=1}^{M} f_{i,j}^{\alpha_{i,j}} \log g_{i,j} : \ M, N \in \mathbb{N}^{\times}, \ f_{i,j}, g_{i,j} \in \mathcal{S}_{+}(X), \ \alpha_{i,j} \in \mathbb{K}, c_i \in \mathbb{F}_{\mathbb{K}} \right\}.$$

Let

$$\mathcal{C}^{\mathbb{K}} = \left\{ \mathcal{C}^{\mathbb{K}}(X) : \ X \subseteq \mathbb{R}^m \text{ subanalytic}, \ m \in \mathbb{N} \right\}.$$

The functions in $\mathcal{C}^{\mathbb{K}}$ are called $\mathbb{K}$-*power-constructible functions*.

Remark 2.3. Notice that $\mathcal{C}^{\mathbb{Q}} = \mathcal{C}$ and if $\mathbb{K} \subseteq \mathbb{R}$, then the functions in $\mathcal{C}^{\mathbb{K}}$ are definable in $\mathbb{R}_{\text{an,exp}}$. If $\mathbb{K} \not\subseteq \mathbb{R}$, then by definition $\mathcal{C}^{\mathbb{K}}(X)$ is a $\mathbb{C}$-algebra. However, if $h = \sum c_i \prod f_{i,j}^{\alpha_{i,j}} \log g_{i,j}$ is such that all the exponents $\alpha_{i,j}$ belong to $\mathbb{R}$, then the real and imaginary parts of h belong to the $\mathbb{R}$-algebra $\mathcal{C}^{\mathbb{R}}(X)$.

In [Sou02, Section 4], the author produces an example of a function f in two variables x and y, defined as a composition of subanalytic functions and irrational powers (in particular, definable in $\mathbb{R}_{\text{an,exp}}$ and even in $\mathbb{R}_{\text{an}}^{\text{pow}}$), such that the parametric integral (with respect to y) of f is not definable in $\mathbb{R}_{\text{an,exp}}$. More precisely the function f is obtained by right-composing a subanalytic function by a suitable irrational power of the variable y. This procedure differs from the one in the above definition, where we left-compose subanalytic functions with irrational powers. Indeed, f is not power-constructible, as our first result (Theorem 2.4 below) is that $\mathcal{C}^{\mathbb{K}}$ is stable under parametric integration.

Theorem 2.4. *Let $h \in \mathcal{C}^{\mathbb{K}}(X \times \mathbb{R}^n)$. There exists $H \in \mathcal{C}^{\mathbb{K}}(X)$ such that*

$$\forall x \in \operatorname{Int}(h; X), \ \int_{\mathbb{R}^n} h(x, y)\, dy = H(x),$$

where

$$\operatorname{Int}(h; X) := \left\{ x \in X : \ y \longmapsto h(x, y) \in L^1(\mathbb{R}^n) \right\}.$$

2.2. **Strong functions.** In the subanalytic and constructible preparation theorems, a special role is played by the so-called *strong functions*: these are bounded subanalytic functions which can be expressed as the composition of a single power series (convergent in a neighbourhood of the closed unit polydisk) with a bounded subanalytic map. In order to define parametric Mellin transforms, we will need a parametric version of strong functions, where the parameter will be the complex number s appearing in the integration kernel of the Mellin transform.

We first give the definition of a subanalytic strong function and then proceed to define its parametric counterpart.

Definition 2.5. For $N \in \mathbb{N}$, we let $\mathcal{S}_c^N(X)$ be the collection of all maps $\psi : X \longrightarrow \mathbb{R}^N$ with components in $\mathcal{S}(X)$, such that $\overline{\psi(X)}$ is contained in the closed polydisk of $\mathbb{R}^N$ centered at zero and of radius 1. We call

$$\mathcal{S}_c(X) = \bigcup_{N \in \mathbb{N}^\times} \mathcal{S}_c^N(X)$$

the collection of all *1-bounded* subanalytic maps defined on X.

The following definition is inspired by [CM12, Definition 3.3] and [CCM+18, Definition 3.6].

Definition 2.6 (Strong functions)**.** We say that $W : X \longrightarrow \mathbb{F}_{\mathbb{K}}$ is an $\mathbb{F}_{\mathbb{K}}$*-valued subanalytic strong function* if there are $N \in \mathbb{N}^\times$, a 1-bounded subanalytic map $\psi : X \longrightarrow \mathbb{R}^N$ and a series $F \in \mathbb{F}_{\mathbb{K}}[\![Z]\!]$ in N variables Z, which converges in a neighbourhood of the closed polydisk D^N centered at zero and of radius $\frac{3}{2}$ in $\mathbb{R}^N$ (we will say for short that F converges *strongly*, see below), such that $W = F \circ \psi$. If furthermore $|F - 1| < \frac{1}{2}$, the function W is called a *strong unit* (see [CCM+18, Remarks 3.7]).

We are now ready to define parametric strong functions: these can be written as certain convergent series composed with 1-bounded subanalytic maps, but the coefficients of the series are now (meromorphic) functions of a complex parameter s.

Definition 2.7. Let $\mathcal{E}$ be the field of meromorphic functions $\xi : \mathbb{C} \longrightarrow \mathbb{C}$ and denote by D^N the closed polydisk of radius $\frac{3}{2}$ and center $0 \in \mathbb{R}^N$.

Given a formal power series $F = \sum_I \xi_I(s) Z^I \in \mathcal{E}[\![Z]\!]$ in N variables Z and with coefficients $\xi_I \in \mathcal{E}$, we say that F *converges strongly* if there exists a closed discrete set $P(F) \subseteq \mathbb{C}$ (called the *set of poles* of F) such that:

- for every $s_0 \in \mathbb{C} \setminus P(F)$, the power series $F(s_0, Z) \in \mathbb{C}[\![Z]\!]$ converges in a neighbourhood of D^N (thus F defines a function on $(\mathbb{C} \setminus P(F)) \times D^N$);
- for every $s_0 \in \mathbb{C}$ there exists $m = m(s_0) \in \mathbb{N}$ such that for all $z_0 \in D^N$, the function $(s, z) \longmapsto (s - s_0)^m F(s, z)$ has a holomorphic extension on some complex neighbourhood of (s_0, z_0)
- $P(F)$ is the set of all $s_0 \in \mathbb{C}$ such that the minimal such $m(s_0)$ is strictly positive.

Remark 2.8. It is easy to see that $P(F)$ coincides with the set of poles of the coefficients ξ_I and that for each $s_0 \in P(F)$ there is an integer $m \in \mathbb{N}$ such that for all I, $\mathrm{ord}_{s_0}(\xi_I) \leq m$.

Definition 2.9 (Parametric strong functions)**.** Given a closed discrete set $P \subseteq \mathbb{C}$, a function $\Phi : (\mathbb{C} \setminus P) \times X \longrightarrow \mathbb{C}$ is called a *parametric strong function* on X if there exist a 1-bounded subanalytic map $\psi \in \mathcal{S}_c^N(X)$ and a strongly convergent series $F = \sum_I \xi_I(s) Z^I \in \mathcal{E}[\![Z]\!]$ with $P(F) \subseteq P$ such that,

$$\forall (s, x) \in (\mathbb{C} \setminus P) \times X,\ \Phi(s, x) = F \circ (s, \psi(x)) = \sum_I \xi_I(s) (\psi(x))^I .$$

Define $\mathcal{A}(X)$ as the collection of all parametric strong functions on X (defined on sets of the form $(\mathbb{C} \setminus P) \times X$, for any closed discrete $P \subseteq \mathbb{C}$). Note that if $X \subseteq \mathbb{R}^0$ then $\mathcal{A}(X) = \mathcal{E}$. We let

$$\mathcal{A} = \{\mathcal{A}(X) :\ X \subseteq \mathbb{R}^m \text{ subanalytic},\ m \in \mathbb{N}\} .$$

Remark 2.10. Since the same $\Phi \in \mathcal{A}(X)$ could be presented by two different series F with different poles, we will say "let $\Phi \in \mathcal{A}(X)$ have no poles outside some closed discrete set

$P \subseteq \mathbb{C}$" to mean that there exist F, ψ such that $\Phi = F \circ (s, \psi)$ and $P(F) \subseteq P$. By the same argument, $\mathcal{A}(X)$ is a $\mathbb{C}$-algebra, up to defining the sum and product on a common domain.

2.3. **Parametric powers and the Mellin transform.** We introduce two parametric integral operator which will be the object of our study.

Definition 2.11.

- For $X \subseteq \mathbb{R}^m$ subanalytic, define

$$\mathcal{P}(\mathcal{S}(X)) = \{P_f : \mathbb{C} \times X \longrightarrow \mathbb{C} \text{ such that } P_f(s,x) = f(x)^s, \text{ for some } f \in \mathcal{S}_+(X)\}.$$

  The *parametric powers of* $\mathcal{S}$ are the functions in the collection

$$\mathcal{P}(\mathcal{S}) = \{\mathcal{P}(\mathcal{S}(X)) : \ X \subseteq \mathbb{R}^m \text{ subanalytic}, \ m \in \mathbb{N}\}.$$

- Let $\mathcal{F} = \{\mathcal{F}(X) : \ X \subseteq \mathbb{R}^m \text{ subanalytic}, \ m \in \mathbb{N}\}$ be a collection of real- or complex-valued functions and $S \subseteq \mathbb{C}$. If $f \in \mathcal{F}(X \times \mathbb{R})$ is such that for all $(s,x) \in S \times X$, $y \longmapsto y^{s-1} f(x,y) \in L^1(\mathbb{R}^{>0})$, define the parametric Mellin transform of f on $S \times X$ as the function

$$\mathcal{M}(f; S \times X)(s,x) = \int_0^{+\infty} y^{s-1} f(x,y)\, \mathrm{d}y, \ \forall (s,x) \in S \times X.$$

  The *parametric Mellin transforms of* $\mathcal{F}$ *on* S are the elements of the collection

$$\mathcal{M}(S; \mathcal{F}) = \{\mathcal{M}(f; S \times X) : \ f \text{ as above, for some } X\}.$$

Our next aim is to define a collection of algebras of functions which is stable under parametric integration and which contains both the parametric powers of $\mathcal{S}$ and the Mellin transforms of $\mathcal{C}^{\mathbb{C}}$ on $\mathbb{C}$ (Definition 2.13). In order to motivate the definition, let us give three simple examples.

Examples 2.12. Let $X \subseteq \mathbb{R}^m$ be subanalytic and $a, b \in \mathcal{S}(X)$ be such that for all $x \in X$, $1 \leq a(x) \leq 2 \leq b(x)$.

(1) Let $\chi_1(x,y)$ be the characteristic function of the set

$$B_1 = \{(x,y) : \ x \in X, \ 0 < y < a(x)\}$$

and consider the subanalytic function

$$f(x,y) = \chi_1(x,y) \frac{a(x)\, b(x)}{a(x)\, b(x) - y} \in \mathcal{S}(X \times \mathbb{R}).$$

Since $1 \leq f(x,y) \leq 2$, the parametric Mellin transform of f is well defined on $S_1 = \{s \in \mathbb{C} : \ \Re(s) > 0\}$, is holomorphic in s and is given by

$$\begin{aligned}\mathcal{M}(f; S_1 \times X) &= \int_0^{a(x)} y^{s-1} \frac{a(x)\, b(x)}{a(x)\, b(x) - y} \mathrm{d}y \\ &= \int_0^{a(x)} y^{s-1} \sum_{k \geq 0} \left(\frac{y}{a(x)\, b(x)} \right)^k \mathrm{d}y.\end{aligned}$$

The series in the above integral converges normally on B_1, hence we can permute sum and integral and write

$$\mathcal{M}\left(f;S_1\times X\right)=\sum_{k\geq0}\left(a\left(x\right)b\left(x\right)\right)^{-k}\int_0^{a(x)}y^{s-1+k}\mathrm{d}y$$
$$=\left(a\left(x\right)\right)^s\sum_{k\geq0}\frac{\left(b\left(x\right)\right)^{-k}}{s+k}.$$

Notice that in this computation we create both the parametric power of a subanalytic function, and a series of functions depending on the complex variable s and on the real variable x. The above series defines a parametric strong function on $\mathbb{C}\times X$, with poles at zero and at the negative integers.

(2) Let $S_2=\{s\in\mathbb{C}:\ \Re\left(s\right)<1\}$ and $\chi_2\left(x,y\right)$ be the characteristic function of the set

$$B_2=\left\{\left(x,y\right):\ x\in X,\ y>a\left(x\right)\right\}.$$

Consider the subanalytic function $g\left(x,y\right)=\chi_2\left(x,y\right)y\left(1+\frac{a(x)}{b(x)y}\right)\in\mathcal{S}\left(X\times\mathbb{R}\right)$. We aim to compute the parametric integral (with respect to the variable y) of the function $y^{-2}\left(g\left(x,y\right)\right)^s$. Since $0\leq\frac{a(x)}{b(x)y}\leq\frac{1}{2}$ on B_2, such an integral exists on $S_2\times X$ and

$$I\left(g;S_2\times X\right):=\int_{a(x)}^{+\infty}y^{s-2}\left(1+\frac{a\left(x\right)}{b\left(x\right)y}\right)^s\mathrm{d}y=\int_{a(x)}^{+\infty}y^{s-2}\sum_k\binom{s}{k}\left(\frac{a\left(x\right)}{b\left(x\right)y}\right)^k\mathrm{d}y$$
$$=\sum_k\binom{s}{k}\left(\frac{a\left(x\right)}{b\left(x\right)}\right)^k\int_{a(x)}^{+\infty}y^{s-2-k}\mathrm{d}y=-\left(a\left(x\right)\right)^{s-1}\sum_k\binom{s}{k}\frac{\left(b\left(x\right)\right)^{-k}}{s-1-k}.$$

Again, the above series defines a parametric strong function on $\mathbb{C}\times X$, with poles at the positive integers.

(3) Let $h\left(s,x,y\right)=f\left(x,y\right)y^{s-1}+y^{-2}\left(g\left(x,y\right)\right)^s$. Direct calculation shows that, letting

$$\mathrm{Int}\left(h;\left(\mathbb{C}\setminus\mathbb{Z}\right)\times X\right):=\left\{\left(s,x\right)\in\left(\mathbb{C}\setminus\mathbb{Z}\right)\times X:\ y\longmapsto h\left(s,x,y\right)\in L^1\left(\mathbb{R}\right)\right\},$$

we have

$$\mathrm{Int}\left(h;\left(\mathbb{C}\setminus\mathbb{Z}\right)\times X\right)=\left\{s\in\mathbb{C}\setminus\mathbb{Z}:\ 0<\Re\left(s\right)<1\right\}\times X.$$

However, the function H defined on $\left(\mathbb{C}\setminus\mathbb{Z}\right)\times X$ by

$$H\left(s,x\right)=\left(a\left(x\right)\right)^s\sum_{k\geq0}\frac{\left(b\left(x\right)\right)^{-k}}{s+k}-\left(a\left(x\right)\right)^{s-1}\sum_k\binom{s}{k}\frac{\left(b\left(x\right)\right)^{-k}}{s-1-k}$$

depends meromorphically on s and can be seen as an interpolation of the integral of h on the whole $\mathbb{C}\times X$.

Given a subanalytic set $X\subseteq\mathbb{R}^m$, recall the definitions of the algebras $\mathcal{A}\left(X\right)$ of parametric strong functions and $\mathcal{C}^{\mathbb{C}}\left(X\right)$ of $\mathbb{C}$-power-constructible functions, and of the collection $\mathcal{P}\left(\mathcal{S}\left(X\right)\right)$ of parametric powers of subanalytic functions (Definitions 2.9, 2.2 and 2.11).

Definition 2.13. If $X \subseteq \mathbb{R}^0$, then define $\mathcal{C}^{\mathcal{M}}(X) = \mathcal{E}$. If $X \subseteq \mathbb{R}^m$, with $m > 0$, then we let $\mathcal{C}^{\mathcal{M}}(X)$ be the $\mathcal{A}(X)$-algebra generated by $\mathcal{C}^{\mathbb{C}}(X) \cup \mathcal{P}(\mathcal{S}(X))$. Every function $h \in \mathcal{C}^{\mathcal{M}}(X)$ can be written on $(\mathbb{C} \setminus P) \times X$ (for some closed discrete $P \subseteq \mathbb{C}$) as a finite sum of *generators* of the form

$$\Phi(s,x) \cdot g(x) \cdot f(x)^s, \tag{2.1}$$

where $g \in \mathcal{C}^{\mathbb{C}}(X)$, $f \in \mathcal{S}_+(X)$ and $\Phi \in \mathcal{A}(X)$ has no poles outside P.

If $h \in \mathcal{C}^{\mathcal{M}}(X)$ can be presented as a sum of generators in which the parametric strong functions have no poles outside some common set $P \subseteq \mathbb{C}$, then we say that h has no poles outside P. We let

$$\mathcal{C}^{\mathcal{M}} = \left\{\mathcal{C}^{\mathcal{M}}(X) : X \subseteq \mathbb{R}^m \text{ subanalytic}, m \in \mathbb{N}\right\}$$

be the collection of algebras of (complex) *parametric power-constructible functions.*

Remark 2.10 also applies to the functions in $\mathcal{C}^{\mathcal{M}}(X)$.

Remark 2.14. If $h \in \mathcal{C}^{\mathcal{M}}(X)$ has no poles outside some closed discrete set P, then for all $s \in \mathbb{C} \setminus P$, $x \longmapsto h(s,x) \in \mathcal{C}^{\mathbb{C}}(X)$ and the dependence on the variables x is piecewise analytic, by o-minimality. Moreover, by definition of $\mathcal{A}(X)$, for all $x \in X$, $s \longmapsto h(s,x)$ is meromorphic on $\mathbb{C}$.

The main goal of this paper is to study the nature of the parametric integrals of functions in $\mathcal{C}^{\mathcal{M}}$. Let $X \subseteq \mathbb{R}^m$ be subanalytic, and consider a function $h \in \mathcal{C}^{\mathcal{M}}(X \times \mathbb{R}^n)$ without poles outside some closed and discrete set $P \subseteq \mathbb{C}$. Then h depends on a complex variable s and on $(m+n)$ real variables (let us call them x, ranging in X, and y, ranging in $\mathbb{R}^n$). We integrate h in the variables y over $\mathbb{R}^n$, whenever the integral exists, and study the nature of the resulting function.

The set of parameters $(s,x) \in (\mathbb{C} \setminus P) \times X$ for which the integral exists is the *integration locus* of h.

Definition 2.15. For $h \in \mathcal{C}^{\mathcal{M}}(X \times \mathbb{R}^n)$ and a closed discrete set $P \subseteq \mathbb{C}$ such that h has no poles outside P, define

$$\mathrm{Int}(h; (\mathbb{C} \setminus P) \times X) := \left\{(s,x) \in (\mathbb{C} \setminus P) \times X : y \longmapsto h(s,x,y) \in L^1(\mathbb{R}^n)\right\}.$$

Our main result is the following.

Theorem 2.16. *Let $h \in \mathcal{C}^{\mathcal{M}}(X \times \mathbb{R}^n)$ be without poles outside some closed discrete set $P \subseteq \mathbb{C}$. There exist a closed discrete set $P' \subseteq \mathbb{C}$ containing P and $H \in \mathcal{C}^{\mathcal{M}}(X)$ with no poles outside P' such that*

$$\forall (s,x) \in \mathrm{Int}(h; (\mathbb{C} \setminus P') \times X),\ H(s,x) = \int_{\mathbb{R}^n} h(s,x,y)\,\mathrm{d}y.$$

Moreover, $P' \setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice.

The above examples and Theorem 2.16 suggest to introduce the following definition.

Definition 2.17. Let $\mathcal{G}(X)$ be a collection of functions $f : \mathbb{C} \times X \longrightarrow \mathbb{C}$, where $X \subseteq \mathbb{R}^m$ is subanalytic and f depends meromorphically on its complex variable s. We say that $\mathcal{G} = \{\mathcal{G}(X) : X \subseteq \mathbb{R}^m, m \in \mathbb{N}\}$ is *stable under generalized parametric Mellin transform* if whenever $f \in \mathcal{G}(X \times \mathbb{R})$ has no poles outside some closed discrete set $P \subseteq \mathbb{C}$, there exist a closed discrete set $P' \subseteq \mathbb{C}$ containing P and $\mathcal{M}_f \in \mathcal{G}(X)$ without poles outside P' such that, if $g(s,x,y) = y^{s-1} f(s,x,y)\, \chi_{(0,+\infty)}(y)$, then

$$\forall (s,x) \in \mathrm{Int}\left(g; (\mathbb{C} \setminus P') \times X\right),\ \mathcal{M}_f(s,x) = \int_0^{+\infty} y^{s-1} f(s,x,y)\, \mathrm{d}y.$$

Corollary 2.18. *$\mathcal{C}^{\mathcal{M}}$ is the smallest system of $\mathcal{A}$-algebras containing $\mathcal{C}^{\mathbb{C}}$ and stable under generalized parametric Mellin transform.*

Proof. By Theorem 2.16, $\mathcal{C}^{\mathcal{M}}$ is such a system. Let us show that it is the smallest.

Let $f \in \mathcal{S}(X)$. Let y be a single variable and let $\chi(x,y)$ be the characteristic function of the set $\{(x,y) :\ 0 < y < |f(x)|\}$ and consider the parametric Mellin transform of the function $s \cdot \chi(x,y)$ on $S = \{s \in \mathbb{C} :\ \Re(s) > 0\}$:

$$\begin{aligned}\mathcal{M}(s,x) &= \int_0^{+\infty} s y^{s-1} \chi(x,y)\, \mathrm{d}y \\ &= s \int_0^{|f(x)|} y^{s-1} \mathrm{d}y = |f(x)|^s .\end{aligned}$$

If $\mathcal{D}$ is a system of $\mathcal{A}$-algebras containing $\mathcal{C}^{\mathbb{C}}$, then $\mathcal{D}$ contains the function $s \cdot \chi(x,y)$, and if $\mathcal{D}$ is stable under the generalized parametric Mellin transform, then $\mathcal{D}$ contains the extension of $\mathcal{M}$ to the whole complex plane. Hence $\mathcal{P}(\mathcal{S}) \subseteq \mathcal{D}$, i.e. $\mathcal{C}^{\mathcal{M}} \subseteq \mathcal{D}$. ☐

2.3.1. *Parametric powers of $\mathbb{K}$-power-subanalytic functions.* We consider several collections, defined via minor variations of the definition of $\mathcal{C}^{\mathcal{M}}$, and which we will prove to be stable under parametric integration.

Let $\mathbb{K} \subseteq \mathbb{C}$ be a subfield.

In Definition 2.9, we replace $\mathcal{E}$ by

$$\mathcal{E}_{\mathbb{K}} := \{\xi \in \mathcal{E} :\ P(\xi) \subseteq \mathbb{K}\}$$

(where $P(\xi)$ is the set of poles of ξ) and we define $\mathcal{A}_{\mathbb{K}}$ accordingly.

We let $\mathcal{C}^{\mathbb{K},\mathcal{M}}(X)$ be the $\mathcal{A}_{\mathbb{K}}(X)$-algebra generated by $\mathcal{C}^{\mathbb{K}}(X) \cup \mathcal{P}(\mathcal{S}(X))$. Every element of $\mathcal{C}^{\mathbb{K},\mathcal{M}}(X)$ can be written as a finite sum of generators of the form (2.1), where now $\Phi \in \mathcal{A}_{\mathbb{K}}(X)$ and $g \in \mathcal{C}^{\mathbb{K}}(X)$.

Next, we define a similar system of algebras which furthermore contains the parametric powers of $\mathbb{K}$-powers of subanalytic functions. For this, given $X \subseteq \mathbb{R}^m$ subanalytic, let

$$\mathcal{P}\left(\mathcal{S}_+^{\mathbb{K}}(X)\right) = \{P_f : \mathbb{C} \times X \longrightarrow \mathbb{C} \text{ such that } P_f(s,x) = f(x)^{\alpha s},\ \text{for some } f \in \mathcal{S}_+(X) \text{ and } \alpha \in \mathbb{K}\}$$

and $\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}(X)$ be the $\mathcal{A}_{\mathbb{K}}(X)$-algebra generated by $\mathcal{C}^{\mathbb{K}}(X) \cup \mathcal{P}\left(\mathcal{S}_+^{\mathbb{K}}(X)\right)$. Every element of $\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}(X)$ can be written as a finite sum of generators of the form

$$\Phi(s,x)\, g(x)\, f_1(x)^{\alpha_1 s} \cdots f_n(x)^{\alpha_n s},$$

where $\Phi \in \mathcal{A}_{\mathbb{K}}(X),\ g \in \mathcal{C}^{\mathbb{K}}(X),\ n \in \mathbb{N},\ \alpha_i \in \mathbb{K}$ and $f_i \in \mathcal{S}_+(X)$.

Let

$$\begin{aligned}\mathcal{C}^{\mathbb{K},\mathcal{M}} &= \left\{\mathcal{C}^{\mathbb{K},\mathcal{M}}(X) :\ X \subseteq \mathbb{R}^m \text{ subanalytic},\ m \in \mathbb{N}\right\}, \\ \mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}} &= \left\{\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}(X) :\ X \subseteq \mathbb{R}^m \text{ subanalytic},\ m \in \mathbb{N}\right\}.\end{aligned}$$

Theorem 2.19. *The statement of Theorem 2.16 also holds if we replace $\mathcal{C}^{\mathcal{M}}$ by either $\mathcal{C}^{\mathbb{K},\mathcal{M}}$ or $\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$ (the closed discrete set P' is now contained in $\mathbb{K}$).*

Argueing as in Corollary 2.18, it follows that $\mathcal{C}^{\mathbb{K},\mathcal{M}}$ is the smallest system of $\mathcal{A}_{\mathbb{K}}$-algebras containing $\mathcal{C}^{\mathbb{K}}$ and stable under parametric Mellin transform. Notice that the collection $\mathcal{C}^{\mathbb{K}}$ of $\mathbb{K}$-power-constructible functions coincides with the collection of all functions in $\mathcal{C}^{\mathbb{K},\mathcal{M}}$ which happen not to depend on the parameter s.

The system of $\mathcal{A}_{\mathbb{K}}$-algebras $\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$ also contains $\mathcal{C}^{\mathbb{K}}$ and is stable under parametric Mellin transform. As a consequence of the proof of Theorem 2.19, we show (see Theorem 6.3 and Remark 6.5) that the system $\mathcal{C}^{\mathcal{P}(\mathbb{C}),\mathcal{M}}$ is strictly larger than $\mathcal{C}^{\mathcal{M}}$.

3. Toolbox

Throughout this paper, $X \subseteq \mathbb{R}^m$ is a subanalytic set which serves as space of parameters (we never integrate with respect to the variables ranging in X). Since all the classes $\mathcal{D}$ of functions defined in Sections 2.1 and 2.3 are stable under multiplication by a subanalytic function, when studying a function $f \in \mathcal{D}(X \times \mathbb{R}^n)$, we are allowed to partition X into subanalytic cells, replace X by one of the cells of the partition and work disjointly in restriction to such a cell. In particular, we may always assume that X is itself a subanalytic cell, and that all cells in $X \times \mathbb{R}^n$ project onto X.

If $\mathcal{D}$ is any of the classes defined in Sections 2.1 and $f \in \mathcal{D}(X \times \mathbb{R}^n)$, we often compute the integral of f with respect to the variables ranging in $\mathbb{R}^n$. If $X \times \mathbb{R}^n$ is partitioned into finitely many subanalytic cells, then only the cells which have nonempty interior in $X \times \mathbb{R}^n$ contribute to the integral. This motivates the following definition.

Definition 3.1. Let $A \subseteq X \times \mathbb{R}$ be a subanalytic cell. We say that A is *open over* X if there are $\varphi_1, \varphi_2 \in \mathcal{S}(X) \cup \{\pm\infty\}$ such that for all $x \in X$, $\varphi_1(x) < \varphi_2(x)$ and

$$A = \{(x, y) : \ x \in X, \ \varphi_1(x) < y < \varphi_2(x)\}.$$

Notation 3.2. For $x \in X$, define $A_x = \{y \in \mathbb{R}^n : \ (x, y) \in A\}$.

Hence, if $f \in \mathcal{D}(A)$, then

$$\mathrm{Int}(f; \mathbb{C} \times X) = \left\{(s, x) \in \mathbb{C} \times X : \ y \longmapsto f(s, x, y) \in L^1(A_x)\right\}.$$

Given a set A, we denote by χ_A the characteristic function of A.

3.1. **Non-compensation arguments.** In this section we prove a result (Proposition 3.4 below) which is a crucial ingredient of the proof of the Stability Theorems 2.4 and 2.16, and of the study the asymptotics of the functions of our classes. The statement of Proposition 3.4 is stronger than the result that we actually need here, since it involves both purely imaginary powers and purely imaginary exponentials. However, the full generality of this result will be used in a forthcoming paper, in which we will study the Fourier transforms of power-constructible functions.

We first recall the definition of continuously uniformly distributed modulo 1 functions, which is a key ingredient in Proposition 3.4 (for the properties and uses of this notion, see [KN74]). In what follows, vol_i stands for the Lebesgue measure in $\mathbb{R}^i$, $i \geq 1$.

Definition 3.3. Let $\{x\} := x - \lfloor x \rfloor$ be the fractional part of the real number x and let $F = (f_1, \ldots, f_\ell) : [0, +\infty) \to \mathbb{R}^\ell$ be any map. If $I_1, \ldots, I_\ell \subseteq \mathbb{R}$ are bounded intervals with nonempty interior, we denote by I the box $\prod_{j=1}^{\ell} I_j$. For $T \geq 0$, let

$$W_{F,I,T} := \{t \in [0, T] : \{F(t)\} \in I\},$$

where $\{F(t)\}$ denotes the tuple $(\{f_1(t)\}, \ldots, \{f_\ell(t)\})$.

The map F is said to be *continuously uniformly distributed modulo* 1, in short *c.u.d. mod* 1, if for every box $I \subseteq [0,1)^\ell$,

$$\lim_{T \to +\infty} \frac{\mathrm{vol}_1 (W_{F,I,T})}{T} = \mathrm{vol}_\ell (I).$$

We use the c.u.d. mod 1 property in the proof of the following proposition. There, we deal with a family of complex exponential functions having as phases the functions in the family $(\sigma_j \log(y) + p_j(y))_{j \in \{1,\ldots,n\}}$. It turns out that in general we cannot extract from this family a c.u.d. mod 1 subfamily, since log is not a c.u.d. mod 1 function (although the family $\sigma_j \log y + p_j(y)$ is, whenever p_j is not constant). To overcome this technical difficulty, we compose $\sigma_j \log y + p_j(y)$ with the change of variables $y = \mathrm{e}^t$, after which we are able to extract a c.u.d. mod 1 subfamily from the family of phases $(\sigma_j t + p_j(\mathrm{e}^t))_{j \in \{1,\ldots,n\}}$.

Proposition 3.4. *Let* $r \geq -1,\ b \geq 1,\ \nu \in \mathbb{N},\ n \in \mathbb{N} \setminus \{0\},\ c_1, \ldots, c_n \in \mathbb{C},\ \sigma_0, \ldots \sigma_n \in \mathbb{R}$ *and* $p_1, \ldots, p_n \in \mathbb{R}[X]$ *be such that* $p_j(0) = 0$ *for* $j = 1, \ldots, n$. *Suppose that* $\sigma_j \log(y) + p_j(y) \neq \sigma_k \log(y) + p_k(y)$ *for* $j \neq k$, *and let*

$$f(y) = y^r (\log y)^\nu \sum_{j=1}^{n} c_j y^{\mathrm{i}\sigma_j} \mathrm{e}^{\mathrm{i}p_j(y)}.$$

The following statements hold.

1. *If* $f \in L^1((b, +\infty))$ *then* $c_j = 0$ *for all* $j = 1, \ldots, n$.
2. *Let* $E(y) = \sum_{j=1}^{n} c_j y^{\mathrm{i}\sigma_j} \mathrm{e}^{\mathrm{i}p_j(y)}$, *where for at least one* $j \in \{1, \ldots, n\}$ *we have* $c_j \neq 0$ *and* $\sigma_j \log(y) + p_j(y) \neq 0$. *There exist* $\varepsilon > 0$ *and a sequence of real numbers* $(y_m)_{m \in \mathbb{N}}$ *which tends to* $+\infty$, *such that for all* $m \geq 0$, $|E(y_m)| \geq \varepsilon$.
3. *There exist* $\delta > 0$ *and two sequences of real numbers* $(y_{1,m})_{m \in \mathbb{N}}$, $(y_{2,m})_{m \in \mathbb{N}}$ *which both tend to* $+\infty$, *such that for all* $m \geq 0$, $|E(y_{1,m}) - E(y_{2,m})| \geq \delta$.

Proof. We may assume that at least one of the functions $g_j(y) = \sigma_j \log(y) + p_j(y)$ is not constant. Indeed, since $p_j(0) = 0$, if g_j is constant then it is zero, hence in this case $n = 1$ and f is not integrable unless $c_1 = 0$. Therefore we may assume without loss of generality that g_1 is not constant, and that $c_1 \neq 0$.

If $f \in L^1((b, +\infty))$, the following integral is finite for all x such that $e^x \geq b$:

$$I(x) := \int_b^{\mathrm{e}^x} |f(y)| \ dy.$$

Performing the change of variables $t = \log(y)$ we obtain

$$I(x) = \int_{\log(b)}^{x} t^\nu \mathrm{e}^{t(r+1)} \left| \sum_{j=1}^{n} c_j \mathrm{e}^{\mathrm{i}\sigma_j t} \mathrm{e}^{\mathrm{i}p_j(\mathrm{e}^t)} \right| \ \mathrm{d}t.$$

Set $\varphi(t)=\sum_{j=1}^{n} c_j \mathrm{e}^{\mathrm{i}\sigma_j t}\mathrm{e}^{\mathrm{i}p_j(\mathrm{e}^t)}$ and $f_j(t)=\sigma_j t+p_j(\mathrm{e}^t)$ for $j=1,\dots,n$. Assume that $f_1,\dots,f_\ell$, for $\ell\le n$, is a basis over $\mathbb{Q}$ of the $\mathbb{Q}$-vector space generated by $f_1,\dots,f_n$. We write

$$f_k = r_{k,1}f_1+\cdots+r_{k,\ell}f_\ell,\ \text{for } k=\ell+1,\dots,n,$$

we denote by ρ_j the least common multiple of the denominators of $r_{\ell+1,j},\dots,r_{n,j}$, and we set $\tilde{f}_j := f_j/2\pi\rho_j$, for $j=1,\dots,\ell$ (note that this family is still $\mathbb{Q}$-linearly independent). Then

$$f_k = 2\pi m_{k,1}\tilde{f}_1+\cdots+2\pi m_{k,\ell}\tilde{f}_\ell,\ \text{for } k=\ell+1,\dots,n,$$

for some $m_{k,1},\dots,m_{k,\ell}\in\mathbb{Z}$, and

$$\varphi(t)=P(\mathrm{e}^{2\pi\mathrm{i}\tilde{f}_1(t)},\dots,\mathrm{e}^{2\pi\mathrm{i}\tilde{f}_\ell(t)}),$$

where $P\in\mathbb{C}\left[X_1,\dots,X_\ell,X_1^{-1},\dots,X_\ell^{-1}\right]$ is a Laurent polynomial.

Note that P contains at least the monomial c_1X_1 (we can always choose f_1 as an element of our basis, since $c_1\neq 0$ and $g_1\neq 0$). Morever since by hypothesis $f_j(t)\neq f_k(t)$, (as functions) for $j\neq k$, the monomials of P cannot cancel out. It follows that P is not constant, and therefore the algebraic set $V:=\{P=0\}$ does not contain the torus $\mathbb{T}:=(S^1)^\ell$. By continuity of P, we can find a real number $\varepsilon>0$ and intervals $A_j^\varepsilon\subset[0,1)$, $j=1,\dots,\ell$, such that $|\varphi(t)|\ge\varepsilon$ on the set

$$W_\varepsilon=\left\{t\ge\log(b):\ \left\{\tilde{f}_j(t)\right\}\in A_j^\varepsilon,\ j=1\dots,\ell\right\}.$$

We claim that the map $F=\left(\tilde{f}_1,\dots,\tilde{f}_\ell\right)$ is c.u.d. mod 1 (which implies in particular that W_ε is nonempty). For this, we use the criterion [KN74, Theorem 9.9], i.e. we show that for any $h\in\mathbb{Z}^\ell$ such that $h\neq 0$,

$$\lim_{T\to+\infty}\frac{1}{T}\int_1^T \mathrm{e}^{2\pi\mathrm{i}\langle h,F(t)\rangle}\,\mathrm{d}t=0.$$

We prove in fact that there exists $T_0\ge 1$ such that

$$J(T)=\int_{T_0}^T \mathrm{e}^{2\pi\mathrm{i}\langle h,F(t)\rangle}\,\mathrm{d}t$$

is bounded from above. For $h\in\mathbb{Z}^\ell$ such that $h\neq 0$, we can write $\langle h,F(t)\rangle=\sigma t+p(\mathrm{e}^t)$, with $\sigma\in\mathbb{R}$ and $p\in X\mathbb{R}[X]$. Since the components of F are $\mathbb{Q}$-linearly independent, $\sigma t+p(\mathrm{e}^t)$ is not identically zero (equivalently, not constant). We can assume that $p\neq 0$, since if not, then $J(T)$ is clearly bounded, and we are done. Let us write

$$\rho(t)=\frac{\langle h,F(t)\rangle}{a_d}=\mathrm{e}^{dt}+\frac{a_{d-1}}{a_d}\mathrm{e}^{(d-1)t}+\ldots+\frac{\sigma}{a_d}t,$$

for some $d\ge 1$, $a_i\in\mathbb{R}$ and $a_d\in\mathbb{R}\setminus\{0\}$. Fix T_0 sufficiently large so that $t\mapsto\rho$ and $t\mapsto\rho'(t)$ are strictly increasing (to $+\infty$) on $[T_0,+\infty)$, and perform the change of variables $u=\rho(t)$ in $J(T)$ to obtain

$$J(T)=\int_{T_0}^T \mathrm{e}^{2\pi\mathrm{i}a_d\rho(t)}\,\mathrm{d}t=\int_{\rho(T_0)}^{\rho(T)}\frac{\mathrm{e}^{2\pi\mathrm{i}a_d u}}{\rho'(\rho^{-1}(u))}\,\mathrm{d}u.$$

By the second mean value theorem for integrals applied to the real part of $J(T)$, we have

$$\Re(J(T)) = \frac{1}{\rho'(T_0)} \int_{\rho(T_0)}^{\tau} \cos(2\pi a_d u)\, \mathrm{d}u,$$

for some $\tau \in (\rho(T_0), \rho(T)]$. This shows that the real part of $J(T)$ is bounded from above, and so is the imaginary part of $J(T)$ by the same computation.

Therefore F is c.u.d. mod 1 and hence, by definition, the set W_ε has infinite measure. Since

$$I(x) \geq \varepsilon \int_{[\log(b),x]\cap W_\varepsilon} t^\nu \mathrm{e}^{(r+1)t}\, \mathrm{d}t$$

and $\nu \geq 0$, $r \geq -1$, this implies that

$$\int_b^{+\infty} f(y)\, \mathrm{d}y = \lim_{x\longrightarrow+\infty} I(x) = +\infty,$$

and proves (1).

To prove (2) and (3) we may still assume that $c_1 \neq 0$ and $g_1 \neq 0$, by our hypothesis on E. In this situation, since we have shown that W_ε has infinite measure, one can find a sequence $(t_m)_{m\in\mathbb{N}}$ which tends to $+\infty$, such that for all $m \geq 0$, $t_m \in W_\varepsilon$, and therefore $|\varphi(t_m)| \geq \varepsilon$. We set, for all $m \in \mathbb{N}$, $y_m = \mathrm{e}^{t_m}$, and we obtain $y_m \to +\infty$ and $E(y_m) = \varphi(t_m)$, which proves (2).

We proceed in the same way to prove (3). Since P is not constant on $\mathbb{T}$, by continuity of P one can find $\delta > 0$ and intervals $A_j^\delta, B_j^\delta \subset [0,1)$, $j = 1,\dots,\ell$, such that $|\varphi(t) - \varphi(t')| \geq \delta$ for any t, t' such that $t \in A^\delta := \{u \in \mathbb{R}, \{\tilde{f}_j(u)\} \in A_j^\delta, j = 1,\dots,\ell\}$ and $t' \in B^\delta := \{u \in \mathbb{R}, \{\tilde{f}_j(u)\} \in B_j^\delta, j = 1,\dots,\ell\}$. But since F is c.u.d mod 1, one can find two sequences $(t_{1,m})_{m\in\mathbb{N}}$ and $(t_{2,m})_{m\in\mathbb{N}}$ tending to $+\infty$, such that for all $m \in \mathbb{N}$, $t_{1,m} \in A^\delta$ and $t_{2,m} \in B^\delta$. Finally, we set $y_{1,m} = \mathrm{e}^{t_{1,m}}$ and $y_{2,m} = \mathrm{e}^{t_{2,m}}$ to obtain (3). $\square$

3.2. **Properties of parametric strong functions.** In this section we give some examples of parametric strong functions and list their properties. The results in this section are stated for $\mathcal{E}$ and $\mathcal{A}$ for simplicity, but they also hold for $\mathcal{E}_\mathbb{K}$ and $\mathcal{A}_\mathbb{K}$.

Examples 3.5. All (finite sum of finite products) of the following functions are parametric strong functions (i.e. they belong to $\mathcal{A}$).

- Any subanalytic strong function (as in Definition 2.6), clearly.
- $(U(x))^s$, where $U \in \mathcal{S}(X)$ is a subanalytic strong unit of the form $U(x) = 1 + F \circ \psi(x)$, with $\psi \in \mathcal{S}_c(X)$, $F \in \mathbb{R}[\![Z]\!]$ and $\sup_{z\in D^N} |F(z)| < 1$ (where D^N is the closed polydisk in $\mathbb{R}^N$ of radius $\frac{3}{2}$). To see this, notice that the series $\tilde{F} = (1 + F(Z))^s = \sum \binom{s}{i} (F(Z))^i \in \mathcal{E}[\![Z]\!]$ is strongly convergent (without poles) and $(U(x))^s = \tilde{F} \circ (s, \psi(x))$.
- Let $B = \{(x,y) \in (2,+\infty) \times \mathbb{R} :\ y > x\}$ and $\Phi(s,x,y) = \sum_{i\geq 2} \xi_i(s) \left(\frac{x}{y}\right)^i \in \mathcal{A}(B)$.

  Then $\varphi(s,x) := \int_{x^2}^{+\infty} \Phi(s,x,y)\, \mathrm{d}y \in \mathcal{A}((2,+\infty))$. To see this, integrate term by term (which is possible, since the series $\Sigma_i \xi_i(s) Z^i$ is strongly convergent) and find that

$\varphi(s,x) = \sum_{i\geq 0} \frac{\xi_{i+2}(s)}{i+1} x^{-i}$, which is again a strongly convergent series with coefficients in $\mathcal{E}$, composed with the 1-bounded subanalytic function $x^{-1} \in \mathcal{S}((2,+\infty))$.

Remark 3.6.

- If $\Phi \in \mathcal{A}(X)$ has no poles outside P, then clearly for every fixed $s \in (\mathbb{C} \setminus P)$, $x \longmapsto \Phi(s,x)$ is a complex-valued subanalytic strong function (in the sense of Definition (2.6)). In particular, up to decomposing X into subanalytic cells, we may suppose that Φ depends analytically on x.
- If $\Phi(s,x) = F \circ (s, \psi(x)) \in \mathcal{A}(X)$ is a parametric strong function then
$$\left\{ \xi_I(s)\, \psi(x)^I :\ I \in \mathbb{N}^N \right\}$$
is a *normally summable* family of functions: the family $\left\{ \sup_{x\in X} \left| \xi_I(s)\, \psi(x)^I \right| \right\} \subseteq [0,1]$ is summable. In particular, if $\tilde{F}$ is obtained from F by taking the sum only over some subset of the support of F and rearranging the terms, then $\tilde{F} \circ (s, \psi(x))$ is a parametric strong function (without poles outside the set $P(F)$).

Remark 3.7. Let (Z,Y) be an $(N+M)$-tuple of variables and $F(s;Z,Y) = \sum_{I,J} \xi_{I,J}(s)\, Z^I Y^J \in \mathcal{E}[\![Z,Y]\!]$ be a strongly convergent series. Then, for all $J \in \mathbb{N}^M$, the series $F_J := \sum_I \xi_{I,J}(s)\, Z^I \in \mathcal{E}[\![Z]\!]$ is strongly convergent. Moreover, for every 1-bounded subanalytic map $c : X \longrightarrow \mathbb{R}^N$, we have $\xi^c_J(s,x) := F_J \circ (s, c(x)) \in \mathcal{A}(X)$. Furthermore the series $F_c := \sum_J \xi^c_J(s,x)\, Y^J \in \mathcal{A}(X)[\![Y]\!]$ is *strongly convergent*, in the sense that the family $\{\xi_{J,x}(s) := \xi^c_J(s,x)\}_{J,x} \subseteq \mathcal{E}$ has a non-accumulating set of common poles with bounded order and the series $\sum_J \xi^c_J(s,x)\, Y^J$ defines a function on $(\mathbb{C}\setminus P) \times X \times D^M$ which is meromorphic in s and analytic in (x,Y).

It follows that, for every 1-bounded map $\gamma : X \longrightarrow \mathbb{R}^M$, the parametric strong function $\Phi(s,x) := F \circ (s, c(x), \gamma(x))$ can also be written as the strongly convergent power series F_c (with suitable parametric strong functions as coefficients), evaluated at $Y = \gamma(x)$. We call $F_c \circ (s, \gamma(x)) = \sum_J \xi^c_J(s,x)\, (\gamma(x))^J$ a *nested presentation* of Φ.

We will often apply the above to the following situation: let $B \subseteq \mathbb{R}^{m+1}$ a subanalytic set such that the projection onto the first m coordinates of B is X. Fix coordinates (x,y), where x is an m-tuple and y is a single variable. Suppose that $(c(x), \gamma(x,y))$ is a 1-bounded subanalytic map on B, where the first component only depends on the variables x. Then the nested presentation of $F \circ (s, c(x), \gamma(x,y)) \in \mathcal{A}(B)$ is of the form

$$F_c \circ (s, \gamma(x,y)) = \sum_J \xi^c_J(s,x)\, (\gamma(x,y))^J, \tag{3.1}$$

where the coefficients ξ^c_J now belong to $\mathcal{A}(X)$.

Remark 3.8. Let s be a fixed real or complex number. Then Examples 3.5, Remarks 3.6 and 3.7 also apply to real- or complex-valued subanalytic strong functions.

3.3. **Subanalytic preparation.**

Let $\mathbb{K} \subseteq \mathbb{C}$ be a subfield and recall that $\mathbb{F}_\mathbb{K}$ is $\mathbb{R}$ if $\mathbb{K} \subseteq \mathbb{R}$ and $\mathbb{C}$ otherwise.

Definition 3.9. Let $X \subseteq \mathbb{R}^m$ be a subanalytic cell and

$$B = \{(x,y) :\ x \in X,\ a(x) < y < b(x)\}, \tag{3.2}$$

where $a, b : X \longrightarrow \mathbb{R}$ are analytic subanalytic functions with $1 \leq a(x) < b(x)$ for all $x \in X$, and b is allowed to be $\equiv +\infty$. We say that B has *bounded y-fibers* if $b < +\infty$ and *unbounded y-fibers* if $b \equiv +\infty$.

- A 1-bounded subanalytic map $\psi : B \longrightarrow \mathbb{R}^{M+2} \in \mathcal{S}_c^{M+2}(B)$ is *y-prepared* if it has the form

$$\psi(x, y) = \left(c(x), \left(\frac{a(x)}{y} \right)^{\frac{1}{d}}, \left(\frac{y}{b(x)} \right)^{\frac{1}{d}} \right), \tag{3.3}$$

  where $d \in \mathbb{N}$.
  If $b \equiv +\infty$, then we will implicitly assume that the last component is missing and hence $\psi : B \longrightarrow \mathbb{R}^{M+1}$.
- An $\mathbb{F}_{\mathbb{K}}$-valued subanalytic strong function $W : B \longrightarrow \mathbb{F}_{\mathbb{K}}$ is *ψ-prepared* if ψ is a y-prepared 1-bounded subanalytic map as in (3.3) and

$$W(x, y) = F \circ \psi(x, y),$$

  for some power series $F \in \mathbb{F}_{\mathbb{K}} [\![Z]\!]$ which converges in a neighbourhood of the ball of radius $\frac{3}{2}$. Notice that W has also a nested presentation (see 3.7) as a strongly convergent power series with coefficients $\mathbb{F}_{\mathbb{K}}$-valued subanalytic strong functions on X, evaluated at $\gamma(x, y) = \left(\left(\frac{a(x)}{y} \right)^{\frac{1}{d}}, \left(\frac{y}{b(x)} \right)^{\frac{1}{d}} \right)$
- A subanalytic function $f \in \mathcal{S}(B)$ is *prepared* if there are $\nu \in \mathbb{Z}$, an analytic function $f_0 \in \mathcal{S}(X)$ and a ψ-prepared real-valued subanalytic strong unit U (for some ψ as in (3.3)) such that

$$f(x, y) = f_0(x)\, y^{\frac{\nu}{d}} U(x, y)$$

Let us recall some notation from [CCM+18, Definitions 3.2, 3.3, 3.4 and 3.8]. In particular, $A \subseteq \mathbb{R}^{m+1}$ will be a cell open over $\mathbb{R}^m$ (it will always be possible to suppose that the base of A is $X \subseteq \mathbb{R}^m$) with analytic subanalytic center θ_A and such that the set $I_A := \{y - \theta_A(x) : (x, y) \in A\}$ is contained in one of the sets $(-\infty, -1), (-1, 0), (0, 1), (1, +\infty)$, as in [CCM+18, Definition 3.4]. We now perform a change of coordinates with the aim of mapping the set I_A to the interval $(1, +\infty)$: there are unique sign conditions $\sigma_A, \tau_A \in \{-1, 1\}$ such that

$$A = \{(x, y) : x \in X, \ a_A(x) < \sigma_A (y - \theta_A(x))^{\tau_A} < b_A(x)\} \tag{3.4}$$

for some analytic subanalytic functions a_A, b_A such that $1 \leq a_A(x) < b_A(x) \leq +\infty$. Let

$$B_A = \{(x, y) : x \in X, \ a_A(x) < y < b_A(x)\} \tag{3.5}$$

and $\Pi_A : B_A \longrightarrow A$ be the bijection

$$\Pi_A(x, y) = (x, \sigma_A y^{\tau_A} + \theta_A(x)), \ \Pi_A^{-1}(x, y) = (x, \sigma_A (y - \theta_A(x))^{\tau_A}). \tag{3.6}$$

We will still denote by Π_A the map $\mathbb{C} \times B_A \ni (s, x, y) \longmapsto (s, \Pi_A(x, y)) \in \mathbb{C} \times A$.

Remark 3.10. By [CCM+18, Definition 3.4(3)], if A is a cell of the form $A = \{(x, y) : x \in X, \ y > f(x)\}$, then $\sigma_A = \tau_A = 1$ and $\theta_A = 0$. Hence in this case $a_A = f$, $b_A = +\infty$ and $B_A = A$.

Proposition 3.11. [LR97][CCM+18, Remark 3.12]*Let $\mathcal{F} \subseteq \mathcal{S}(X \times \mathbb{R})$ be a finite collection of subanalytic functions. There is a cell decomposition of $\mathbb{R}^{m+1}$ compatible with X such that for each cell A that is open over $\mathbb{R}^m$ (which we may suppose to be of the form* (3.4)*) and for every $h \in \mathcal{F}$, $h \circ \Pi_A$ is prepared on B_A.*

4. Preparation of (parametric) power-constructible functions

Let $\mathbb{K} \subseteq \mathbb{C}$ be a subfield and recall that $\mathbb{F}_{\mathbb{K}}$ is $\mathbb{R}$ if $\mathbb{K} \subseteq \mathbb{R}$ and $\mathbb{C}$ otherwise. In this section, y will be a single variable. For each of the classes introduced in Sections 2.1 and 2.3, we will give a *prepared presentation* of its elements, with respect to the last subanalytic variable (denoted by y).

4.1. Preparation of power-constructible functions.

Definition 4.1. Let B be as in (3.2). A generator T of the $\mathbb{F}_{\mathbb{K}}$-algebra $\mathcal{C}^{\mathbb{K}}(B)$ is called *prepared* if

$$T(x,y) = G_0(x)\, y^{\frac{\eta}{d}} (\log y)^{\mu} W(x,y), \tag{4.1}$$

where $G_0 \in \mathcal{C}^{\mathbb{K}}(X)$, $\eta \in \mathbb{K}$, $\mu \in \mathbb{N}$ and W is a ψ-prepared $\mathbb{F}_{\mathbb{K}}$-valued subanalytic strong function, for some 1-bounded ψ as in (3.3).

It follows from Remark 3.8 that, in the notation of (3.1), if B has bounded y-fibers (i.e. $b < +\infty$), then W can be written as

$$\sum_{m,n} \xi^c_{m,n}(x) \left(\frac{a(x)}{y}\right)^{\frac{m}{d}} \left(\frac{y}{b(x)}\right)^{\frac{n}{d}}, \tag{4.2}$$

and if B has unbounded y-fibers (i.e. $b \equiv +\infty$), then W can be written as

$$\sum_{k} \xi^c_{k}(x) \left(\frac{a(x)}{y}\right)^{\frac{k}{d}}. \tag{4.3}$$

Proposition 4.2. *Let $\mathcal{F} \subseteq \mathcal{C}^{\mathbb{K}}(X \times \mathbb{R})$ be a finite collection of $\mathbb{K}$-power-constructible functions. Then there is a cell decomposition of $\mathbb{R}^{m+1}$ compatible with X such that for each cell A that is open over $\mathbb{R}^m$ (which we may suppose to be of the form* (3.4)*) and each $h \in \mathcal{F}$, $h \circ \Pi_A$ is a finite sum of prepared generators of the form* (4.1)*.*

Proof. The proof is a straightforward refinement of the proofs of [CM12, Corollary 3.5] and [CCM+18, Proposition 3.10]: one prepares first all the subanalytic data appearing in h, by Proposition 3.11, and then observes the effect of applying log or a power $\eta \in \mathbb{K}$ to a subanalytic prepared function. In particular, notice that if $U(x,y)$ is a ψ-prepared subanalytic strong unit, then U^{η} is again a ψ-prepared $\mathbb{F}_{\mathbb{K}}$-valued subanalytic strong unit. $\square$

4.2. Preparation of parametric strong functions.

Let $\mathbb{K} \subseteq \mathbb{C}$ be a subfield and refer to the definitions of $\mathcal{E}_{\mathbb{K}}, \mathcal{A}_{\mathbb{K}}$ in Section 2.3.1.

Definition 4.3. Let B be as in (3.2). A parametric strong function $\Phi \in \mathcal{A}_{\mathbb{K}}(B)$ is called *ψ-prepared* (where ψ is as in (3.3)) if there exists a strongly convergent series $F = \sum \xi_I(s) Z^I \in \mathcal{E}_{\mathbb{K}}[\![Z]\!]$ such that

$$\forall (s,x,y) \in (\mathbb{C} \setminus P(F)) \times B,\ \Phi(s,x,y) = F \circ (s, \psi(x,y)). \tag{4.4}$$

Notice that if Φ is ψ-prepared, then Φ has a nested presentation (see Remark 3.7) as a power series with coefficients in $\mathcal{A}_{\mathbb{K}}(X)$, evaluated at $\gamma(x,y) = \left(\left(\frac{a(x)}{y}\right)^{\frac{1}{d}}, \left(\frac{y}{b(x)}\right)^{\frac{1}{d}}\right)$:

$$\forall (s,x,y) \in \mathbb{C} \setminus P(F) \times B,\ \Phi(s,x,y) = \sum_{m,n} \xi^c_{m,n}(s,x) \left(\frac{a(x)}{y}\right)^{\frac{m}{d}} \left(\frac{y}{b(x)}\right)^{\frac{n}{d}}, \tag{4.5}$$

where $\xi^c_{m,n}(s,x) = \sum_J \xi_{J,m,n}(s)(c(x))^J \in \mathcal{A}_{\mathbb{K}}(X)$.

Remark 4.4. Let $\Phi \in \mathcal{A}_{\mathbb{K}}(B)$ be ψ-prepared, as above. If B has unbounded y-fibers (i.e. $b \equiv +\infty$ in (3.2)), recall that

$$\psi(x,y) = \left(c(x), \left(\frac{a(x)}{y}\right)^{\frac{1}{d}}\right), \tag{4.6}$$

hence the nested ψ-prepared form of Φ is

$$\forall (s,x,y) \in (\mathbb{C} \setminus P(F)) \times B,\ \Phi(s,x,y) = \sum_{k} \xi^c_{k}(s,x) \left(\frac{a(x)}{y}\right)^{\frac{k}{d}}, \tag{4.7}$$

where $\xi^c_{k}(s,x) = \sum_J \xi_{J,k}(s)(c(x))^J \in \mathcal{A}_{\mathbb{K}}(X)$.

Lemma 4.5. *Let $\mathcal{F} \subseteq \mathcal{A}_{\mathbb{K}}(X \times \mathbb{R})$ be a finite set of functions Φ which have no poles outside some closed discrete set $P \subseteq \mathbb{K}$. Then there is a cell decomposition of $\mathbb{R}^{m+1}$ compatible with X such that for each cell A that is open over $\mathbb{R}^m$ (which we may suppose to be of the form* (3.4)*), each $\Phi \circ \Pi_A$ is ψ-prepared on $(\mathbb{C} \setminus P) \times B_A$ (for some y-prepared 1-bounded subanalytic map ψ as in* (3.3)*).*

Proof. We will consider the case of a single function Φ for simplicity of notation (the general case is obtained by taking as Φ the product of the functions in $\mathcal{F}$). Write $\Phi = G \circ (s,\eta)$, where $G = \sum_I \varphi_I(s) T^I \in \mathcal{E}[\![T]\!]$ is a strongly convergent series in N variables T and $\eta = (\eta_1, \ldots, \eta_N) : X \times \mathbb{R} \longrightarrow \mathbb{R}^N$ is a 1-bounded subanalytic map.

Apply subanalytic preparation (Proposition (3.11)) to the components of η. This yields a cell decomposition of $X \times \mathbb{R}$ such that, if A is a cell of the form (3.4), then the components of $\eta \circ \Pi_A$ are $\hat{\psi}$-prepared B_A, where $\hat{\psi}(x,y) = \left(\hat{c}(x), \left(\frac{a_A(x)}{y}\right)^{\frac{1}{d}}, \left(\frac{y}{b_A(x)}\right)^{\frac{1}{d}}\right)$ is a y-prepared strongly subanalytic map:

$$\eta_j \circ \Pi_A(x,y) = c_j(x)\, y^{\frac{\ell_j}{d}} v_j(x,y) \quad (1 \leq j \leq N),$$

where $c_j \in \mathcal{S}(X)$ is analytic, ℓ_j is an integer and v_j is a $\hat{\psi}$-prepared strong unit. By rescaling the unit, we may furthermore assume that $\left|c_j(x)\, y^{\frac{\ell_j}{d}}\right| \leq 1$ on the closure of B_A. Partition

$$\begin{aligned} \{1, \ldots, N\} &= \bigcup_{* \in \{<,=,>\}} J_* \\ &= \bigcup_{* \in \{<,=,>\}} \left\{ j : \ \frac{\ell_j}{d} * 0 \right\} \end{aligned}$$

and notice that the subanalytic map $\tilde{c} := (\tilde{c}_1, \dots, \tilde{c}_N)$ given by

$$\tilde{c}_j(x) := \begin{cases} c_j(x) \cdot (a_A(x))^{\frac{\ell_j}{d}} & (j \in J_<) \\ c_j(x) & (j \in J_=) \\ c_j(x) \cdot (b_A(x))^{\frac{\ell_j}{d}} & (j \in J_>) \end{cases}$$

is 1-bounded. Hence,

$$\eta_j \circ \Pi_A(x,y) = \begin{cases} \tilde{c}_j(x) \left(\frac{a_A(x)}{y}\right)^{-\frac{\ell_j}{d}} v_j(x,y) & (j \in J_<) \\ \tilde{c}_j(x)\, v_j(x,y) & (j \in J_=) \\ \tilde{c}_j(x) \left(\frac{y}{b_A(x)}\right)^{\frac{\ell_j}{d}} v_j(x,y) & (j \in J_>) \end{cases}$$

and, for $I = (i_1, \dots, i_N) \in \mathbb{N}^N$,

$$(\eta_j \circ \Pi_A(x,y))^{i_j} = \tilde{c}_j(x)^{i_j} f_{I,j}(x,y),$$

where

$$f_{I,j}(x,y) = \begin{cases} \left(\frac{a_A(x)}{y}\right)^{-\frac{\ell_j}{d} i_j} (v_j(x,y))^{i_j} & (j \in J_<) \\ (v_j(x,y))^{i_j} & (j \in J_=) \\ \left(\frac{y}{b_A(x)}\right)^{\frac{\ell_j}{d} i_j} (v_j(x,y))^{i_j} & (j \in J_>) \end{cases}.$$

Notice that the $f_{I,j}$ are $\hat{\psi}$-prepared subanalytic strong functions, hence so is their product $f_I(x,y) := \prod_{j \leq N} f_{I,j}(x,y)$. Therefore, there is a strongly convergent power series with coefficients in $\mathbb{F}_{\mathbb{K}}$

$$F_I = \sum_{K,m,n} d^I_{K,m,n} \tilde{Z}^K Y_1^m Y_2^n \in \mathbb{F}_{\mathbb{K}} \left[\!\left[\tilde{Z}, Y_1, Y_2 \right]\!\right]$$

such that $f_I(x,y) = F_I \circ \hat{\psi}(x,y)$.

Therefore, on B_A we can write

$$\begin{aligned} &\Phi \circ \Pi_A(s,x,y) = \\ &= \sum_{I=(i_1,\dots,i_N)} \varphi_I(s) \left(\eta \circ \Pi_A(x,y)\right)^I \\ &= \sum_{I=(i_1,\dots,i_N)} \varphi_I(s) \left(\tilde{c}(x)\right)^I f_I(x,y) \\ &= \sum_{I=(i_1,\dots,i_N)} \varphi_I(s) \left(\tilde{c}(x)\right)^I \sum_{K,m,n} d^I_{K,m,n} \left(\hat{c}(x)\right)^K \left(\frac{a_A(x)}{y}\right)^{\frac{m}{d}} \left(\frac{y}{b_A(x)}\right)^{\frac{n}{d}} \\ &= \sum_{I,K,m,n} d^I_{K,m,n} \varphi_I(s) \left(\tilde{c}(x)\right)^I \left(\hat{c}(x)\right)^K \left(\frac{a_A(x)}{y}\right)^m \left(\frac{y}{b_A(x)}\right)^n. \end{aligned}$$

Now, if we let $\tilde{I} = (I, K)$ and

$$\xi_{\tilde{I},m,n}(s) = d^I_{K,m,n} \varphi_I(s),$$

then the family $\left\{\xi_{\tilde{I},m,n}\right\}$ is strong and the series

$$F=\sum_{\tilde{I},m.n}\xi_{\tilde{I},m,n}\left(s\right)Z^{\tilde{I}}Y_1^mY_2^n\in\mathcal{E}_{\mathbb{K}}\left[\!\left[Z,Y_1,Y_2\right]\!\right]$$

is strongly convergent (with $P\left(F\right)=P\left(G\right)$). Let $c\left(x\right)=\left(\tilde{c}\left(x\right),\hat{c}\left(x\right)\right)$. Then, in the notation of (3.3), on B_A we have

$$\Phi\circ\Pi_A\left(s,x,y\right)=F\circ\left(s,\psi\left(x,y\right)\right),$$

so $\Phi\circ\left(s,\Pi_A\left(x,y\right)\right)$ is ψ-prepared on B_A, as required. □

4.3. **Preparation of parametric power-constructible functions.** In this section we let $\mathcal{D}$ be either $\mathcal{C}^{\mathbb{K},\mathcal{M}}$ or $\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$ (see Section 2.3.1).

Definition 4.6. Let B be as in (3.2) and $P\subseteq\mathbb{K}$ be a closed discrete set. A generator $T\in\mathcal{D}\left(B\right)$ with no poles outside P is *prepared* if for all $(s,x,y)\in(\mathbb{C}\setminus P)\times B$,

$$T\left(s,x,y\right)=G_0\left(s,x\right)y^{\frac{\ell s+\eta}{d}}\left(\log y\right)^{\mu}\Phi\left(s,x,y\right),\tag{4.8}$$

where $G_0\in\mathcal{D}\left(X\right)$, $\ell,\eta\in\mathbb{K}$, $\mu\in\mathbb{N}$ and $\Phi\in\mathcal{A}_{\mathbb{K}}\left(B\right)$ is a ψ-prepared parametric strong function (see Definition 4.3). If $\mathcal{D}=\mathcal{C}^{\mathbb{K},\mathcal{M}}$, then we require that $\ell\in\mathbb{Z}$.

Proposition 4.7. *Let $P\subseteq\mathbb{K}$ be a closed discrete set and $h\in\mathcal{D}\left(X\times\mathbb{R}\right)$ have no poles outside P. Then there is a cell decomposition of $\mathbb{R}^{m+1}$ compatible with X such that for each cell A that is open over $\mathbb{R}^m$ (which we may suppose to be of the form (3.4)), $h\circ\Pi_A$ is a finite sum of prepared generators on $(\mathbb{C}\setminus P)\times B_A$.*

Proof. Suppose first that $\mathcal{D}=\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$. Write h as a finite sum of generators of the form

$$T\left(s,x,y\right)=\Phi\left(s,x,y\right)\cdot g\left(x,y\right)\cdot f_1\left(x,y\right)^{\alpha_1 s}\cdot\ldots\cdot f_n\left(x,y\right)^{\alpha_n s},$$

with $\Phi\in\mathcal{A}_{\mathbb{K}}\left(X\times\mathbb{R}\right)$, $g\in\mathcal{C}^{\mathbb{K}}\left(X\times\mathbb{R}\right)$, $f_i\in\mathcal{S}_+\left(X\times\mathbb{R}\right)$, $\alpha_i\in\mathbb{K}$. Apply Proposition 3.11 simultaneously to all the f_i and to all the subanalytic data in all the Φ and g appearing in the generators. This yields a cell decomposition of $X\times\mathbb{R}$ such that on each cell A with center θ_A, there is a y-prepared subanalytic map ψ as in (3.3) such that, after composing with Π_A all the subanalytic functions considered above are prepared. In particular, each of the f_j appearing in the parametric power, after composing with Π_A, has the form

$$\tilde{f}_j\left(x\right)y^{\frac{\ell_j}{d}}U_j\left(x,y\right),$$

where $\tilde{f}_j\in\mathcal{S}_+\left(X\right)$, $\ell_j\in\mathbb{Z}$ and $U_j\in\mathcal{S}\left(B_A\right)$ is a ψ-prepared subanalytic strong unit. Hence, by the second of Examples 3.5, $\Xi_j\left(s,x,y\right):=\left|U_j\left(x,y\right)\right|^s\in\mathcal{A}_{\mathbb{K}}\left(B_A\right)$ and is ψ-prepared.

Apply Proposition 4.2 to prepare each $g\circ\Pi_A$, which can be hence written as a finite sum of terms of the form

$$g_j\left(x\right)y^{\frac{\eta_j}{d}}\left(\log y\right)^{\nu_j}W_j\left(x,y\right),$$

where $\nu_j\in\mathbb{N}$, $\eta_j\in\mathbb{K}$, $g_j\in\mathcal{C}^{\mathbb{K}}\left(X\right)$ is analytic and W_j is an $\mathbb{F}_{\mathbb{K}}$-valued ψ-prepared subanalytic strong function on B_A.

Apply Lemma 4.5 to ψ-prepare each $\Phi\circ\Pi_A$ on B_A as $F_j\circ(s,\psi(x,y))$, where ψ has now some extra components depending only on the variables x. Notice that this does not affect the preparation work already done.

Finally, define $G_j(s,x) = \tilde{f}_j(x)^{\alpha_j s} g_j(x)$ and $\Phi_j(s,x,y) = F_j \circ (s, \psi(x,y)) \cdot W_j(x,y) \cdot \Xi_j(s,x,y)$. Then clearly $G_j \in \mathcal{C}^{\mathbb{K},\mathcal{M}}(X)$ and $\Phi_j \in \mathcal{A}_{\mathbb{K}}(B_A)$ is ψ-prepared, with no poles outside P, hence we have written $h \circ \Pi_A$ as a finite sum of terms of the form

$$G_j(s,x) \cdot y^{\frac{\alpha_j \ell_j s + \eta_j}{d}} \cdot (\log y)^{\nu_j} \Phi_j(s,x,y)$$

and we are done.

If $\mathcal{D} = \mathcal{C}^{\mathbb{K},\mathcal{M}}$, then repeat the above proof with $n = \alpha_1 = 1$. □

5. Integration of prepared (parametric) power-constructible generators

In this section we let $\mathcal{D}$ be either $\mathcal{C}^{\mathbb{K},\mathcal{M}}$ or $\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$.

Given a cell $B \subseteq \mathbb{R}^{m+1}$, we study the integrability, and compute the integral, of a prepared generator of $\mathcal{D}(B)$.

Let B be as in (3.2) and $T \in \mathcal{D}(B)$ be a prepared generator with no poles outside P (for some discrete closed set $P \subseteq \mathbb{K}$). We aim to study the nature of the parametric integral

$$\int_{a(x)}^{b(x)} T(s,x,y)\,\mathrm{d}y, \tag{5.1}$$

for all $(s,x) \in (\mathbb{C} \setminus P) \times X$ such that $y \longmapsto T(s,x,y) \in L^1(B_x)$.

We prove that there exist a closed discrete set $P' \supseteq P$ and a function $H \in \mathcal{D}(X)$ with no poles outside P' such that the above integral coincides with H.

We start by recalling the classical formula to compute the antiderivative of any power-log monomial in y.

Lemma 5.1. *Let $\ell, \gamma \in \mathbb{K}$, $d, \mu \in \mathbb{N}$ with $\ell, d \neq 0$. Let $s \in \mathbb{C}$ such that $\ell s + \gamma \neq -d$. Then*

$$\int y^{\frac{\ell s+\gamma}{d}} (\log y)^{\mu}\,\mathrm{d}y = \sum_{i=0}^{\mu} c_{\mu,i} (\log y)^{i} \frac{y^{\frac{\ell s+\gamma+d}{d}}}{(\ell s+\gamma+d)^{\mu+1-i}}, \tag{5.2}$$

where $c_{\mu,i} = (-1)^{\mu-i} \frac{\mu!}{i!} d^{\mu+1-i}$.

5.1. **Cells with bounded y-fibers.** Recall that B is as in (3.2) and suppose that $b < +\infty$. Let $T \in \mathcal{D}(B)$ be a prepared generator (as in (4.8)) without poles outside some closed discrete set $P \subseteq \mathbb{K}$. We study the integrability of T on B: since B has bounded y-fibers, the function $y \longmapsto T(s,x,y)$ extends to a continuous function on the boundary of B_x, hence the integral

$$\int_{a(x)}^{b(x)} T(s,x,y)\,\mathrm{d}y$$

is finite. Let us compute it.

Let

$$P' := \begin{cases} P \cup \{s :\ \ell s + \eta \in \mathbb{Z}\} & \text{if } \ell \neq 0, \\ P & \text{if } \ell = 0. \end{cases}$$

There are several cases to consider.

• If $\ell \neq 0$, then, for $(s,x) \in (\mathbb{C} \setminus P') \times X$, we deduce from Lemma 5.1 and normal convergence that

$$
\begin{aligned}
&\int_{a(x)}^{b(x)} T(s,x,y)\,\mathrm{d}y \\
&= \sum_{i=0}^{\mu}\sum_{m,n} \frac{c_{\mu,i} G_0(s,x)\,\xi_{m,n}(s,x)}{(\ell s+\eta+d-m+n)^{\mu+1-i}} \frac{(a(x))^{\frac{m}{d}}}{(b(x))^{\frac{n}{d}}} \cdot \left[y^{\frac{\ell s+\eta+d-m+n}{d}} (\log y)^i \right]_{a(x)}^{b(x)} \\
&= \sum_{i=0}^{\mu} c_{\mu,i} G_0(s,x)\,(b(x))^{\frac{\ell s+\eta+d}{d}} (\log b(x))^i \sum_{m,n} \frac{\xi_{m,n}(s,x)\left(\frac{a(x)}{b(x)}\right)^{\frac{m}{d}}}{(\ell s+\eta+d-m+n)^{\mu+1-i}} \\
&- \sum_{i=0}^{\mu} c_{\mu,i} G_0(s,x)\,(a(x))^{\frac{\ell s+\eta+d}{d}} (\log a(x))^i \sum_{m,n} \frac{\xi_{m,n}(s,x)\left(\frac{a(x)}{b(x)}\right)^{\frac{n}{d}}}{(\ell s+\eta+d-m+n)^{\mu+1-i}}
\end{aligned}
\tag{5.3}
$$

As a consequence of the Dominated Convergence Theorem, the fact that $\forall x \in X,\ 1 \leq a(x) < b(x) < +\infty$ and the results in Section 3.2, the expressions

$$
\sum_{m,n} \frac{\xi_{m,n}(s,x)\left(\frac{a(x)}{b(x)}\right)^{\frac{m}{d}}}{(\ell s+\eta+d-m+n)^{\mu+1-i}}, \qquad \sum_{m,n} \frac{\xi_{m,n}(s,x)\left(\frac{a(x)}{b(x)}\right)^{\frac{n}{d}}}{(\ell s+\eta+d-m+n)^{\mu+1-i}}
$$

define functions in $\mathcal{A}(X)$ without poles outside P'.

• If $\ell = 0$ and $\eta \notin \mathbb{Z}$, then the above equation holds for all $(s,x) \in (\mathbb{C}\setminus P)\times X$, since the denominator does not vanish.

• If $\ell = 0$ and $\eta \in \mathbb{Z}$, then we split Φ into the sum of two (still strongly convergent) series, by isolating the indices which contribute, in T, to the power y^{-1}:

$$
\begin{aligned}
&\Phi(s,x,y) = \Phi_{=}(s,x,y) + \Phi_{\neq}(s,x,y) \\
&= \sum_{\substack{m,n:\\ m=\eta+d+n}} \xi_{m,n}(s,x) \left(\frac{a(x)}{y}\right)^{\frac{m}{d}} \left(\frac{y}{b(x)}\right)^{\frac{n}{d}} + \sum_{\substack{m,n:\\ m\neq\eta+d+n}} \xi_{m,n}(s,x) \left(\frac{a(x)}{y}\right)^{\frac{m}{d}} \left(\frac{y}{b(x)}\right)^{\frac{n}{d}} \\
&= y^{-\frac{\eta+d}{d}} (a(x))^{\frac{\eta+d}{d}} \sum_{n} \xi_{n+\eta+d,n}(s,x) \left(\frac{a(x)}{b(x)}\right)^{\frac{n}{d}} + \sum_{\substack{m,n:\\ m\neq\eta+d+n}} \xi_{m,n}(s,x) \left(\frac{a(x)}{y}\right)^{\frac{m}{d}} \left(\frac{y}{b(x)}\right)^{\frac{n}{d}}.
\end{aligned}
$$

The integral of $T_{\neq}(s,x,y) := G_0(s,x)\,y^{\frac{\eta}{d}} (\log y)^{\mu}\,\Phi_{\neq}(s,x,y)$ is computed as in the previous cases, and the denominators never vanish.

As for $T_{=}(s,x,y) := G_0(s,x)\,y^{\frac{\eta}{d}} (\log y)^{\mu}\,\Phi_{=}(s,x,y)$, for $(s,x) \in (\mathbb{C}\setminus P)\times X$, we have

$$
\begin{aligned}
\int_{a(x)}^{b(x)} T_{=}(s,x,y)\,\mathrm{d}y &= G_0(s,x)\,(a(x))^{\frac{\eta+d}{d}} \sum_{n} \xi_{n+\eta+d,n}(s,x) \left(\frac{a(x)}{b(x)}\right)^{\frac{n}{d}} \frac{(\log b(x))^{\mu+1}}{\mu+1} \\
&\quad - G_0(s,x)\,(a(x))^{\frac{\eta+d}{d}} \sum_{n} \xi_{n+\eta+d,n}(s,x) \left(\frac{a(x)}{b(x)}\right)^{\frac{n}{d}} \frac{(\log a(x))^{\mu+1}}{\mu+1}.
\end{aligned}
$$

Hence we have shown that there is $H \in \mathcal{D}(X)$ without poles outside some closed discrete set $P' \supseteq P$, such that

$$\forall (s,x) \in (\mathbb{C} \setminus P') \times X,\ H(s,x) = \int_{a(x)}^{b(x)} T(s,x,y)\, \mathrm{d}y.$$

Remark 5.2. If $\ell = 0$ then H has no new singularities. If $\ell \neq 0$, let $\sigma \in P' \setminus P$. Since for all $(x,y) \in B$, the function $s \longmapsto T(s,x,y)$ is holomorphic and bounded in a neighbourhood of σ, by differentiation under the integral sign, the integral $\int_{a(x)}^{b(x)} T(s,x,y)\, \mathrm{d}y$ is also holomorphic in a neighbourhood of σ. Since such an integral coincides with H on a deleted neighbourhood of σ and $s \longmapsto H(s,x)$ is meromorphic, σ is not a pole of H but a removable singularity. Hence,

$$\begin{aligned} H_\sigma(x) &:= \lim_{s \longrightarrow \sigma} H(s,x) = \lim_{s \longrightarrow \sigma} \int_{a(x)}^{b(x)} T(s,x,y)\, \mathrm{d}y \\ &= \int_{a(x)}^{b(x)} \lim_{s \longrightarrow \sigma} T(s,x,y)\, \mathrm{d}y = \int_{a(x)}^{b(x)} T(\sigma,x,y)\, \mathrm{d}y. \end{aligned}$$

The rightmost integral can be computed in a similar way as we did above for the case $\ell = 0, \eta \in \mathbb{Z}$ (where now we split the series according to the condition $m = \ell\sigma + \eta + d + n$) and the computation clearly shows that $H_\sigma \in \mathcal{C}^{\mathbb{K}}(X)$.

Finally, notice that every $\sigma \in P' \setminus P$ has the form $\sigma = \frac{\nu_0 - \eta - d}{\ell}$ for some $\nu_0 \in \mathbb{Z}$, so that if ℓ and/or η are in $\mathbb{K}$, then so is σ.

Hence, we have proven the following statement.

Proposition 5.3. *Let B be as in (3.2) with $b < +\infty$, $\mathbb{K} \subseteq \mathbb{C}$ be a subfield and let $\mathcal{D}$ be either $\mathcal{C}^{\mathbb{K},\mathcal{M}}$ or $\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$. Let $T \in \mathcal{D}(B)$ be a prepared generator with no poles outside P (for some discrete closed set $P \subseteq \mathbb{K}$), as in Definition 4.6. Let*

$$P' = P \cup \{s \in \mathbb{C} :\ \ell s + \eta \in \mathbb{Z}\} \subseteq \mathbb{K}.$$

Then

$$\mathrm{Int}(T; (\mathbb{C} \setminus P) \times X) = (\mathbb{C} \setminus P) \times X$$

and there exist a function $H \in \mathcal{D}(X)$ without poles outside P' such that

$$\forall (s,x) \in (\mathbb{C} \setminus P') \times X,\ H(s,x) = \int_{a(x)}^{b(x)} T(s,x,y)\, \mathrm{d}y.$$

Moreover, for all $\sigma \in P' \setminus P$ there is a function $H_\sigma \in \mathcal{C}^{\mathbb{K}}(X)$ such that

$$\forall x \in X,\ H_\sigma(x) = \int_{a(x)}^{b(x)} T(\sigma,x,y)\, \mathrm{d}y$$

and $\forall x \in X$, the function $s \longmapsto H(s,x)$ can be holomorphically extended at $s = \sigma$ by setting $H(\sigma,x) = H_\sigma(x)$.

Remark 5.4. The proposition also applies to any finite sum of prepared generators on the bounded cell B, with P' a finite union of closed and discrete sets and $P' \setminus P$ contained in a finitely generated $\mathbb{Z}$-lattice.

5.2. **Cells with unbounded y-fibers.** We now introduce a type of function in $\mathcal{D}(X\times\mathbb{R})$ which has a particularly simple expression in the last variable y.

Definition 5.5. Let $A\subseteq X\times\mathbb{R}$ be a subanalytic cell which is open over X (see Definition 3.1). A function $h\in\mathcal{D}(A)$ without poles outside some closed discrete set $P\subseteq\mathbb{K}$ is *Puiseux in y* if there are $\ell,\eta\in\mathbb{K},\ d\in\mathbb{N}\setminus\{0\},\ \mu\in\mathbb{N}$ and a collection $\{g_k(s,x)\}_{k\in\mathbb{N}}\subseteq\mathcal{D}(X)$ such that for all $s\in\mathbb{C}\setminus P$, the series of functions

$$\varphi(s,x,y):=\sum_k g_k(s,x)\,y^{-\frac{k}{d}}$$

converges normally on A and $\forall(x,y)\in A,\ \mathbb{C}\setminus P\ni s\longmapsto\varphi(s,x,y)$ is holomorphic, and

$$h(s,x,y)=\varphi(s,x,y)\,y^{\frac{\ell s+\eta}{d}}(\log y)^{\mu}=\sum_k g_k(s,x)\,y^{\frac{\ell s+\eta-k}{d}}(\log y)^{\mu}. \tag{5.4}$$

We call the tuple (ℓ,η,d,μ) the *Puiseux data* of h.

Remark 5.6. Let B be as in (3.2) and $T\in\mathcal{D}(B)$ be a prepared generator (for some y-prepared 1-bounded subanalytic map ψ as in (3.3)). If B has unbounded y-fibers, then T is Puiseux in y.

We now turn our attention to prepared generators of $\mathcal{D}(B)$, where, in the definition (3.2) of B, we have $b\equiv+\infty$. More generally, in what follows we will suppose that $T\in\mathcal{D}(B)$ is a finite sum of prepared generators (where ψ is as in (4.6)), sharing the same Puiseux data and without poles outside some closed discrete set $P\subseteq\mathbb{K}$. Hence, for some $\ell,\eta\in\mathbb{K},\ \mu\in\mathbb{N}$, T has the form

$$\begin{aligned}T(s,x,y)&=\sum_{j\le N}T_j(s,x,y)\\&=\sum_{j\le N}G_j(s,x)\,y^{\frac{\ell s+\eta}{d}}(\log y)^{\mu}\sum_k\xi_{j,k}(s,x)\left(\frac{a(x)}{y}\right)^{\frac{k}{d}}\\&=y^{\frac{\ell s+\eta}{d}}(\log y)^{\mu}\sum_k h_k(s,x)\left(\frac{a(x)}{y}\right)^{\frac{k}{d}},\end{aligned}\tag{5.5}$$

where $h_k=\sum_{j\le N}G_j\xi_{j,k}\in\mathcal{D}(X)$.

First, we describe $\operatorname{Int}(T;(\mathbb{C}\setminus P)\times X)$. Let $m_k(s,y)=y^{\frac{\ell s+\eta-k}{d}}(\log y)^{\mu}$ and notice that, since $a(x)\ge1$ and since for all $s\in\mathbb{C}$ the real parts of the exponent of y in m_k and $m_{k'}$ are different if $k\ne k'$,

$$\operatorname{Int}(T;(\mathbb{C}\setminus P)\times X)=\bigcap_{k\in\mathbb{N}}\operatorname{Int}(h_k m_k;(\mathbb{C}\setminus P)\times X).$$

- If $\ell\ne0$ then

$$\begin{aligned}\operatorname{Int}(h_k m_k;(\mathbb{C}\setminus P)\times X)=&\ \{s\in\mathbb{C}\setminus P:\ \Re(\ell s+\eta)+d-k<0\}\times X\\&\cup\{(s,x)\in(\mathbb{C}\setminus P)\times X:\ \Re(\ell s+\eta)+d-k\ge0\wedge h_k(s,x)=0\}\end{aligned}$$

and hence, if

(5.6)
$$S_0 = \{s \in \mathbb{C} : \ \Re(\ell s + \eta) + d < 0\} \text{ and } S_i = \{s \in \mathbb{C} : \ i - 1 \leq \Re(\ell s + \eta) + d < i\} \ (i \geq 1),$$

then

$$\operatorname{Int}(T; (\mathbb{C} \setminus P) \times X) = (S_0 \times X) \cup \bigcup_{i \geq 1} \left\{(s,x) \in (S_i \setminus P) \times X : \ \bigwedge_{k<i} h_k(s,x) = 0\right\}. \tag{5.7}$$

- If $\ell = 0$ then

$$\operatorname{Int}(h_k m_k; (\mathbb{C} \setminus P) \times X) = \begin{cases} (\mathbb{C} \setminus P) \times X & \text{if } \Re(\eta) + d - k < 0 \\ \{(s,x) \in (\mathbb{C} \setminus P) \times X : \ h_k(s,x) = 0\} & \text{if } \Re(\eta) + d - k \geq 0 \end{cases}$$

and hence, if $k_0 = \lfloor \Re(\eta) \rfloor + d$, then

$$\operatorname{Int}(T; (\mathbb{C} \setminus P) \times X) = \left\{(s,x) \in (\mathbb{C} \setminus P) \times X : \ \bigwedge_{k \leq k_0} h_k(s,x) = 0\right\}. \tag{5.8}$$

Let

$$P' = \begin{cases} P \cup \{s \in \mathbb{C} : \ \Re(\ell s + \eta) + d \in \mathbb{N}\} & \text{if } \ell \neq 0 \\ P & \text{if } \ell = 0 \end{cases} \subseteq \mathbb{K}. \tag{5.9}$$

Notice that $(P' \setminus P) \cap S_0 = \emptyset$.

Our next aim is to show that there exists $H \in \mathcal{D}(X)$, with no poles outside P' such that H coincides with the integral of T on its integration locus.

In the notation of Lemma 5.1, let

$$H_k(s,x) = -(a(x))^{\frac{\ell s + \eta + d}{d}} \sum_{i \leq \mu} c_{\mu,i} (\log a(x))^i \frac{h_k(s,x)}{(\ell s + \eta + d - k)^{\mu+1-i}},$$

and define

$$H(s,x) = \begin{cases} \displaystyle\sum_{k \geq 0} H_k(s,x) & \text{if } \ell \neq 0 \\ \displaystyle\sum_{k > k_0} H_k(s,x) & \text{if } \ell = 0 \end{cases}.$$

By the results in Section 3.2, $H \in \mathcal{D}(X)$ and has no poles outside P', and by Lemma 5.1,

$$\forall (s,x) \in \operatorname{Int}(T; (\mathbb{C} \setminus P') \times X), \ \int_{a(x)}^{+\infty} T(s,x,y) \, \mathrm{d}y = H(s,x).$$

If $\ell = 0$ then H has no new singularities, whereas if $\ell \neq 0$ then the new singularities are located in $(\mathbb{C} \setminus S_0) \times X$ and are in general not removable.

Remark 5.7. If $\mathcal{D} = \mathcal{C}^{\mathbb{K},\mathcal{M}}$ then the sets S_i $(i \geq 1)$ in (5.6) are vertical strips in the complex plane of fixed width $\frac{1}{\ell}$. The points $\sigma \in P' \setminus P$ lie on the boundaries of such strips and their imaginary part is equal to $\dfrac{\Im(\eta)}{\ell}$. If $\mathcal{D} = \mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$, where $\mathbb{K} \not\subseteq \mathbb{R}$, then $\ell \in \mathbb{K}$ and the sets S_i are parallel (not necessarily vertical) strips of fixed width. The points $\sigma \in P' \setminus P$ again lie on the boundaries of such strips and satisfy the equation $\Re(\ell)\Im(\sigma)+\Im(\ell)\Re(\sigma)+\Im(\eta)=0$. In both cases, the set $P'\setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice and hence P' is closed and discrete.

Hence, we have proven the following result.

Proposition 5.8. *Let B be as in (3.2) with $b=+\infty$, $\mathbb{K}\subseteq\mathbb{C}$ be a subfield and let $\mathcal{D}$ be either $\mathcal{C}^{\mathbb{K},\mathcal{M}}$ or $\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$. Let $T\in\mathcal{D}(B)$ be a finite sum of prepared generators sharing the same Puiseux data, as in (5.5), with no poles outside P (for some discrete closed set $P\subseteq\mathbb{K}$). Then $\mathrm{Int}(T;(\mathbb{C}\setminus P)\times X)$ is described as in (5.7) (if $\ell\neq 0$) or in (5.8) (if $\ell=0$) and, for P' as in (5.9), there exists a function $H\in\mathcal{D}(X)$ without poles outside P' such that*

$$\forall(s,x)\in\mathrm{Int}(T;(\mathbb{C}\setminus P)\times X),\ H(s,x)=\int_{a(x)}^{+\infty}T(s,x,y)\,\mathrm{d}y.$$

6. Stability under integration of (parametric) power-constructible functions

This section is devoted to the proof of the results of stability under parametric integration in Section 2.

For the rest of this section we let $\mathcal{D}$ be either $\mathcal{C}^{\mathbb{K},\mathcal{M}}$ or $\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$.

We will first prove stability under integration when we integrate with respect to a single variable y. In this case, we can also give a description of the integration locus. The strategy is the following: we prepare the function we want to integrate with respect to the variable y. This produces a cell decomposition such that on each cell, in the new coordinates the function is a sum of prepared generators. If the cell has bounded y-fibers, then the function is integrable everywhere in restriction to such a cell, and we have already shown (see Remark 5.4) that the integral can be expressed as a function of $\mathcal{D}$. If the cell has unbounded y-fibers and the prepared generators all share the same Puiseux data, then we know how to conclude by the results of the previous section. It remains to consider the case of a sum of generators who have different Puiseux data. Such data induce a partition of $\mathbb{C}$ into areas (see (5.6)) which are involved in the description of the integrability locus. In order to deal with different Puiseux data, we introduce the notion of non-accumulating grid.

Definition 6.1. Given $N,d\in\mathbb{N}^{\times}$ and $\{(\ell_i,\eta_i):\ 0\leq i\leq N\}\subseteq\mathbb{K}^2$, define

$$\begin{aligned}
\Xi_{i,0,-}&=\emptyset,\\
\Xi_{i,0,\circ}&=\{s\in\mathbb{C}:\ \Re(\ell_i s+\eta_i)+d<0\}\quad(i\leq N),\\
\Xi_{i,j,-}&=\{s\in\mathbb{C}:\ \Re(\ell_i s+\eta_i)+d=j-1\}\quad\left(i\leq N,\ j\in\mathbb{N}^{\times}\right),\\
\Xi_{i,j,\circ}&=\{s\in\mathbb{C}:\ j-1<\Re(\ell_i s+\eta_i)+d<j\}\quad\left(i\leq N,\ j\in\mathbb{N}^{\times}\right).
\end{aligned}$$

A collection of sets (partitioning $\mathbb{C}$) of the form

$$\mathcal{G}=\{\Xi_{i,j,\star}:\ i\leq N,\ j\in\mathbb{N},\ \star\in\{-,\circ\}\}$$

is called a *non-accumulating grid* of *data* $\{N,d,(\ell_0,\eta_0),\ldots,(\ell_N,\eta_N)\}$. Note that if $\ell_i=0$ then $\forall j\in\mathbb{N},\forall\star\in\{-,\circ\}$, $\Xi_{i,j,\star}$ is either empty or the whole $\mathbb{C}$.

A $\mathcal{G}$-*cell* is a nonempty subset $\Sigma \subseteq \mathbb{C}$ such that

$$\forall \Xi \in \mathcal{G},\ \Xi \cap \Sigma = \emptyset \text{ or } \Sigma \subseteq \Xi, \text{ and } \Sigma = \bigcap \{\Xi \in \mathcal{G} :\ \Sigma \subseteq \Xi\}.$$

We let $\mathcal{P}(\mathcal{G})$ be the collection of all $\mathcal{G}$-cells. The $\mathcal{G}$-cells are convex and form a partition of $\mathbb{C}$. Each $\mathcal{G}$-cell either has empty interior (an isolated point, a segment or a line) or is an open subset of $\mathbb{C}$ containing an open ball of radius ε, for some $\varepsilon = \varepsilon(\mathcal{G}) > 0$ depending only on $\mathcal{G}$ (hence the word "non-accumulating"). Given a $\mathcal{G}$-cell Σ, there are functions $j_\Sigma : \{0, \ldots, N\} \longrightarrow \mathbb{N}$ and $\star_\Sigma : \{0, \ldots, N\} \longrightarrow \{-, \circ\}$ such that $\Sigma = \bigcap_{i \le N} \Xi_{i, j_\Sigma(i), \star_\Sigma(i)}$.

If all the ℓ_i are in $\mathbb{R}^\times$, the we say that $\mathcal{G}$ is a *vertical* non-accumulating grid. In this case, the cells with empty interior are points or vertical lines, and the open cells are vertical strips of width $\ge \varepsilon$, for some $\varepsilon = \varepsilon(\mathcal{G}) > 0$.

Example 6.2. Let $N, d \in \mathbb{N}^\times$. For $i \le N$, let T_i be a sum of prepared generators on an unbounded cell, sharing the same Puiseux data $(\ell_i, \eta_i, d, \mu_i)$ (as in (5.5), see Remark 5.6), without poles outside some closed discrete set $P \subseteq \mathbb{K}$. Consider the non-accumulating grid of data $\{N, d, (\ell_0, \eta_0), \ldots, (\ell_N, \eta_N)\}$ and let $\Sigma = \bigcap_{i \le N} \Xi_{i, j_\Sigma(i), \star_\Sigma(i)} \in \mathcal{P}(\mathcal{G})$ be a $\mathcal{G}$-cell. Then

$$\mathrm{Int}(T_i; (\Sigma \setminus P) \times X) = \left\{ (s, x) :\ s \in \Sigma \setminus P,\ \bigwedge_{k < j_\Sigma(i)} g_{i,k}(s, x) = 0 \right\},$$

where $g_{i,k} \in \mathcal{D}(X)$ are the coefficients in the series expansion (5.4) of T_i. It follows that, if we rename

$$\left\{ g_k^\Sigma :\ k \in J_\Sigma \right\} = \left\{ g_{i,k} :\ i \le N,\ k < j_\Sigma(i) \right\},$$

then

$$\bigcap_{i \le N} \mathrm{Int}(T_i; (\Sigma \setminus P) \times X) = \left\{ (s, x) :\ s \in \Sigma \setminus P,\ \bigwedge_{k \in J_\Sigma} g_k^\Sigma(s, x) = 0 \right\}.$$

Theorem 6.3. *Let $\mathbb{K} \subseteq \mathbb{C}$ be a subfield and let $\mathcal{D}$ be either $\mathcal{C}^{\mathbb{K}, \mathcal{M}}$ or $\mathcal{C}^{\mathcal{P}(\mathbb{K}), \mathcal{M}}$. Let $P \subseteq \mathbb{K}$ be a closed discrete set and $h \in \mathcal{D}(X \times \mathbb{R})$ be with no poles outside P. There exist a closed discrete set $P' \subseteq \mathbb{K}$, containing P and contained in a finitely generated $\mathbb{Z}$-lattice, and a function $H \in \mathcal{D}(X)$ without poles outside P' such that*

$$\forall (s, x) \in \mathrm{Int}(h; (\mathbb{C} \setminus P') \times X), \quad \int_{\mathbb{R}} h(s, x, y)\, \mathrm{d}y = H(s, x).$$

Moreover, there exists a non-accumulating grid $\mathcal{G}$ as in Definition 6.1 such that

$$\mathrm{Int}(h; (\mathbb{C} \setminus P') \times X) = \bigcup_{\Sigma \in \mathcal{P}(\mathcal{G})} \left\{ (s, x) :\ s \in \Sigma \setminus P',\ \bigwedge_{k \in J_\Sigma} g_k^\Sigma(s, x) = 0 \right\}, \tag{6.1}$$

for a suitable finite set J_Σ and suitable $g_k^\Sigma \in \mathcal{D}(X)$ without poles outside P.

Proof. Apply Proposition 4.7 to h to find a cell decomposition of $\mathbb{R}^{m+1}$ such that on each cell B_A as in (3.5), $h \circ \Pi_A$ is a finite sum of prepared generators (for some y-prepared 1-bounded subanalytic map ψ_A). We may suppose that X itself is a cell and we concentrate on the collection $\mathcal{X}$ of all the cells of the decomposition which have X as a base, and which are open over $\mathbb{R}^m$. Since $\operatorname{Int}(h; (\mathbb{C} \setminus P) \times X) = \bigcap_{A \in \mathcal{X}} \operatorname{Int}(h \cdot \chi_A; (\mathbb{C} \setminus P) \times X)$ and

$$\forall (s,x) \in \operatorname{Int}(h; (\mathbb{C} \setminus P) \times X), \quad \int_{\mathbb{R}} h(s,x,y)\,\mathrm{d}y = \sum_{A \in \mathcal{X}} \int_{\mathbb{R}} h(s,x,y) \cdot \chi_A(x,y)\,\mathrm{d}y,$$

it is enough to prove the theorem for the functions $h \cdot \chi_A$.

For $A \in \mathcal{X}$, we can write

$$h \circ \Pi_A(s,x,y) = \sum_{i \le M_A} \tilde{T}_i^A(s,x,y),$$

where each $\tilde{T}_i^A \in \mathcal{D}(B_A)$ is a prepared generator. Recall the notation in (3.4) and note that

$$\frac{\partial \Pi_A}{\partial y}(x,y) = \sigma_A \tau_A y^{\tau_A - 1}.$$

Define

$$T_i^A(s,x,y) := \sigma_A \tau_A y^{\tau_A - 1} \tilde{T}_i^A(s,x,y).$$

Then,

$$\operatorname{Int}\left(\tilde{T}_i^A \circ \Pi_A^{-1}; (\mathbb{C} \setminus P) \times X\right) = \operatorname{Int}\left(T_i^A; (\mathbb{C} \setminus P) \times X\right)$$

and $\forall (s,x) \in \operatorname{Int}(h \cdot \chi_A; (\mathbb{C} \setminus P) \times X)$,

$$\begin{aligned}\int_{\mathbb{R}} h(s,x,y) \cdot \chi_A(x,y)\,\mathrm{d}y &= \int_{a_A(x)}^{b_A(x)} h \circ \Pi_A(s,x,y) \cdot \frac{\partial \Pi_A}{\partial y}(x,y)\,\mathrm{d}y \\ &= \int_{a_A(x)}^{b_A(x)} \sum_{i \le M_A} T_i^A(s,x,y)\,\mathrm{d}y.\end{aligned}$$

If B_A has bounded y-fibers, then by Proposition 5.3 and Remark 5.4,

$$\operatorname{Int}\left(T_i^A; (\mathbb{C} \setminus P) \times X\right) = (\mathbb{C} \setminus P) \times X$$

and there are a closed discrete set $P'_A \subseteq \mathbb{K}$ (containing P and contained in a finitely generated $\mathbb{Z}$-lattice) and functions $H_i^A \in \mathcal{D}(X)$ without poles outside P'_A, such that

$$\forall (s,x) \in (\mathbb{C} \setminus P'_A) \times X, \quad \sum_{i \le M_A} H_i^A(s,x) = \int_{\mathbb{R}} h(s,x,y) \cdot \chi_A(s,x)\,\mathrm{d}y.$$

If B_A has unbounded y-fibers, then consider the prepared generators $\tilde{T}_i^A$ (which are Puiseux in y, of Puiseux data $(\ell'_i, \eta'_i, d, \mu'_i)$). Suppose that there are $i \ne j \le M_A$ such that $\ell'_i = \ell'_j$, $\mu'_i = \mu'_j$ and $\eta'_i - \eta'_j = \nu \in \mathbb{N}$. Write

$$\begin{aligned}\tilde{T}_j^A(s,x,y) &= \sum_k \tilde{g}_{j,k}(s,x)\, y^{\frac{\ell'_j s + \eta'_j - k}{d}} (\log y)^{\mu'_j} \\ &= \sum_k h_{j,k}(s,x)\, y^{\frac{\ell'_i s + \eta'_i - k}{d}} (\log y)^{\mu'_i},\end{aligned}$$

where

$$h_{j,k}(s,x)=\begin{cases}0 & \text{if } k<\nu\\ \tilde{g}_{j,k-\nu} & \text{if } k\geq\nu\end{cases}.$$

Now $\tilde{T}_i^A$ and $\tilde{T}_j^A$ share the same Puiseux data (and so do T_i^A ans T_j^A). Hence, by summing together all generators which share the same Puiseux data, we may write

$$\sum_{i\leq M_A}\tilde{T}_i^A(s,x,y)=\sum_{i\leq N_A}\tilde{T}_i(s,x,y),$$

where $N_A\in\mathbb{N}$ and, if $T_i=\sigma_A\tau_A y^{\tau_A-1}\tilde{T}_i$,

$$T_i(s,x,y)=\sum_k g_{i,k}(s,x)\,y^{\frac{\ell_i s+\eta_i-k}{d}}(\log y)^{\mu_i}\in\mathcal{D}(B_A) \tag{6.2}$$

is a finite sum of prepared generators on the unbounded cell B_A sharing the same Puiseux data (ℓ_i,η_i,d,μ_i). Moreover, $\forall i\neq j\leq N_A$, $(\ell_i,\eta_i,\mu_i)\neq(\ell_j,\eta_j,\mu_j)$ and if $(\ell_i,\mu_i)=(\ell_j,\mu_j)$ then $\eta_i-\eta_j\notin\mathbb{Z}$. Let

$$P_A'=P\cup\{s\in\mathbb{C}:\ \exists i\leq N_A \text{ s.t. } \ell_i\neq 0 \text{ and } \ell_i s+\eta_i+d\in\mathbb{N}\}.$$

Apply Proposition 5.8 to each T_i and find $H_i\in\mathcal{D}(X)$ without poles outside P_A' such that

$$\forall(s,x)\in\mathrm{Int}(T_i;(\mathbb{C}\setminus P_A')\times X),\ H_i(s,x)=\int_{a_A(x)}^{+\infty}T_i(s,x,y)\,\mathrm{d}y.$$

Clearly, $\bigcap_{i\leq N_A}\mathrm{Int}(T_i;(\mathbb{C}\setminus P)\times X)\subseteq\mathrm{Int}(h\cdot\chi_A;(\mathbb{C}\setminus P)\times X)$ and

$$\forall(s,x)\in\bigcap_i\mathrm{Int}(T_i;(\mathbb{C}\setminus P_A')\times X),\ \int_{\mathbb{R}}h(s,x,y)\cdot\chi_A(x,y)\,\mathrm{d}y=H_0+\cdots+H_N.$$

Recall that the description of the above integrability locus is given in Example 6.2, with respect to the non-accumulating grid $\mathcal{G}_A$ of data $\{N_A,d,(\ell_0,\eta_0),\ldots,(\ell_{N_A},\eta_{N_A})\}$. We would hence be done if we could show that the integrability locus of $h\cdot\chi_A$ coincided with the intersection of the integrability loci of the T_i. This is the case, outside a closed discrete set, as we now show.

Let

$$P_A''=\{s\in\mathbb{C}:\ \exists i\neq j\leq N_A \text{ s.t. } \mu_i=\mu_j,\ \ell_i\neq\ell_j \text{ and } (\ell_i-\ell_j)s+(\eta_i-\eta_j)\in\mathbb{Z}\} \tag{6.3}$$

and notice that $P_A''\subseteq\mathbb{K}$ is contained in a finitely generated $\mathbb{Z}$-lattice. Note that $\forall s\in\mathbb{C}\setminus P_A''$, the tuples

$$\left(\frac{\ell_i s+\eta_i-k}{d},\mu_i\right)\qquad 1\leq i\leq N_A,\ k\in\mathbb{N}$$

are pairwise distinct.

We now show that $\mathrm{Int}(h\cdot\chi_A;(\mathbb{C}\setminus P_A'')\times X)=\bigcap_i\mathrm{Int}(T_i;(\mathbb{C}\setminus P_A'')\times X)$.

Let $\Sigma = \bigcap_{i\leq N} \Xi_{i,j_\Sigma(i),\star_\Sigma(i)}$ be a $\mathcal{G}_A$-cell, in the notation of Example 6.2, and let $(s_0, x_0) \in \mathrm{Int}\,(h \cdot \chi_A; (\Sigma \setminus P''_A) \times X)$. For all $(s,x,y) \in (\Sigma \setminus P''_A) \times B_A$, write

$$\begin{aligned}\sum_{i=1}^{N_A} T_i\,(s,x,y) &= \left(\sum_{i=1}^{N_A} \sum_{k=0}^{j_\Sigma(i)-1} g_{i,k}\,(s,x)\, y^{\frac{\ell_i s+\eta_i-k}{d}} (\log y)^{\mu_i} \right) + \left(\sum_{i=1}^{N_A} \sum_{k\geq j_\Sigma(i)} g_{i,k}\,(s,x)\, y^{\frac{\ell_i s+\eta_i-k}{d}} (\log y)^{\mu_i} \right) \\ &= h^\Sigma_{A,1}\,(s,x,y) + h^\Sigma_{A,2}\,(s,x,y)\end{aligned}$$

and notice that $\mathrm{Int}\left(h^\Sigma_{A,2}; \Sigma \times X\right) = \Sigma \times X$, so $(s_0,x_0) \in \mathrm{Int}\left(h^\Sigma_{A,1}; (\Sigma \setminus P''_A) \times X\right)$. Rename the (finitely many) terms appearing in the double sum defining $h^\Sigma_{A,1}$ as

$$\left\{ g^\Sigma_j\,(s,x)\, y^{\alpha_j s+\beta_j} (\log y)^{\nu_j} \right\}_{j\in J_\Sigma}$$

and let

$$a_j = \Re\,(\alpha_j s_0 + \beta_j)\,,\ b_j = \Im\,(\alpha_j s_0 + \beta_j)\,.$$

Recall that $(a_j, b_j) \neq (a_{j'}, b_{j'})$ whenever $\nu_j = \nu_{j'}$, since $s_0 \notin P''_A$. Let (a_0, ν_0) be the lexicographic maximum of the set $\{(a_j,\nu_j) :\ j \in J_\Sigma\}$ and let $J_0 = \{j \in J_\Sigma :\ (a_j,\nu_j) = (a_0,\nu_0)\}$. Write

$$h^\Sigma_{A,1}\,(s_0,x_0,y) = y^{a_0} (\log y)^{\nu_0} \sum_{j\in J_0} g^\Sigma_j\,(s_0,x_0)\, y^{\mathrm{i}b_j} + \sum_{j\in J_\Sigma\setminus J_0} g^\Sigma_j\,(s_0,x_0)\, y^{a_j+\mathrm{i}b_j} (\log y)^{\nu_j}\,.$$

Since $(s_0,x_0) \in \mathrm{Int}\left(h^\Sigma_{A,1}; (\Sigma \setminus P''_A) \times X\right)$, it follows from Proposition 3.4 (in the case where all the polynomials p_j are identically zero) that $\bigwedge_{j\in J_0} g^\Sigma_j\,(s_0,x_0) = 0$. By repeating this procedure with the index set $J_\Sigma \setminus J_0$, we end up obtaining that

$$\bigwedge_{j\in J_\Sigma} g^\Sigma_j\,(s_0,x_0) = 0,$$

i.e. $(s_0,x_0) \in \bigcap_{i\leq N_A} \mathrm{Int}\,(T_i; (\Sigma \setminus P''_A) \times X)$.

Summing up, if we define $P'' = \bigcup \{P''_A :\ B_A \text{ unbounded}\}$, $\mathcal{G} := \bigcup \{\mathcal{G}_A :\ B_A \text{ unbounded}\}$ and $P' := \bigcup_{A\in\mathcal{X}} P'_A \cup P''$, then the proof of the theorem is complete. □

Remark 6.4. In the previous proof, if $\sigma \in P''_A$, then we rewrite the functions $T_i\,(\sigma,x,y)$ by regrouping the terms with the same exponents. We obtain thus new functions $T_{i,\sigma} \in \mathcal{C}^{\mathbb{C}}\,(X \times \mathbb{R})$ (seen as functions in $\mathcal{C}^{\mathcal{M}}\,(X \times \mathbb{R})$ which happen not to depend on s) to which Proposition 5.8 applies and such that, if $h_\sigma\,(x,y) = h\,(\sigma,x,y) \cdot \chi_A\,(x,y)$, then

$$\mathrm{Int}\,(h_\sigma; X) = \bigcap_i \mathrm{Int}\,(T_{i,\sigma}; X)\,.$$

Moreover, if $\sigma \in P''_A \setminus P'_A$ then σ is not a singularity of either of the H_i and, since the computation of the integral is done integrating term-by-term, it is still the case that

$$\int_{\mathbb{R}} h\,(\sigma,x,y) \cdot \chi_A\,(x,y)\,\mathrm{d}y = H_0\,(\sigma,x) + \cdots + H_N\,(\sigma,x)\,.$$

Remark 6.5. The non-accumulating grid $\mathcal{G}$ in Theorem 6.3 is vertical in all but the case $\mathcal{D} = \mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$, with $\mathbb{K} \not\subseteq \mathbb{R}$. This implies in particular that the system $\mathcal{C}^{\mathcal{P}(\mathbb{C}),\mathcal{M}}$ is strictly larger that the system $\mathcal{C}^{\mathcal{M}}$: for example, if $h \in \mathcal{C}^{\mathcal{P}(\mathbb{C}),\mathcal{M}}\,(X \times \mathbb{R})$ is a finite sum of generators which are Puiseux in y on some cell A with unbounded y-fibers (see Definition 5.5), where the real and imaginary parts of the exponents ℓ appearing in the Puiseux data are all nonzero, then the integration locus of h in (6.1) is based on a non-accumulating grid which is not vertical. Hence h cannot be an element of $\mathcal{C}^{\mathcal{M}}$.

We now conclude the proof of Theorem 2.19, using Fubini's Theorem.

Proof. We argue by induction on $n \in \mathbb{N}^{\times}$. If $n = 1$ then it is Theorem 6.3. We prove the case $n+1$: let y be an n-tuple of variables and let z be a single variable, and consider $h \in \mathcal{D}\left(X \times \mathbb{R}^{n+1}\right)$ without poles outside some closed discrete set P. By Fubini's Theorem, for all $(s,x) \in \mathrm{Int}\left(h; (\mathbb{C} \setminus P) \times X\right)$, the set

$$E_{(s,x)} := \left\{y \in \mathbb{R}^n :\ (s,x,y) \in \mathrm{Int}\left(h; (\mathbb{C} \setminus P) \times X \times \mathbb{R}^n\right)\right\}$$

is such that $\mathbb{R}^n \setminus E_{(s,x)}$ has measure zero and

$$\iint_{\mathbb{R}^{n+1}} h\left(s,x,y,z\right) \mathrm{d}y \wedge \mathrm{d}z = \int_{E_{(s,x)}} \left[\int_{\mathbb{R}} h\left(s,x,y,z\right) \mathrm{d}z\right] \mathrm{d}y.$$

By Theorem 6.3, applied to h as an element of $\mathcal{D}\left(\left(X \times \mathbb{R}^n\right) \times \mathbb{R}\right)$, there exist a set $P_1 \subseteq \mathbb{K}$ (containing P and contained in a finitely generated $\mathbb{Z}$-lattice) and a function $H_1 \in \mathcal{D}\left(X \times \mathbb{R}^n\right)$ without poles outside P_1 such that

$$\forall\left(s,x,y\right) \in \mathrm{Int}\left(h; (\mathbb{C} \setminus P_1) \times X \times \mathbb{R}^n\right),\ H_1\left(s,x,y\right) = \int_{\mathbb{R}} h\left(s,x,y,z\right) \mathrm{d}z.$$

We now apply the inductive hypothesis to H_1 and find that there exist $P' \subseteq \mathbb{K}$ (containing P_1 and contained in a finitely generated $\mathbb{Z}$-lattice) and a function $H \in \mathcal{D}\left(X\right)$ without poles outside P' such that

$$\forall\left(s,x\right) \in \mathrm{Int}\left(H_1; (\mathbb{C} \setminus P') \times X\right),\ H\left(s,x\right) = \int_{\mathbb{R}^n} H_1\left(s,x,y\right) \mathrm{d}y.$$

Let $(s,x) \in \mathrm{Int}\left(h; (\mathbb{C} \setminus P') \times X\right)$. Since H_1 is defined on the whole $(\mathbb{C} \setminus P') \times X \times \mathbb{R}^n$ and $\mathbb{R}^n \setminus E_{(s,x)}$ has measure zero,

$$\iint_{\mathbb{R}^{n+1}} h\left(s,x,y,z\right) \mathrm{d}y \wedge \mathrm{d}z = \int_{\mathbb{R}^n} H_1\left(s,x,y\right) \mathrm{d}y.$$

In particular, $(s,x) \in \mathrm{Int}\left(H_1; (\mathbb{C} \setminus P') \times X\right)$ and

$$\iint_{\mathbb{R}^{n+1}} h\left(s,x,y,z\right) \mathrm{d}y \wedge \mathrm{d}z = H\left(s,x\right).$$

□

Remark 6.6. The proof of Theorem 2.4 is obtained as a special case of that of Theorem 2.19, where all the functions involved happen not to depend on the variable s.

We conclude this section with some further remarks about the classes $\mathcal{C}^{\mathbb{K}}, \mathcal{C}^{\mathbb{K},\mathcal{M}}, \mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$ considered here. Again, we let $\mathcal{D}$ be either $\mathcal{C}^{\mathbb{K},\mathcal{M}}$ or $\mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$.

Remarks 6.7.

(1) Let $S \subseteq \mathbb{C}$ be open and define $\mathcal{D}_S\left(X\right) := \left\{h \restriction S \times X :\ h \in \mathcal{D}\left(X\right)\right\}$. Clearly, Theorem 2.19 also holds for $\mathcal{D}_S$.

(2) $\mathcal{D}$ is stable under right-composition with meromorphic functions, in the following sense. Let $\xi \in \mathcal{E}_{\mathbb{K}}$ and $S, S' \subseteq \mathbb{C}$ open such that $\xi(S) = S'$. If $h \in \mathcal{D}_{S'}(X)$ then $(s,x) \longmapsto h(\xi(s), x) \in \mathcal{D}_S(X)$.

(3) $\mathcal{D}$ and $\mathcal{C}^{\mathbb{K}}$ are stable under right-composition with subanalytic maps, in the following sense. Let $X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^n$ be subanalytic and $\varphi : X \longrightarrow Y$ be a map with components in $\mathcal{S}(X)$. If $h \in \mathcal{D}(Y)$ and $g \in \mathcal{C}^{\mathbb{K}}(Y)$ then $(s,x) \longmapsto h(s, \varphi(x)) \in \mathcal{D}(X)$ and $g \circ \varphi \in \mathcal{C}^{\mathbb{K}}(X)$.

Finally, for $h \in \mathcal{D}(X \times \mathbb{R})$ without poles outside some closed discrete set $P \subseteq \mathbb{K}$, we describe (uniformly in the parameters (s,x)) the behaviour of h when $y \longrightarrow +\infty$. For this, we apply Proposition 4.7 to prepare h and we concentrate on the unique cell A (with base X) which has vertical unbounded fibers. By Remark 3.10, Π_A is the identity and $A = B_A = \{(x,y) : \ x \in X, \ y > a(x)\}$.

Arguing as in the proof of Theorem 6.3 (the case of a cell with unbounded y-fibers) we can write, $\forall (s,x,y) \in (\mathbb{C} \setminus P) \times A$,

$$h(s,x,y) = \sum_{i \leq N} T_i(s,x,y),$$

where each T_i is Puiseux in y, as in (6.2). Moreover, by enlarging P to contain the "collision set" defined in (6.3), we may suppose that $\forall s \in \mathbb{C} \setminus P$, the tuples

$$\left(\frac{\ell_i s + \eta_i - k}{d}, \mu_i\right) \qquad i \leq N, \ k \in \mathbb{N} \tag{6.4}$$

are pairwise distinct. Recall that $\ell_i, \eta_i \in \mathbb{K}$ and $d, \mu_i \in \mathbb{N}$.

Fix an enumeration $\mathbb{N} \ni j \longmapsto (i(j), k(j)) \in \{0, \ldots, N\} \times \mathbb{N}$, so that we may rewrite (6.4) as

$$(\lambda_j(s), \nu_j) = \left(\frac{\ell_{i(j)} s + \eta_{i(j)} - k(j)}{d}, \mu_{i(j)}\right).$$

Define

$$a_j(s) = \Re(\lambda_j(s)) = \frac{\Re\left(\ell_{i(j)} s + \eta_{i(j)}\right) - k(j)}{d}, \ b_j(s) = \Im(\lambda_j(s)) = \frac{\Im\left(\ell_{i(j)} s + \eta_{i(j)}\right)}{d}.$$

Notice that $b_j(s)$ takes at most $N+1$ different values, for every fixed s. Hence, we may write h as the sum of a uniformly summable family of functions as follows:

$$h(s,x,y) = \sum_j h_j(s,x)\, y^{a_j(s) + \mathrm{i} b_j(s)} (\log y)^{\nu_j}, \tag{6.5}$$

where $h_j \in \mathcal{D}(X)$.

In a forthcoming paper, we will use (6.5) to show that $\mathcal{C}^{\mathbb{K}}$ is stable under taking pointwise limits and that neither of the classes $\mathcal{C}^{\mathbb{K}}, \mathcal{C}^{\mathbb{K},\mathcal{M}}, \mathcal{C}^{\mathcal{P}(\mathbb{K}),\mathcal{M}}$ contains the Fourier transforms of all subanalytic functions.

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