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Raf Cluckers, Georges Comte, Tamara Servi

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Abstract. We enrich the class of power-constructible functions, introduced in [CCRS23], to a class $\mathcal{C}^{\mathcal{M},\mathcal{F}}$ of algebras of functions which contains all complex powers of subanalytic functions, their parametric Mellin and Fourier transforms, and which is stable under parametric integration. By describing a set of generators of a special prepared form we deduce information on the asymptotics and on the loci of integrability of the functions of $\mathcal{C}^{\mathcal{M},\mathcal{F}}$. We furthermore identify a subclass $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ of $\mathcal{C}^{\mathcal{M},\mathcal{F}}$ which is the smallest class containing all power-constructible functions and stable under parametric Fourier transforms and right-composition with subanalytic maps. This class is also stable under parametric integration, under taking pointwise and L^p-limits, and under parametric Fourier-Plancherel transforms. Finally, we give a full asymptotic expansion in the power-logarithmic scale, uniformly in the parameters, for functions in $\mathcal{C}^{\mathbb{C},\mathcal{F}}$.

Contents

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1. Introduction

Understanding integrals is at the heart of many mathematical problems, and often brings together challenges from both geometry and analysis. Indeed, integration is a transcendental process usually applied to functions naturally arising from basic geometric problems, and as such, having remarkable properties one aims to preserve. The present work is in the same spirit as Liouville's theorem on elementary integrals and its recent variants by Pila and Tsimerman (see [PT22]); it concerns rich classes of functions whose parametric integrals are of a somewhat similar nature as the original functions.

To be more accurate, two types of problems may be considered in this spirit.

The first consists in describing a class of functions, possibly the smallest one, stable under parametric integration and containing a given class of functions. For instance, in the context of real o-minimal geometry, this kind of problem has been addressed for the class of semialgebraic and subanalytic functions. Indeed, in [LR98, CLR00, CM11, CM12] it has been proved that the class $\mathcal{C}$ of constructible functions (that is to say the functions which are polynomials in globally subanalytic functions and their logarithms) form the smallest class of real-valued functions which contains all globally subanalytic functions and which is stable under parametric integration. In [Kai13] a proper subclass of $\mathcal{C}$ is introduced. This class is based on Nash functions (and their anti-derivatives) and turns out to be a small class of functions stable under parametric integration and containing the semialgebraic functions. Furthermore this class is suitable for studying families of periods as parametric integrals in the viewpoint of [KZ01] (see [Kai23]). In a similar spirit, we fully describe here the smallest class $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ of functions which contains all complex powers and complex exponentials (of module one) of globally subanalytic functions, and stable under parametric integration (a natural framework for studying families of exponential periods, see [KZ01, Section 4.3]).

The second type of problems, addressed here for the class $\mathcal{C}^{\mathcal{M},\mathcal{F}}$, still consists in describing a class of functions containing a given class of functions and stable under parametric integration, but we additionally require our class to be stable under other analytic key operations like Fourier and Mellin transforms. A hard part of this program consists then in finding the geometric properties preserved by the parametric integration process and our analytic transformations. The challenge here comes from the fact that by the action of these analytic transformations we leave the convenient framework of o-minimal geometry by introducing, via Mellin transforms, the (meromorphic) dependence on a complex parameter s.

Finally, let us note that several formalisms of motivic (and uniform p-adic) integration have a similar flavor and set-up. Such classes can then, for example, be used to define tame classes of distributions, which are at the same time stable under Fourier transform and analytically (wave front) holonomic [AC20, ACRS23].

In this work, our starting point is the class $\mathcal{C}^{\mathbb{C}}$ of power-constructible functions, defined and studied in [CCRS23], which extends $\mathcal{C}$ by including complex powers of globally subanalytic functions. This class includes complex-valued oscillatory functions, hence we leave the realm of o-minimality, but many tame geometric and analytic properties are preserved, such as stability under parametric integration and well-understood (convergent) power-logarithmic asymptotics. In [CCMRS18] we studied the parametric Fourier transforms of constructible functions (thus equally leaving the realm of real geometry) and described a class containing such transforms and stable under parametric integration. In [CCRS23] we studied Mellin transforms of power-constructible functions and showed that stability under parametric integration is preserved.

In the current paper we combine the action of parametric Fourier and Mellin transforms on the class $\mathcal{C}^{\mathbb{C}}$ of power-constructible functions. We define a system $\mathcal{C}^{\mathcal{M},\mathcal{F}}$ of $\mathbb{C}$-algebras containing all such transforms and stable under parametric integration (see Definition 2.19 and Theorem 2.21). We describe a set of generators of a particular prepared form which allows us to prove the stability under parametric integration and deduce information about the asymptotics at infinity in a chosen variable of the functions of the class. We furthermore identify a subclass $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ (see Definition 2.7) of $\mathcal{C}^{\mathcal{M},\mathcal{F}}$ which is the smallest class containing $\mathcal{C}^{\mathbb{C}}$ and stable under parametric Fourier transform and right-composition with subanalytic maps, and give, for the functions of this class, asymptotic expansions in the power-logarithmic scale (Theorem 7.6), in a chosen variable y and uniformly in the other variables x (which serve as parameters and range in a given globally subanalytic set). We also deduce the stability of the class $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ under taking pointwise and L^p-limits, and under the Fourier-Plancherel transform (Theorems 7.11, 8.7, 8.8).

The main geometric tools for achieving this program come from o-minimality (see [Dri99]) and, more precisely, from the geometry of subanalytic sets and functions. They consist in resolution results in the form of preparation theorems, in the spirit of [Par94], [LR98] or [Mil06]. The key analytic tool we use is the theory of continuously uniformly distributed modulo one functions (c.u.d. mod 1, for short), building on [Wey16, KN74, CCMRS18], and its uniform variants. One of the deep challenges comes from the oscillatory nature of functions in $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ and $\mathcal{C}^{\mathcal{M},\mathcal{F}}$, which imposes a careful study of the integration loci (see Definition 2.3 and Theorem 6.5). This is where the interaction between the theory of c.u.d. mod 1 functions and the geometry of subanalytic sets comes into play.

The paper is organized as follows.

In Section 2 we introduce the classes $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ and $\mathcal{C}^{\mathcal{M},\mathcal{F}}$, and state our main results (Theorems 2.9 and 2.21). The class $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ is a collection of functions defined on globally subanalytic sets, the class $\mathcal{C}^{\mathcal{M},\mathcal{F}}$ is a collection of functions that also depend on a complex parameter s, which is only allowed to range in a vertical open strip with bounded width. However, it is possible to extend a function on a given strip to a larger strip (Proposition 2.20).

In Section 3, we choose suitable generators for $\mathcal{C}^{\mathcal{M},\mathcal{F}}$ as an abelian group, which allow us to prove the extension result (see Section 3.2).

In Section 4 we identify two special types of generators, *strongly integrable* and *monomial* (see Definition 4.2), and show that their parametric integrals still belong to the class $\mathcal{C}^{\mathcal{M},\mathcal{F}}$ (Corollary 4.6 and Proposition 4.8). In Section 4.2 we give the proof of Theorems 2.9 and 2.21, assuming Theorem 6.5, which is a precise form of Theorem 2.21 when integrating over only one variable.

Sections 5 and 6 are devoted to the proof of Theorem 6.5, which requires both subanalytic resolution of singularities and preparation techniques, and non-compensation results based on the theory of c.u.d. mod 1 functions.

In Section 7 we study the asymptotics of the functions in the power-logarithmic scale and prove the stability of the class $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ under pointwise limits (Theorems 7.6 and 7.11).

In Section 8 we prove the L^p-completeness of the class $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ and its stability under the parametric Fourier-Plancherel transform (Theorems 8.7 and 8.8).

2. Context, definitions and main results

A subset X of $\mathbb{R}^m$ is globally subanalytic if it is the image under the canonical projection from $\mathbb{R}^{m+n}$ to $\mathbb{R}^m$ of a globally semianalytic subset of $\mathbb{R}^{m+n}$ (i.e. a subset $Y \subseteq \mathbb{R}^{m+n}$ such that, in a neighborhood of every point of $\mathbb{P}^1(\mathbb{R})^{m+n}$, Y is described by finitely many analytic equations and inequalities). Equivalently, X is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$ (see for example [DD88]). Thus, the logarithm $\log : (0,+\infty) \longrightarrow \mathbb{R}$ and the power map $x^y : (0,+\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ are functions whose graph is not subanalytic, but they are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ (see for example [DMM94]).

Throughout this paper $X \subseteq \mathbb{R}^m$ will be a globally subanalytic set (from now on, just "*subanalytic set*", for short). Denote by $\mathcal{S}(X)$ the collection of all subanalytic functions on X, i.e. all the functions of domain X whose graph is a subanalytic set, and let $\mathcal{S}_+(X) = \{f \in \mathcal{S}(X) : \ f(X) \subseteq (0,+\infty)\}$.

Notation 2.1. Whenever we fix, for every $m \in \mathbb{N}$ and $X \subseteq \mathbb{R}^m$ subanalytic, a collection $\mathcal{G}(X)$ of real- or complex-valued functions defined on X, we denote by $\mathcal{G}$ the *system* of all collections $\mathcal{G}(X)$. For instance, $\mathcal{S}$ is the system of collections of all subanalytic functions defined on subanalytic sets:

$$\mathcal{S} = \{\mathcal{S}(X) : \ X \subseteq \mathbb{R}^m \text{ subanalytic}, \ m \in \mathbb{N}\}.$$

Definition 2.2. For $X \subseteq \mathbb{R}^m$ subanalytic, define

$$\begin{aligned} \mathcal{S}_+^{\mathbb{C}}(X) &= \{f^\alpha : \ f \in \mathcal{S}_+(X), \ \alpha \in \mathbb{C}\}, \\ \log \mathcal{S}_+(X) &= \{\log f : \ f \in \mathcal{S}_+(X)\}, \\ \mathrm{e}^{\mathrm{i}\mathcal{S}}(X) &= \left\{\mathrm{e}^{\mathrm{i}f} : \ f \in \mathcal{S}(X)\right\}. \end{aligned}$$

A function defined on X and taking its values in $\mathbb{C}$ is called a *complex-valued subanalytic function* if its real and imaginary parts are in $\mathcal{S}(X)$. For example, if $f \in \mathcal{S}(X)$ is bounded (i.e. for all $x \in X$, $|f(x)| \leq M$, for some $M > 0$), then $\mathrm{e}^{\mathrm{i}f}$ is a complex-valued subanalytic function. If such a bounded f is furthermore strictly positive (i.e. $f \in \mathcal{S}_+(X)$) and bounded away from zero (i.e. for all $x \in X$, $f(x) \geq m$, for some $m > 0$), then $\log f$ is a real-valued subanalytic function and for all $\alpha \in \mathbb{C}$, f^α is a complex-valued subanalytic function.

Definition 2.3. Let $\mathcal{G}$ be a system as in Notation 2.1. For $h \in \mathcal{G}(X \times \mathbb{R}^n)$, the *integration locus* of h on X is the set

$$\mathrm{Int}(h; X) = \left\{x \in X : \ y \longmapsto h(x,y) \in L^1(\mathbb{R}^n)\right\}.$$

We say that $\mathcal{G}$ is *stable under parametric integration* if for all $h \in \mathcal{G}(X \times \mathbb{R}^n)$ there exists $H \in \mathcal{G}(X)$ such that

$$\forall x \in \operatorname{Int}(h; X), \ H(x) = \int_{\mathbb{R}^n} h(x, y)\, \mathrm{d}y.$$

Finally, define $\mathcal{G}(X \times \mathbb{R}^n)_{\text{int}} = \{h \in \mathcal{G}(X \times \mathbb{R}^n) : \ \operatorname{Int}(h; X) = X\}$.

Thus, for example, $h \in \mathcal{G}((X \times \mathbb{R}) \times \mathbb{R})_{\text{int}}$ means that for all $(x, y) \in X \times \mathbb{R}$, $t \longmapsto h(x, y, t) \in L^1(\mathbb{R})$, whereas $h \in \mathcal{G}(X \times \mathbb{R}^2)_{\text{int}}$ means that for all $x \in X$, $(y, t) \longmapsto h(x, y, t) \in L^1(\mathbb{R}^2)$.

Next, we introduce the parametric Fourier transform acting on a system $\mathcal{G}$ as in Notation 2.1.

Definition 2.4. Let $h \in \mathcal{G}(X \times \mathbb{R})_{\text{int}}$. Define the *parametric Fourier transform of* h as the function

$$\mathcal{F}[h] : X \times \mathbb{R} \ni (x, t) \longmapsto \int_{\mathbb{R}} h(x, y)\, \mathrm{e}^{-2\pi \mathrm{i} t y} \mathrm{d}y$$

and the *fixed frequency parametric Fourier transform of* h as the function obtained from $\mathcal{F}[h]$ by fixing $t = -\frac{1}{2\pi}$, i.e.

$$\mathfrak{f}[h] : X \ni x \longmapsto \int_{\mathbb{R}} h(x, y)\, \mathrm{e}^{\mathrm{i} y} \mathrm{d}y.$$

Notation 2.5. The letter χ will be used for characteristic functions. Thus, if $A \subseteq \mathbb{R}^n$, then χ_A will be the characteristic function of the set A.

We will often work in restriction to subanalytic cells $A \subseteq X \times \mathbb{R}$, for some $X \subseteq \mathbb{R}^m$ subanalytic. If $x \in X$, then A_x denotes the fiber of A over x, i.e. the set $\{y \in \mathbb{R} : \ (x, y) \in A\}$. As X serves as a space of parameters (we will never integrate with respect to the variables x ranging in X), we are allowed to partition X into subanalytic cells, replace X by one of the cells of the partition and work disjointly in restriction to such a cell. In particular, we may always assume that X is itself a subanalytic cell, and that all cells in $X \times \mathbb{R}$ project onto X. Moreover, we will always concentrate on cells A which are *open over* X (see [CCRS23, Definition 3.1]), as these are the only cells whose fibers give a nonzero contribution when integrating a function defined on $X \times \mathbb{R}$ with respect to its last variable.

2.1. **Fourier transforms of power-constructible functions.**

In [CCMRS18] we constructed the smallest system of $\mathbb{C}$-algebras containing $\mathcal{S} \cup \mathrm{e}^{\mathrm{i}\mathcal{S}}$ and stable under parametric integration. Such a system contains in particular the parametric Fourier transforms of all subanalytic functions. The first aim of this paper is to extend such a construction to describe the smallest system $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ containing $\mathcal{S}_+^{\mathbb{C}} \cup \mathrm{e}^{\mathrm{i}\mathcal{S}}$ and stable under parametric integration. For this, our starting point is the system $\mathcal{C}^{\mathbb{C}}$ of *power-constructible functions* defined in [CCRS23]. Let us recall its definition and main properties.

Theorem 2.6 ([CCRS23, Definition 2.2 and Theorem 2.4])**.** *For $X \subseteq \mathbb{R}^m$ subanalytic, let $\mathcal{C}^{\mathbb{C}}(X)$ be the $\mathbb{C}$-algebra generated by $\mathcal{S}_+^{\mathbb{C}}(X) \cup \log \mathcal{S}_+(X)$. The system $\mathcal{C}^{\mathbb{C}}$ of power-constructible functions is the smallest system of $\mathbb{C}$-algebras containing $\mathcal{S}_+^{\mathbb{C}}$ and stable under parametric integration.*

A natural candidate for the smallest system containing $\mathcal{S}_+^{\mathbb{C}} \cup \mathrm{e}^{\mathrm{i}\mathcal{S}}$ and stable under parametric integration would be the system $\mathcal{C}^{\mathbb{C},\mathrm{i}\mathcal{S}}$ of $\mathbb{C}$-algebras $\mathcal{C}^{\mathbb{C},\mathrm{i}\mathcal{S}}(X)$ generated by $\mathcal{C}^{\mathbb{C}}(X) \cup \mathrm{e}^{\mathrm{i}\mathcal{S}}(X)$. However, we will show (see Corollary 7.8) that such a system is not stable under parametric integration. This motivates the following definition.

Definition 2.7. Consider the fixed frequency parametric Fourier operator $\mathfrak{f}$ acting on $\mathcal{C}^{\mathbb{C}}$:

$$\mathfrak{f}[g](x) = \int_{\mathbb{R}} g(x,y)\, \mathrm{e}^{\mathrm{i}y} \mathrm{d}y \quad \left(g \in \mathcal{C}^{\mathbb{C}}(X \times \mathbb{R})_{\mathrm{int}}\right).$$

Define

$$\mathcal{C}^{\mathbb{C},\mathcal{F}}(X) = \left\{\mathfrak{f}[g] : \ g \in \mathcal{C}^{\mathbb{C}}(X \times \mathbb{R})_{\mathrm{int}}\right\}.$$

Remark 2.8. Notice that $\mathcal{C}^{\mathbb{C},\mathcal{F}}(X)$ is a $\mathbb{C}$-module and that $1 = \mathfrak{f}\left[\frac{\mathrm{i}}{2}\chi_{[\pi,2\pi]}\right]$. In particular, $\mathcal{C}^{\mathbb{C}}(X) \subseteq \mathcal{C}^{\mathbb{C},\mathcal{F}}(X)$. At this stage, it is not clear whether $\mathcal{C}^{\mathbb{C},\mathcal{F}}(X)$ is a $\mathbb{C}$-algebra.

Note also that $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ is stable under right-composition with subanalytic maps: if $X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^n$ are subanalytic sets, $G : Y \longrightarrow X$ is a subanalytic map and $h \in \mathcal{C}^{\mathbb{C},\mathcal{F}}(X)$, then $h \circ G \in \mathcal{C}^{\mathbb{C},\mathcal{F}}(Y)$.

Our first result is the following.

Theorem 2.9. *The system $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ is stable under parametric integration. It is a system of $\mathbb{C}$-algebras, and indeed the smallest such system containing $\mathcal{S}_+^{\mathbb{C}} \cup \mathrm{e}^{\mathrm{i}\mathcal{S}}$ and stable under parametric integration. It is also the smallest such system containing $\mathcal{C}^{\mathbb{C}}$ and stable under the parametric Fourier transform and right-composition with subanalytic maps.*

Subsequently, we derive results on asymptotic expansions, pointwise limits, L^p-limits, and the Fourier-Plancherel transform for the class $\mathcal{C}^{\mathbb{C},\mathcal{F}}$. Such results are stated and proven in Sections 7 and 8.

2.2. **Parametric Mellin and Fourier transforms of power-constructible functions.** We now turn our attention to the Mellin transform.

Definition 2.10. Let $\Sigma \subseteq \mathbb{C}$ be an open set. For $h \in \mathcal{C}^{\mathbb{C}}(X \times [0,+\infty))$ such that for all $s \in \Sigma$ and for all $x \in X$, the function $y \longmapsto y^{s-1}h(x,y)$ belongs to $L^1([0,+\infty))$, the *parametric Mellin transform of h on Σ* is the function

$$\mathcal{M}_{\Sigma}[h] : \Sigma \times X \ni (s,x) \longmapsto \int_0^{+\infty} y^{s-1} h(x,y)\, \mathrm{d}y.$$

In [CCRS23] we studied the parametric Mellin transforms of power-constructible functions: we constructed a system $\mathcal{C}^{\mathcal{M}}$ containing such transforms and stable under parametric integration (see Definition 2.17 and Theorem 2.18 below). The second aim of this work is to construct a system containing both the parametric Mellin transforms and the parametric Fourier transforms of power-constructible functions, and stable under parametric integration. As the Mellin transform introduces a new complex variable s, the domains of the functions we consider will be suitable subsets of $\mathbb{C}\times\mathbb{R}^m$, rather than just subsets of $\mathbb{R}^m$. The notions of integration locus, parametric integral transform and stability under parametric integration need to be made precise in this new context, which is what we do next.

In what follows, we will consider several collections of functions defined on sets of the form $\Sigma \times X$, where Σ is a suitable subset of $\mathbb{C}$ and X is a subanalytic subset of $\mathbb{R}^m$, for some

$m \in \mathbb{N}$. We will study the action of some integral operators on these collections of functions, and, more generally, the nature of the parametric integrals of such functions. Let us fix some notation.

Definition 2.11. An open vertical strip of bounded width in $\mathbb{C}$ is a set of the form

$$\Sigma = \{s \in \mathbb{C} : \ p < \Re(s) < q\},$$

where $p, q \in \mathbb{R}$ and $p < q$. For short, we will say that Σ is a *strip*.

Notation 2.12. Given a strip $\Sigma \subseteq \mathbb{C}$ and a subanalytic set $X \subseteq \mathbb{R}^m$, let $\mathcal{D}_\Sigma(X)$ be a collection of complex-valued functions such that for all $h \in \mathcal{D}_\Sigma(X)$ there is a closed discrete set $P \subseteq \mathbb{C}$ such that the domain of h contains $(\Sigma \setminus P) \times X$ (we say that h *has no poles outside* P). Denote by $\mathcal{D}_\Sigma$ the system $\{\mathcal{D}_\Sigma(X) : \ X \subseteq \mathbb{R}^m \text{ subanalytic}, \ m \in \mathbb{N}\}$.

Suppose furthermore that the collection $\{\mathcal{D}_\Sigma : \ \Sigma \subseteq \mathbb{C} \text{ strip}\}$ has the *extension property*: for every subanalytic set $X \subseteq \mathbb{R}^m$, given any two strips Σ, Σ' such that $\Sigma \subseteq \Sigma'$ and a closed discrete set $P \subseteq \mathbb{C}$ and $h \in \mathcal{D}_\Sigma(X)$ without poles outside P, there exists $h' \in \mathcal{D}_{\Sigma'}(X)$ without poles outside P such that $h' \restriction (\Sigma \setminus P) \times X = h$. Define $\mathcal{D}(X)$ as the direct limit of $\{\mathcal{D}_\Sigma(X) : \ \Sigma \subseteq \mathbb{C} \text{ strip}\}$ and $\mathcal{D} = \{\mathcal{D}(X) : \ X \subseteq \mathbb{R}^m \text{ subanalytic}, \ m \in \mathbb{N}\}$. For $h \in \mathcal{D}(X)$ and a closed discrete set $P \subseteq \mathbb{C}$, we say that h has no poles outside P if this is the case for some representative of h on each strip Σ.

Definition 2.13. Given $h \in \mathcal{D}_\Sigma(X \times \mathbb{R}^n)$ without poles outside some closed discrete set $P \subseteq \mathbb{C}$, define the *integration locus* of h as

$$\mathrm{Int}(h; (\Sigma \setminus P) \times X) = \left\{(s, x) \in (\Sigma \setminus P) \times X : \ y \longmapsto h(s, x, y) \in L^1(\mathbb{R}^n)\right\}$$

and we set

$$\begin{aligned}\mathcal{D}_\Sigma(X \times \mathbb{R}^n)_{\text{int}} = &\{h \in \mathcal{D}_\Sigma(X \times \mathbb{R}^n) : \ \mathrm{Int}(h; (\Sigma \setminus P) \times X) = (\Sigma \setminus P) \times X, \\ &\text{for some closed discrete } P \subseteq \mathbb{C}\}.\end{aligned}$$

We consider the following parametric integral transforms acting on $\mathcal{D}$, where the word *generalized* refers to the fact that, unlike the case of the corresponding classical transforms, we allow the operator to act on functions for which the integral transform is not everywhere defined.

Definition 2.14. Let $\mathcal{D}$ be as in Notation 2.12 and $h \in \mathcal{D}_\Sigma(X \times \mathbb{R})$ be without poles outside some closed discrete set $P \subseteq \mathbb{C}$.

- Let χ_+ be the characteristic function of the half-line $[0, +\infty)$ and

  $$\widetilde{h}(s, x, y) = \chi_+(y)\, y^{s-1} h(s, x, y).$$

  The *generalized parametric Mellin transform* of h is the function defined on $\mathrm{Int}\left(\widetilde{h}; (\Sigma \setminus P) \times X\right)$ given by

  $$\mathcal{M}[h](s, x) = \int_0^{+\infty} y^{s-1} h(s, x, y)\, \mathrm{d}y.$$

  The integration kernel of this transform is the function $(s, y) \longmapsto \chi_+(y)\, y^{s-1}$.

- The *generalized parametric Fourier transform* of h is the function defined on $\mathrm{Int}\,(h; (\Sigma \setminus P) \times X) \times \mathbb{R}$ given by
$$\mathcal{F}\,[h]\,(s,x,t) = \int_{\mathbb{R}} h\,(s,x,y)\,\mathrm{e}^{-2\pi \mathrm{i} t y}\mathrm{d}y.$$
The integration kernel is the function $(t,y) \longmapsto \mathrm{e}^{-2\pi \mathrm{i} t y}$.
- The *generalized fixed frequency parametric Fourier transform* of h is the function defined on $\mathrm{Int}\,(h; (\Sigma \setminus P) \times X)$ given by
$$\mathfrak{f}\,[h]\,(s,x) = \mathcal{F}\,[h]\left(s,x,-\frac{1}{2\pi}\right) = \int_{\mathbb{R}} h\,(s,x,y)\,\mathrm{e}^{\mathrm{i}y}\mathrm{d}y.$$
The integration kernel is the function $y \longmapsto \mathrm{e}^{\mathrm{i}y}$.

For each of these operators, the elements of the pairs (s,x) for which the parametric transform of h is defined are called the *parameters* of the transform.

Definition 2.15. Let $\mathcal{D}$ be as in Notation 2.12.

- $\mathcal{D}$ is *stable under the generalized parametric Mellin transform* if for all Σ and X, for all $h \in \mathcal{D}_\Sigma\,(X \times \mathbb{R})$ without poles outside some closed discrete set $P \subseteq \mathbb{C}$ there are a closed discrete set $P' \subseteq \mathbb{C}$ such that $P \subseteq P' \subseteq \mathbb{C}$, and a function $H \in \mathcal{D}_\Sigma\,(X)$ without poles outside P' such that, if $\widetilde{h}\,(s,x,y) = \chi_+\,(y)\,y^{s-1}h\,(s,x,y)$, then
$$\forall\,(s,x) \in \mathrm{Int}\left(\widetilde{h}; (\Sigma \setminus P') \times X\right),\ H\,(s,x) = \mathcal{M}\,[h]\,(s,x)\,.$$
- $\mathcal{D}$ is *stable under the generalized parametric Fourier transform* if for all Σ and X, for all $h \in \mathcal{D}_\Sigma\,(X \times \mathbb{R})$ without poles outside some closed discrete set $P \subseteq \mathbb{C}$ there are a closed discrete set $P' \subseteq \mathbb{C}$ such that $P \subseteq P' \subseteq \mathbb{C}$, and a function $H \in \mathcal{D}_\Sigma\,(X \times \mathbb{R})$ without poles outside P' such that
$$\forall\,(s,x,t) \in \mathrm{Int}\,(h; (\Sigma \setminus P') \times X) \times \mathbb{R},\ H\,(s,x,t) = \mathcal{F}\,[h]\,(s,x,t)\,.$$

Definition 2.16. A system $\mathcal{D}$ as in Notation 2.12 is *stable under parametric integration* if for every strip $\Sigma \subseteq \mathbb{C}$ and every subanalytic set $X \subseteq \mathbb{R}^m$, given $h \in \mathcal{D}_\Sigma\,(X \times \mathbb{R}^n)$ without poles outside some closed discrete set $P \subseteq \mathbb{C}$ there exists a closed discrete set $P' \subseteq \mathbb{C}$ such that $P \subseteq P'$ and $P' \setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice, and there exists a function $H \in \mathcal{D}_\Sigma\,(X)$ without poles outside P' such that
$$\forall\,(s,x) \in \mathrm{Int}\,(h; (\Sigma \setminus P') \times X)\,,\ H\,(s,x) = \int_{\mathbb{R}^n} h\,(s,x,y)\,\mathrm{d}y.$$

2.2.1. *Parametric power-constructible functions.* Recall the following definitions and results from [CCRS23].

Definition 2.17.

- (1-bounded subanalytic maps) For $N \in \mathbb{N}$, we let $\mathcal{S}_c^N\,(X)$ be the collection of all maps $\psi : X \longrightarrow \mathbb{R}^N$ with components in $\mathcal{S}\,(X)$, such that $\overline{\psi\,(X)}$ is contained in the closed polydisk of $\mathbb{R}^N$ centered at zero and of radius 1. The members of the collection $\mathcal{S}_c\,(X) = \bigcup_{N \in \mathbb{N}^\times} \mathcal{S}_c^N\,(X)$ are called *1-bounded* subanalytic maps defined on X.

- (Strongly convergent series) Let $\mathcal{E}$ be the field of meromorphic functions $\xi : \mathbb{C} \longrightarrow \mathbb{C}$ and denote by D^N the closed polydisk of radius $\frac{3}{2}$ and center $0 \in \mathbb{R}^N$. Given a formal power series $F = \sum_I \xi_I(s) Z^I \in \mathcal{E}[\![Z]\!]$ in N variables Z and with coefficients $\xi_I \in \mathcal{E}$, we say that F *converges strongly* if there exists a closed discrete set $P(F) \subseteq \mathbb{C}$ (called the *set of poles* of F) such that:
  - for every $s_0 \in \mathbb{C} \setminus P(F)$, the power series $F(s_0, Z) \in \mathbb{C}[\![Z]\!]$ converges in a neighbourhood of D^N (thus F defines a function on $(\mathbb{C} \setminus P(F)) \times D^N$);
  - for every $s_0 \in \mathbb{C}$ there exists $m = m(s_0) \in \mathbb{N}$ such that for all $z_0 \in D^N$, the function $(s, z) \longmapsto (s - s_0)^m F(s, z)$ has a holomorphic extension on some complex neighbourhood of (s_0, z_0);
  - $P(F)$ is the set of all $s_0 \in \mathbb{C}$ such that the minimal such $m(s_0)$ is strictly positive.
- (**Parametric strong functions**) Given a closed discrete set $P \subseteq \mathbb{C}$, a function $\Phi : (\mathbb{C} \setminus P) \times X \longrightarrow \mathbb{C}$ is called a *parametric strong function* on X if there exist a 1-bounded subanalytic map $\psi \in \mathcal{S}_c^N(X)$ and a strongly convergent series $F = \sum_I \xi_I(s) Z^I \in \mathcal{E}[\![Z]\!]$ with $P(F) \subseteq P$ such that,
$$\forall (s, x) \in (\mathbb{C} \setminus P) \times X,\ \Phi(s, x) = F \circ (s, \psi(x)) = \sum_I \xi_I(s) (\psi(x))^I .$$
Define $\mathcal{A}(X)$ as the collection of all parametric strong functions on X. If $\Phi \in \mathcal{A}(X)$ has no poles outside $P \subseteq \mathbb{C}$, then for all $s \in \mathbb{C} \setminus P,\ x \longmapsto \Phi(s, x)$ is bounded. If furthermore for all $s \in \mathbb{C} \setminus P,\ x \longmapsto \Phi(s, x)$ is bounded away from zero then we call Φ a *parametric strong unit*. A parametric strong function which happens not to depend on the variable s is called a *subanalytic strong function*.
- (Parametric powers) For $X \subseteq \mathbb{R}^m$ subanalytic, define the *parametric powers of* $\mathcal{S}$ on X as the functions in the collection

$$\mathcal{P}(\mathcal{S}_+(X)) = \{P_f : \mathbb{C} \times X \longrightarrow \mathbb{C} \text{ such that } P_f(s, x) = f(x)^s,\ \text{for some } f \in \mathcal{S}_+(X)\}.$$

- (**Parametric power-constructible functions**) If $X \subseteq \mathbb{R}^0$, then define $\mathcal{C}^{\mathcal{M}}(X) = \mathcal{E}$. If $X \subseteq \mathbb{R}^m$, with $m > 0$, then we let $\mathcal{C}^{\mathcal{M}}(X)$ be the $\mathcal{A}(X)$-algebra generated by $\mathcal{C}^{\mathbb{C}}(X) \cup \mathcal{P}(\mathcal{S}_+(X))$. The system $\mathcal{C}^{\mathcal{M}}$ is the collection of algebras of *parametric power-constructible functions*. Every function $h \in \mathcal{C}^{\mathcal{M}}(X)$ can be written on $(\mathbb{C} \setminus P) \times X$ (for some closed discrete $P \subseteq \mathbb{C}$) as a closed discrete sum of *generators* of the form

$$\Phi(s, x) \cdot g(x) \cdot f(x)^s , \tag{2.1}$$

where $g \in \mathcal{C}^{\mathbb{C}}(X),\ f \in \mathcal{S}_+(X)$ and $\Phi \in \mathcal{A}(X)$ has no poles outside P. Here the word "parametric" refers to the variable $s \in \mathbb{C}$ seen as a new complex parameter (alongside the real parameters $x \in X$).

The functions in $\mathcal{C}^{\mathcal{M}}$ have a domain of the form $(\mathbb{C} \setminus P) \times X$. We are interested in studying functions defined on domains of the form $(\Sigma \setminus P) \times X$, where Σ is a strip. For this, we define $\mathcal{C}^{\mathcal{M}}_{\Sigma}(X)$ as the collection of all restrictions to $\Sigma \times X$ of functions in $\mathcal{C}^{\mathcal{M}}(X)$ and thus form the systems $\mathcal{A}, \mathcal{P}(\mathcal{S}_+)$ and $\mathcal{C}^{\mathcal{M}}$, proceeding as in Notation 2.12 (note that, since the functions in these collections are defined on the whole of $\mathbb{C}$ and not just on strips, the two definitions of $\mathcal{C}^{\mathcal{M}}$ coincide). With this notation we immediately derive from [CCRS23] the following result.

Theorem 2.18 ([CCRS23, Theorem 2.16 and Corollary 2.18])**.** *The system $\mathcal{C}^{\mathcal{M}}$ is stable under parametric integration. Moreover, $\mathcal{C}^{\mathcal{M}}$ is the smallest system of $\mathcal{A}$-algebras containing $\mathcal{C}^{\mathbb{C}}$ and stable under the generalized parametric Mellin transform.*

2.2.2. *Parametric Fourier transforms of parametric power-constructible functions.* Our next goal is to define a system containing both the parametric Fourier and the parametric Mellin transforms of power-constructible functions.

Definition 2.19. Let $\Sigma \subseteq \mathbb{C}$ be a strip. Consider the fixed frequency parametric Fourier operator $\mathfrak{f}$ acting on $\mathcal{C}_{\Sigma}^{\mathcal{M}}$:

$$\mathfrak{f}[h](s,x) = \int_{\mathbb{R}} h(s,x,y)\,\mathrm{e}^{\mathrm{i}y}\mathrm{dy} \quad \left(h \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R})_{\mathrm{int}}\right).$$

If h has no poles outside some closed discrete set $P \subseteq \mathbb{C}$, then so does $\mathfrak{f}[h]$. Define

$$\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X) = \left\{\mathfrak{f}[h] : \ h \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R})_{\mathrm{int}}\right\}.$$

It is a $\mathcal{C}_{\Sigma}^{\mathcal{M}}(X)$-module.

We will show in Section 3 that the functions in the above collection can be extended to the whole complex plane, in the sense of Notation 2.12:

Proposition 2.20. *The collection $\left\{\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X) : \ \Sigma \text{ strip}\right\}$ has the extension property.*

Thanks to the above proposition, we may define the system

$$\mathcal{C}^{\mathcal{M},\mathcal{F}} = \left\{\mathcal{C}^{\mathcal{M},\mathcal{F}}(X) : \ X \subseteq \mathbb{R}^m \text{ subanalytic}, \ m \in \mathbb{N}\right\}.$$

Our main stability result is the following.

Theorem 2.21. *The system $\mathcal{C}^{\mathcal{M},\mathcal{F}}$ is stable under parametric integration. It is a system of $\mathbb{C}$-algebras, containing $\mathcal{C}^{\mathbb{C}} \cup \mathrm{e}^{\mathrm{i}\mathcal{S}}$, and stable under generalized parametric Mellin and Fourier transforms.*

3. Generators of $\mathcal{C}^{\mathcal{M},\mathcal{F}}$ and proof of the extension result

3.1. **Generators of $\mathcal{C}^{\mathcal{M},\mathcal{F}}$.** In this section we choose a set of generators for $\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X)$ as an additive group, of a special form, which is suitable for proving Proposition 2.20 and Theorem 2.21.

First, we recall some definitions from [CCRS23].

Definition 3.1. Let $X \subseteq \mathbb{R}^m$ be a subanalytic cell and

$$B = \{(x,y) : \ x \in X, \ a(x) < y < b(x)\}, \tag{3.1}$$

where $a, b : X \longrightarrow \mathbb{R}$ are analytic subanalytic functions with $1 \leq a(x) < b(x)$ for all $x \in X$, and b is allowed to be $\equiv +\infty$. We say that B has *bounded y-fibers* if $b < +\infty$ and *unbounded y-fibers* if $b \equiv +\infty$.

- A 1-bounded subanalytic map $\psi : B \longrightarrow \mathbb{R}^{M+2} \in \mathcal{S}_c^{M+2}(B)$ is *y-prepared* if it has the form

$$\psi(x,y) = \left(c(x), \left(\frac{a(x)}{y}\right)^{\frac{1}{d}}, \left(\frac{y}{b(x)}\right)^{\frac{1}{d}}\right), \tag{3.2}$$

where $d \in \mathbb{N} \setminus \{0\}$.
If $b \equiv +\infty$, then we will implicitly assume that the last component is missing and hence $\psi : B \longrightarrow \mathbb{R}^{M+1}$.

- A subanalytic strong unit $U \in \mathcal{S}(B)$ is *ψ-prepared* if there exists a strongly convergent series $F \in \mathbb{R}[\![Z]\!]$ such that $U = F \circ \psi$. If B has unbounded y-fibers, then the *nested ψ-prepared form* of U is

$$U(x,y) = \sum_k b_k(x) \left(\frac{a(x)}{y}\right)^{\frac{k}{d}}, \tag{3.3}$$

where the subanalytic functions b_k are bounded and b_0 does not vanish on X.

- A parametric strong function $\Phi \in \mathcal{A}_\Sigma(B)$ is *ψ-prepared* if there exists a strongly convergent series $F = \sum \xi_I(s) Z^I \in \mathcal{E}[\![Z]\!]$ such that

$$\forall (s,x,y) \in (\Sigma \setminus P(F)) \times B,\ \Phi(s,x,y) = F \circ (s, \psi(x,y)). \tag{3.4}$$

If B has unbounded y-fibers, then the *nested ψ-prepared form* of Φ is

$$\Phi(s,x,y) = \sum_k \xi_k(s,x) \left(\frac{a(x)}{y}\right)^{\frac{k}{d}},\ \text{where } \xi_k(s,x) \in \mathcal{A}_\Sigma(X). \tag{3.5}$$

- A subanalytic function $\varphi \in \mathcal{S}(B)$ is *prepared* if there are $\omega \in \mathbb{Z}$, an analytic function $\varphi_0 \in \mathcal{S}(X)$ and a ψ-prepared subanalytic strong unit U such that

$$\varphi(x,y) = \varphi_0(x)\, y^{\frac{\omega}{d}} U(x,y). \tag{3.6}$$

In order to choose suitable generators for $\mathcal{C}^{\mathcal{M},\mathcal{F}}$, we first need to introduce two additional classes of functions.

Definition 3.2. Let $\Sigma \subseteq \mathbb{C}$ be a strip and $X \subseteq \mathbb{R}^m$ be a subanalytic set. Let B be as in (3.1).

- Let $\mathcal{C}^{\mathcal{M},\mathrm{i}\mathcal{S}}_\Sigma(X)$ be the additive group generated by the functions of the form

$$g\mathrm{e}^{\mathrm{i}\varphi} \quad \left(g \in \mathcal{C}^{\mathcal{M}}_\Sigma(X), \varphi \in \mathcal{S}(X)\right). \tag{3.7}$$

It is a $\mathbb{C}$-algebra.

- A *transcendental element* is a function of the form

$$(\Sigma \setminus P) \times X \ni (s,x) \longmapsto \gamma(s,x) = \int_{\mathbb{R}} \chi_B(x,y)\, y^{\lambda(s)} (\log y)^\mu\, \Phi(s,x,y)\, \mathrm{e}^{\sigma \mathrm{i} y} \mathrm{d}y,$$

where $\sigma \in \{+,-\}, \mu \in \mathbb{N}, \Phi$ is a ψ-prepared parametric strong function (as in (3.4), with ψ as in (3.2)) without poles outside the closed discrete set $P \subseteq \mathbb{C}$ and $\lambda(s) = \dfrac{\ell s + \eta}{d}$, for some $\ell \in \mathbb{Z}, \eta \in \mathbb{C}$ and the same d appearing in (3.2). If B has unbounded y-fibers, then we require that for all $s \in \Sigma,\ \Re(\lambda(s)) < -1$.
We let $\Gamma_\Sigma(X)$ be the collection of all transcendental elements on $\Sigma \times X$.
Thus, a *generator* (as an additive group) of the $\mathcal{C}^{\mathcal{M},\mathrm{i}\mathcal{S}}_\Sigma(X)$-module generated by the set $\Gamma_\Sigma(X)$ is a function of the form

$$T = g\mathrm{e}^{\mathrm{i}\varphi}\gamma \quad \left(g \in \mathcal{C}^{\mathcal{M}}_\Sigma(X), \varphi \in \mathcal{S}(X), \gamma \in \Gamma_\Sigma(X)\right). \tag{3.8}$$

Notice that $1 = \mathfrak{f}\left[\frac{\mathrm{i}}{2}\chi_{[\pi,2\pi]}\right] \in \Gamma_\Sigma(X)$. In particular, (3.7) is an instance of (3.8).

Lemma 3.3. *Let T be a generator as in (3.8), without poles outside some closed discrete set $P \subseteq \mathbb{C}$. There exists $h \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R})_{\text{int}}$ without poles outside P such that $T = \mathfrak{f}[h]$. In particular,*

$$\mathcal{C}_{\Sigma}^{\mathcal{M},\mathrm{i}\mathcal{S}}(X), \Gamma_{\Sigma}(X) \subseteq \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X).$$

Proof. Let B be as in (3.1) and

$$G(s,x,y) = \chi_B(x,y)\, g(s,x)\, y^{\lambda(s)} (\log y)^{\mu}\, \Phi(s,x,y).$$

Then $G \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R})_{\text{int}}$ and

$$T(s,x) = \int_{\mathbb{R}} G(s,x,y)\, \mathrm{e}^{\mathrm{i}(\varphi(x)+\sigma y)} \mathrm{d}y.$$

Thus, by a change of variables, $T = \mathfrak{f}[h]$ with $h(s,x,y) := \sigma G(s,x,\sigma(y-\varphi(x))) \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R})_{\text{int}}$. Notice that h has no poles outside P. $\square$

Lemma 3.4. *Let $h \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X)$. There are a closed discrete set $P \subseteq \mathbb{C}$ and finitely many generators $T_1, \dots, T_m$ as in (3.8) such that $h, T_1, \dots, T_m$ have no poles outside P and $h = \sum T_j$. In particular, $\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X)$ can also be described as the $\mathcal{C}_{\Sigma}^{\mathcal{M},\mathrm{i}\mathcal{S}}(X)$-module generated by the set $\Gamma_{\Sigma}(X)$, and the functions of the form (3.8) are generators of $\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X)$ as an additive group.*

Recall the notation established right after Definition 3.9 in [CCRS23].

Notation 3.5. Let $A \subseteq X \times \mathbb{R}$ be a subanalytic cell which is open over and projects onto X (see Notation 2.5), let θ_A be its center, so that the set $\{y - \theta_A(x) : (x,y) \in A\}$ is contained in one of the sets $(-\infty,-1), (-1,0), (0,1), (1,+\infty)$, as in [CCMRS18, Definition 3.4]. There are unique sign conditions $\sigma_A, \tau_A \in \{-1,1\}$ such that

$$A = \{(x,y) : x \in X,\ a_A(x) < \sigma_A (y - \theta_A(x))^{\tau_A} < b_A(x)\} \tag{3.9}$$

for some analytic subanalytic functions a_A, b_A such that $1 \leq a_A(x) < b_A(x) \leq +\infty$. Let

$$B_A = \{(x,y) : x \in X,\ a_A(x) < y < b_A(x)\} \tag{3.10}$$

and $\Pi_A : B_A \longrightarrow A$ be the bijection

$$\Pi_A(x,y) = (x, \sigma_A y^{\tau_A} + \theta_A(x)),\ \Pi_A^{-1}(x,y) = (x, \sigma_A (y - \theta_A(x))^{\tau_A}). \tag{3.11}$$

We will still denote by Π_A the map $\mathbb{C} \times B_A \ni (s,x,y) \longmapsto (s, \Pi_A(x,y)) \in \mathbb{C} \times A$.

Remark 3.6. By [CCMRS18, Definition 3.4(3)], if A is a cell of the form $A = \{(x,y) : x \in X,\ y > f(x)\}$ with $f \in \mathcal{S}(X)$ and $f \geq 1$, then $\sigma_A = \tau_A = 1$ and $\theta_A = 0$. Hence in this case $a_A = f,\ b_A = +\infty$ and $B_A = A$.

Proof of Lemma 3.4. Write $h = \mathfrak{f}[g]$, for some $g \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R})_{\text{int}}$ and apply the parametric power-constructible Preparation Theorem [CCRS23, Proposition 4.7] to g: this yields a cell decomposition of $X \times \mathbb{R}$ and by linearity of the integral we may concentrate on a cell A which is open over X.

Using Notation 3.5, if $\tau_A = -1$ then the set $\{|y-\theta_A(x)| : (x,y) \in A\}$ is contained in $(0,1)$, so that $(x,y) \longmapsto \mathrm{e}^{\mathrm{i}(y-\theta_A(x))}$ is a complex-valued subanalytic function. Hence, in this case we may write

$$\int_{A_x} g \restriction A(s,x,y)\,\mathrm{e}^{\mathrm{i}y}\mathrm{d}y = \mathrm{e}^{\mathrm{i}\theta_A(x)} \int_{A_x} g \restriction A(s,x,y)\,\mathrm{e}^{\mathrm{i}(y-\theta_A(x))}\mathrm{d}y.$$

As the integrand on the right hand side is a parametric power-constructible function, by [CCRS23, Theorem 2.16 and Remark 6.7] there are a closed discrete set $P \subseteq \mathbb{C}$ (containing the poles of g) and a parametric power-constructible function $G \in \mathcal{C}_\Sigma^{\mathcal{M}}(X)$ without poles outside P such that $\mathfrak{f}[g \restriction A] = \mathrm{e}^{\mathrm{i}\theta_A} G$.

If $\tau_A = 1$ then we apply the change of variables Π_A under the sign of integral and, using [CCRS23, Proposition 4.7], we write $g \circ \Pi_A$ as a finite sum of prepared generators as in [CCRS23, Equation (4.8)]:

$$\begin{aligned}
\int_{A_x} g \restriction A(s,x,y)\,\mathrm{e}^{\mathrm{i}y}\mathrm{d}y &= \int_{a_A(x)}^{b_A(x)} g \circ \Pi_A(s,x,y)\,\frac{\partial \Pi_A}{\partial y}(x,y)\,\mathrm{e}^{\mathrm{i}(\sigma_A y + \theta_A(x))}\mathrm{d}y \\
&= \mathrm{e}^{\mathrm{i}\theta_A(x)} \int_{a_A(x)}^{b_A(x)} \sum_i G_i(s,x)\, y^{\lambda_i(s)} (\log y)^{\mu_i}\, \Phi_i(s,x,y)\,\mathrm{e}^{\mathrm{i}\sigma_A y}\mathrm{d}y \\
&= \sum_i \mathrm{e}^{\mathrm{i}\theta_A(x)} G_i(s,x)\,\gamma_i(s,x).
\end{aligned}$$

Summing up, we have written h as a finite sum of generators without poles outside some closed discrete set $P \subseteq \mathbb{C}$. $\square$

Thus, from now on we will refer to the functions of the form (3.8) as *generators* of $\mathcal{C}_\Sigma^{\mathcal{M},\mathcal{F}}(X)$.

3.2. **Proof of Proposition 2.20.** Let $\Sigma, \Sigma' \subseteq \mathbb{C}$ be strips such that $\Sigma \subseteq \Sigma'$ and $h \in \mathcal{C}_\Sigma^{\mathcal{M},\mathcal{F}}(X)$ without poles outside some closed discrete set $P \subseteq \mathbb{C}$. Write h as a finite sum of generators of the form (3.8), which is possible by Lemma 3.4. The problem is that the integrands of the transcendental elements appearing in the generators are integrable on $(\Sigma \setminus P) \times X$ but might not be on the whole $(\Sigma' \setminus P) \times X$ (the only issue here is the integrability of a power of y at $+\infty$, as the parametric strong functions are bounded on the whole plane $\mathbb{C}$). In this case, we need to rewrite the transcendental elements as sums of generators on $(\Sigma' \setminus P) \times X$. With this in mind, we may suppose that h itself is a transcendental element with unbounded y-fibers of the form

$$h(s,x) = \int_{a(x)}^{+\infty} y^{\lambda(s)} (\log y)^{\mu}\, \Phi(s,x,y)\,\mathrm{e}^{\sigma \mathrm{i}y}\mathrm{d}y,$$

with $a \in \mathcal{S}(X)$ such that for all $x \in X$, $a(x) \geq 1$, and that the set $S_0 = \{s \in \Sigma' : \Re(\lambda(s)) < -1\}$ is a proper subset of Σ'. It follows that the above integral is not finite on $(\Sigma' \setminus S_0) \times X$. Using the strong convergence of the series defining Φ and (3.5), we may rewrite, for some

$k_0 \geq 0$,

$$h(s,x) = \sum_{k \geq k_0} \xi_k^c(s,x)(a(x))^{\frac{k}{d}} \int_{a(x)}^{+\infty} y^{\frac{\ell s+\eta-k}{d}} (\log y)^\mu \mathrm{e}^{\sigma \mathrm{i} y} \mathrm{d}y$$
$$= \sum_{k \geq k_0} g_k(s,x) \int_{a(x)}^{+\infty} y^{\lambda_k(s)} (\log y)^\mu \mathrm{e}^{\sigma \mathrm{i} y} \mathrm{d}y$$

As the real part of the exponent $\lambda_k(s)$ decreases as k increases and as Σ' has bounded width, there are only finitely many power-log monomials which are not integrable for all $s \in \Sigma'$. Let us concentrate on one such critical power-log monomial and use integration by parts (where we integrate the exponential and derive the power-log monomial):

$$\forall (s,x) \in S_0 \times X, \int_{a(x)}^{+\infty} y^{\lambda_k(s)} (\log y)^\mu \mathrm{e}^{\sigma \mathrm{i} y} \mathrm{d}y$$
$$= \sigma \mathrm{i} \mathrm{e}^{\sigma \mathrm{i} a(x)} (a(x))^{\lambda_k(s)} (\log(a(x)))^\mu - \int_{a(x)}^{+\infty} y^{\lambda_k(s)-1} (\log y)^{\mu-1} (\lambda_k(s) \log y + \mu) \mathrm{e}^{\sigma \mathrm{i} y} \mathrm{d}y.$$

Note that the right-hand side of the above equality is actually defined on a strictly larger set than $S_0 \times X$, namely on the set $S_1 \times X$, where $S_1 = \{s \in \Sigma' : \Re(\lambda_k(s)) < 0\}$. Since Σ' has bounded width, there is an integer $N_k \in \mathbb{N}$ such that $\Sigma' = \{s \in \Sigma' : \Re(\lambda_k(s)) < N_k - 1\}$ and if we repeat the above procedure N_{k_0} times for each critical monomial, then we rewrite h as a sum of generators such that the transcendental elements are well defined on the whole $(\Sigma' \setminus P) \times X$.

□

4. Strongly integrable and monomial generators

The aim of this section is to prove Theorems 2.9 and 2.21, assuming a central result, Theorem 6.5, the proof of which requires extensive work carried out in the next two sections. We start by dealing with integrals of some specific functions in our class $\mathcal{C}^{\mathcal{M},\mathcal{F}}$.

4.1. **Special generators of $\mathcal{C}^{\mathcal{M},\mathcal{F}}$.** We lay the foundations for the proofs of Theorems 2.9 and 2.21 by treating some special cases to which we will reduce later (in Sections 5 and 6). More precisely, we identify two special types of generators for $\mathcal{C}^{\mathcal{M},\mathcal{F}}$, strongly integrable generators and monomial generators, for which we show that their parametric integrals lie in $\mathcal{C}^{\mathcal{M},\mathcal{F}}$. In Section 6, Proposition 6.4 will provide a reduction to such special generators.

To illustrate the main ideas of this section, we start with two examples of explicit integration of very simple generators.

Examples 4.1. Let $\Sigma = \{s : -2 < \Re(s) < 1\}$ and $B = \{(x,y) : x \in X, y > a(x)\}$, for some analytic $a \in \mathcal{S}(X)$ such that for all $x \in X$, $a(x) \geq 1$.

(1) Let $D = \{(x,y,t) : (x,y) \in B, t > \widetilde{a}(x,y)\}$, for some analytic $\widetilde{a} \in \mathcal{S}(B)$ such that for all $(x,y) \in B$, $\widetilde{a}(x,y) \geq 1$, and $\Phi \in \mathcal{A}_\Sigma(D)$ be a parametric strong function without poles outside some closed discrete set $P \subseteq \mathbb{C}$. If

$$g(s,x,y,t) = y^{-s-3} t^{s-2} \Phi(s,x,y,t) \chi_D(x,y,t),$$

then for all $(s,x,y)\in(\Sigma\setminus P)\times X\times\mathbb{R},\ t\longmapsto g(s,x,y,t)\in L^1(\mathbb{R})$, so $h=\mathfrak{f}[g]\in\mathcal{C}_\Sigma^{\mathcal{M},\mathcal{F}}(X\times\mathbb{R})$ is well defined on $(\Sigma\setminus P)\times X\times\mathbb{R}$.
We claim that $g\in\mathcal{C}_\Sigma^{\mathcal{M}}(X\times\mathbb{R}^2)_{\text{int}}$ and that there exist a closed discrete set $P'\supseteq P$ and a function $H\in\mathcal{C}_\Sigma^{\mathcal{M},\mathcal{F}}(X)$ without poles outside P' such that

$$\forall(s,x)\in(\Sigma\setminus P')\times X,\ H(s,x)=\int_{\mathbb{R}}h(s,x,y)\,\mathrm{d}y.$$

To see this, note that, since Φ is bounded, there is a constant $C>0$ such that

$$\forall(s,x,y)\in(\Sigma\setminus P)\times B,\ \int_1^{+\infty}t^{s-2}\left|\Phi(s,x,y,t)\right|\mathrm{d}t<C.$$

It follows that

$$\forall(s,x,y)\in(\Sigma\setminus P)\times B,\ \mathfrak{f}[|g|]\leq y^{-s-3}\int_1^{+\infty}t^{s-2}\left|\Phi(s,x,y,t)\right|\mathrm{d}t\leq Cy^{-s-3},$$

so $y\longmapsto\mathfrak{f}[|g|](s,x,y)\in L^1(\mathbb{R})$ and by Tonelli's Theorem, $g\in\mathcal{C}_\Sigma^{\mathcal{M}}(X\times\mathbb{R}^2)_{\text{int}}$. Hence, by Fubini's Theorem

$$\int_{\mathbb{R}}h(s,x,y)\,\mathrm{d}y=\int_{\mathbb{R}}t^{s-2}\mathrm{e}^{\mathrm{i}t}\left[\int_{\mathbb{R}}y^{-s-3}\Phi(s,x,y,t)\,\chi_D(x,y,t)\,\mathrm{d}y\right]\mathrm{d}t$$

and the integrand $\tilde{g}$ in the inner integral belongs to $\mathcal{C}_\Sigma^{\mathcal{M}}(D)$ and is integrable with respect to y. By Theorem 2.18, there are a closed discrete set $P'\subseteq\mathbb{C}$ containing P and a function $G\in\mathcal{C}_\Sigma^{\mathcal{M}}(X\times\mathbb{R})$ without poles outside P' such that

$$\forall(s,x)\in(\Sigma\setminus P')\times X,\ G(s,x,t)=t^{s-2}\int_{\mathbb{R}}y^{-s-3}\Phi(s,x,y,t)\,\chi_D(x,y,t)\,\mathrm{d}y.$$

Notice also that $G\in\mathcal{C}_\Sigma^{\mathcal{M}}(X\times\mathbb{R})_{\text{int}}$ and that $H=\mathfrak{f}[G]$ proves the claim.

(2) Consider $g(s,x,y)=y^s\chi_B(x,y)\in\mathcal{C}_\Sigma^{\mathcal{M}}(X\times\mathbb{R})$ and $T(s,x,y)=g(s,x,y)\,\mathrm{e}^{\mathrm{i}y}\in\mathcal{C}_\Sigma^{\mathcal{M},\mathcal{F}}(X\times\mathbb{R})$. As

$$\mathrm{Int}(g;\Sigma\times X)=\{s\in\Sigma:\ \Re(s)<-1\}\times X\neq\Sigma\times X,$$

we cannot apply the operator $\mathfrak{f}$ to g in order to express the integral of T with respect to y. However, we claim that there exists a function $H\in\mathcal{C}_\Sigma^{\mathcal{M},\mathcal{F}}(X)$ such that

$$\forall(s,x)\in\mathrm{Int}(T;\Sigma\times X),\ H(s,x)=\int_{a(x)}^{+\infty}y^s\mathrm{e}^{\mathrm{i}y}\mathrm{d}y. \tag{4.1}$$

To show this, let us integrate by parts $y^s\mathrm{e}^{\mathrm{i}y}$ twice, where we integrate the exponential and derive the parametric power:

$$\begin{aligned}\int y^s\mathrm{e}^{\mathrm{i}y}\mathrm{d}y&=y^s\frac{\mathrm{e}^{\mathrm{i}y}}{\mathrm{i}}-\frac{1}{\mathrm{i}}\int sy^{s-1}\mathrm{e}^{\mathrm{i}y}\mathrm{d}y\\&=\mathrm{i}y^s\mathrm{e}^{\mathrm{i}y}+sy^{s-1}\mathrm{e}^{\mathrm{i}y}-s(s-1)\int y^{s-2}\mathrm{e}^{\mathrm{i}y}\mathrm{d}y.\end{aligned}$$

Define

$$H(s,x)=-\mathrm{i}\,(a(x))^s\,\mathrm{e}^{\mathrm{i}a(x)}-s\,(a(x))^{s-1}\,\mathrm{e}^{\mathrm{i}a(x)}-s(s-1)\int_{a(x)}^{+\infty}y^{s-2}\mathrm{e}^{\mathrm{i}y}\mathrm{d}y.$$

Notice that H is well defined on $\Sigma \times X$, because the real part of the exponent in the integrand is always < -1 on Σ, and that $H \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X)$, because the last term is obtained by applying the operator $\mathfrak{f}$ to $\tilde{g}(s,x,y) = s(s-1)y^{s-2}\chi_B(s,y) \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R})_{\mathrm{int}}$. Note also that H satisfies (4.1), since $\mathrm{Int}(T; \Sigma \times X) = \{s \in \Sigma : \Re(s) < -1\} \times X$, and on this part of the space the exponents of the parametric powers y^s, y^{s-1} have negative real part.

The techniques illustrated in these two examples can be generalized and used to integrate generators of $\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$ of a rather simple form.

Recall that, by Lemma 3.3, every generator of $\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$ can be written as $\mathfrak{f}[h]$, for some $h \in \mathcal{C}_{\Sigma}^{\mathcal{M}}((X \times \mathbb{R}) \times \mathbb{R})_{\mathrm{int}}$.

Definition 4.2. Let $T \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$ be a generator.

- T is *strongly integrable* if T can be written as $\mathfrak{f}[h]$, for some $h \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R}^2)_{\mathrm{int}}$. If B is a cell as in (3.1), we say that T is strongly integrable *on* B if $T\chi_B$ is strongly integrable.
- T is *monomial in (its last variable)* y if T has the form

$$T(s,x,y) = f(s,x)\, y^{\lambda(s)} (\log y)^{\mu}\, \mathrm{e}^{\mathrm{i}Q(x,y)}, \tag{4.2}$$

where $f \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X), \mu \in \mathbb{N}, \lambda(s) = \frac{\ell s + \eta}{d}$ for some $\ell \in \mathbb{Z},\ \eta \in \mathbb{C},\ d \in \mathbb{N} \setminus \{0\}$ and $Q \in \mathcal{S}(X)\left[y^{\frac{1}{d}}\right]$ is a polynomial in the variable $y^{\frac{1}{d}}$ with coefficients subanalytic functions of x. The tuple (d, ℓ, η, μ, Q) is called the *monomial data* of T.

Remark 4.3. Let $T = g\mathrm{e}^{\mathrm{i}\varphi}\gamma \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$ be a generator and write γ as $\int_{\mathbb{R}} G(s,x,y,t)\,\mathrm{e}^{\mathrm{i}t}\mathrm{d}t$ for some appropriate $G \in \mathcal{C}_{\Sigma}^{\mathcal{M}}((X \times \mathbb{R}) \times \mathbb{R})_{\mathrm{int}}$. Suppose that $gG \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R}^2)_{\mathrm{int}}$. Then T is strongly integrable. To see this, proceed as in Lemma 3.3 and write T as $\mathfrak{f}[h]$. It is clear that $h \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R}^2)_{\mathrm{int}}$.

Our next aim is to integrate a single generator which is of either of the forms in Definition 4.2.

Proposition 4.4. *Let $g \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R}^2)_{\mathrm{int}}$ be without poles outside some closed discrete set $P \subseteq \mathbb{C}$ and $\varphi \in \mathcal{S}(X \times \mathbb{R}^2)$. There exist a closed discrete set $P' \subseteq \mathbb{C}$ containing P and a function $H \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X)$ without poles outside P' such that*

$$\forall (s,x) \in (\Sigma \setminus P') \times X,\ H(s,x) = \int_{\mathbb{R}^2} g(s,x,y,t)\,\mathrm{e}^{\mathrm{i}\varphi(x,y,t)}\mathrm{d}y \wedge \mathrm{d}t.$$

Moreover, the set $P' \setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice.

Proof. Up to decomposing $X \times \mathbb{R}^2$ into subanalytic cells, we may suppose that, on each cell A of base X and open over $\mathbb{R}^m$, either φ does not depend on (y,t), or for one of these two variables (say, y), the function $y \longmapsto \varphi(x,y,t)$ is C^1 and strictly monotonic.

In the first case we factor the exponential out of the integral and we apply Theorem 2.18 to g. In the second case, up to applying the subanalytic change of variables $(x,y,t) \longmapsto (x, \varphi(x,y,t), t)$ and multiplying by the Jacobian of its inverse, we may suppose that $\varphi(x,y,t) =$

y on A. Hence, by Fubini's Theorem,

$$\int_A g(s,x,y,t)\,\mathrm{e}^{\mathrm{i}\varphi(x,y,t)}\mathrm{d}y \wedge \mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}y}\left[\int_{\mathbb{R}} \chi_A(x,y,t)\,g(s,x,y,t)\,\mathrm{d}t\right]\mathrm{d}y.$$

Again by Theorem 2.18, applied to the integrand inside the square brackets, the right-hand side of the above equation is of the form $\mathfrak{f}[\tilde{g}]$, for some suitable $\tilde{g} \in \mathcal{C}^{\mathcal{M}}_{\Sigma}(X \times \mathbb{R})$ without poles outside some closed discrete set $P' \supseteq P$. We conclude by linearity of the integral, taking the sum over the cells of the decomposition. $\square$

Corollary 4.5. *$\mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X)$ is a $\mathbb{C}$-algebra.*

Proof. Is suffices to show that if $h_1, h_2 \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X)$ then $h_1 \cdot h_2 \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X)$. Write $h_i = \mathfrak{f}[g_i]$, for some $g_i \in \mathcal{C}^{\mathcal{M}}_{\Sigma}(X \times \mathbb{R})_{\mathrm{int}}$. By Fubini's Theorem, $(s,x,y,t) \longmapsto g_1(s,x,y) \cdot g_2(s,x,t) \in \mathcal{C}^{\mathcal{M}}_{\Sigma}(X \times \mathbb{R}^2)_{\mathrm{int}}$ so Proposition 4.4 applies. $\square$

Proposition 4.4 allows us to integrate strongly integrable generators.

Corollary 4.6. *Let $T \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X \times \mathbb{R})$ be a strongly integrable generator without poles outside some closed discrete set $P \subseteq \mathbb{C}$. There exist a closed discrete set $P' \subseteq \mathbb{C}$ containing P, such that $P' \setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice, and a function $H \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X)$ without poles outside P' such that*

$$\forall (s,x) \in (\Sigma \setminus P') \times X,\ H(s,x) = \int_{\mathbb{R}} T(s,x,y)\,dy.$$

Proof. Immediate from Fubini's Theorem and Proposition 4.4. $\square$

Next, we consider a monomial generator and interpolate its integral on a given cell by a function of the class $\mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}$.

Lemma 4.7. *Let B be a cell as in (3.1) with bounded y-fibers and let $T \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X \times \mathbb{R})$ be a generator which is monomial in y (as in (4.2)), without poles outside some closed discrete set $P \subseteq \mathbb{C}$. Then T is strongly integrable on B.*

Proof. For all $(s,x) \in (\Sigma \setminus P) \times X$, $y \longmapsto |T(s,x,y)\,\chi_B(x,y)|$ extends to a continuous function on $[a(x), b(x)]$. Hence, by Remark 4.3 we are done. $\square$

Proposition 4.8. *Let B be a cell as in (3.1) with unbounded y-fibers and let $T \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X \times \mathbb{R})$ be a generator which is monomial in y (as in (4.2)), without poles outside some closed discrete set $P \subseteq \mathbb{C}$. Then*

$$\mathrm{Int}(T\chi_B; (\Sigma \setminus P) \times X) = \{(s,x) :\ \Re(\lambda(s)) < -1 \vee (\Re(\lambda(s)) \geq -1 \wedge f(s,x) = 0)\}.$$

Moreover there are a closed discrete set $P' \subseteq \mathbb{C}$ containing P, such that $P' \setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice, and a function $H \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X)$ without poles outside P' such that

$$\forall (s,x) \in \mathrm{Int}(T\chi_B; (\Sigma \setminus P') \times X),\ H(s,x) = \int_{a(x)}^{+\infty} T(s,x,y)\,dy.$$

Proof. The statement on the integration locus is immediate.

Write $Q(x,y)=\sum_{i\leq n} b_i(x)\, y^{\frac{i}{d}}$. We may suppose that $n>0$, because otherwise we are done by Theorem 2.18. By o-minimality, we may suppose that for all $x\in X$, $b_n(x)\neq 0$. By Lemma 4.7 and definable choice, we may suppose that for all $x\in X$, $y\longmapsto Q(x,y)$ is monotonic (say, strictly increasing) on $(a(x),+\infty)$. Hence we may write $Q(x,y)=b_n(x)\, y^{\frac{n}{d}}(1+\varepsilon_1(x,y))$, where $\varepsilon_1\in\mathbb{R}\{x\}\left[y^{-\frac{1}{d}}\right]$ with $\varepsilon_1(x,0)=0$, and the compositional inverse has the form $\phi(x,z)=c(x)\, z^{\frac{d}{n}}(1+\varepsilon_2(x,z))$, for some analytic $c\in\mathcal{S}(X)$ and $\varepsilon_2\in\mathbb{R}\left\{x,z^{-\frac{1}{n}}\right\}$ with $\varepsilon_2(x,0)=0$. Note that $\dfrac{\partial\phi}{\partial z}(x,z)=\dfrac{d}{n}c(x)\, z^{\frac{d}{n}-1}(1+\varepsilon_3(x,z))$, for some $\varepsilon_3\in\mathbb{R}\left\{x,z^{-\frac{1}{n}}\right\}$ with $\varepsilon_3(x,0)=0$. Hence, on $\operatorname{Int}(T\chi_B;(\Sigma\setminus P)\times X)$ we may write

$$\int_{a(x)}^{+\infty} T(s,x,y)\,\mathrm{d}y=\tilde{f}(s,x)\int_{Q(x,a(x))}^{+\infty} z^{\tilde{\lambda}(s)}\left[\frac{d}{n}\log z+G(x,z)\right]^{\mu} u(s,x,z)\,\mathrm{e}^{\mathrm{i}z}\mathrm{d}z,$$

where $\tilde{f}(s,x)=f(s,x)\frac{d}{n}(c(x))^{\lambda(s)+1}$, $\tilde{\lambda}(s)=\frac{d}{n}\lambda(s)+\frac{d}{n}-1$, $G(x,z)=\log(c(x))+\log(1+\varepsilon_2(x,z))$ and $u(s,x,z)=(1+\varepsilon_2(x,z))^{\lambda(s)}(1+\varepsilon_3(x,z))$ is a parametric strong unit. Note that, for all $x\in X$, G can be expanded as a power series (with nonzero constant term) in the variable $z^{-\frac{1}{n}}$. By expanding the power μ of the square bracket, we may rewrite the above integral as a finite sum of terms where the integrand has the form $z^{\tilde{\lambda}(s)}\mathrm{e}^{\mathrm{i}z}\cdot(\log z)^{\nu}\, U_{\nu}(s,x,z)$, for some $\nu\leq\mu$ and parametric strong unit U_{ν} which can be expanded as a series in the variable $z^{-\frac{1}{n}}$. It follows that there are finitely many monomials in the integrand of the form $z^{\Re(\tilde{\lambda}(s))-\frac{k}{n}}(\log z)^{\nu}$ for which the integral is not finite. Argueing as in the proof of Proposition 2.20, we find the function H in the statement by integration by parts. $\square$

4.2. **Overview of the proofs of Theorems 2.9 and 2.21.** The proof of Theorem 2.21, which will be completed in Section 6, is organized as follows: we show that, given $h\in\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X\times\mathbb{R})$ without poles outside some closed discrete set $P\subseteq\mathbb{C}$, the domain $X\times\mathbb{R}$ can be partitioned into subanalytic cells such that on each cell A, up to a subanalytic change of variables, h can be written as a finite sum of generators which are either strongly integrable or monomial in the last variable (Proposition 6.4). The results of the current section provide, for each such generator T_i, a description of the integration locus and a function $H_i\in\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X)$ without poles outside some closed discrete set $P'\supseteq P$, which coincides with the integral of T_i on its integration locus. Next, we show that, up to possibly enlarging the closed discrete set P', the integration locus of $h\restriction A$ is the intersection of the integration loci of the generators T_i (this is done using a non-compensation argument proven in [CCRS23, Proposition 3.4]). Thus the sum of the functions H_i interpolates the integral of $h\restriction A$ on its integration locus (Theorem 6.5). Theorem 2.21 follows from Theorem 6.5 by Fubini's Theorem (with an argument spelled out in detail in [CCRS23, pp. 31-32]), which shows that we can iterate the argument above integrating with respect to one variable at the time.

The proof of Theorem 2.9 is just a special case of that of Theorem 2.21, where all the functions involved happen not to depend on the variable s. In particular, that $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ is a system of $\mathbb{C}$-algebras containing $\mathrm{e}^{\mathrm{i}\mathcal{S}}$ follows from Corollary 4.5 and Lemma 3.3. It follows from stability under parametric integration and the definition of $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ that it is the smallest system of $\mathbb{C}$-algebras containing $\mathcal{S}_{+}^{\mathbb{C}}\cup\mathrm{e}^{\mathrm{i}\mathcal{S}}$ and stable under parametric integration, and the smallest such system containing $\mathcal{C}^{\mathbb{C}}$ and stable under the parametric Fourier transform and right-composition with subanalytic maps.

5. Preparation

With the aim of proving Proposition 6.4, in this section we write every function of the class $\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X\times\mathbb{R})$, piecewise and up to a subanalytic change of variables, as a finite sum of generators of a special *prepared form* which gives some information on their integration locus. This builds further (and relies) on preparation results for subanalytic functions from [Par94, Par01, LR98, Mil06]

Notation 5.1. Let $B\subseteq X\times\mathbb{R}$ be as in (3.1) Consider a cell $D\subseteq B\times\mathbb{R}$ of the form

$$D=\left\{(x,y,t):\ (x,y)\in B,\ \widetilde{a}(x,y)<t<\widetilde{b}(x,y)\right\}, \tag{5.1}$$

where $\widetilde{a},\widetilde{b}:B\longrightarrow\mathbb{R}$ are analytic subanalytic functions with $1\leq\widetilde{a}(x,y)<\widetilde{b}(x,y)$ for all $(x,y)\in B$, and $\widetilde{b}$ is allowed to be $\equiv+\infty$. We say that D has *bounded t-fibers* if $\widetilde{b}<+\infty$ and *unbounded t-fibers* if $\widetilde{b}\equiv+\infty$.

Suppose furthermore that $\widetilde{a},\widetilde{b},\widetilde{b}-\widetilde{a}$ have the following prepared form:

$$\begin{aligned} &\widetilde{a}(x,y)=a_0(x)\,y^{\frac{\alpha}{d}}u_a(x,y), &&\widetilde{b}(x,y)=b_0(x)\,y^{\frac{\beta}{d}}u_b(x,y),\\ &\widetilde{b}(x,y)-\widetilde{a}(x,y)=d_0(x)\,y^{\frac{\Delta}{d}}u_d(x,y), \end{aligned} \tag{5.2}$$

where $\alpha,\beta,\Delta\in\mathbb{N},\ a_0,b_0,d_0\in\mathcal{S}(X)$ are analytic and $u_a,u_b,u_d\in\mathcal{S}(B)$ are ψ-prepared subanalytic strong units (for ψ as in (3.2)). If $\widetilde{b}=+\infty$ we stipulate that $b_0=d_0=+\infty,\ u_b=u_d=1$ and $\beta=\Delta=0$.

Define

$$\Psi(x,y,t)=\left(\psi(x,y),\left(\frac{a_0(x)\,y^{\frac{\alpha}{d}}}{t}\right)^{\frac{1}{d}},\left(\frac{t}{b_0(x)\,y^{\frac{\beta}{d}}}\right)^{\frac{1}{d}}\right), \tag{5.3}$$

where if D has unbounded t-fibers we omit the last component. Note that Ψ is a 1-bounded subanalytic map on D.

Definition 5.2. A generator $T\in\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X\times\mathbb{R})$ without poles outside some closed discrete set $P\subseteq\mathbb{C}$ is *prepared on B* if for all $(s,x,y)\in(\Sigma\setminus P)\times B$,

$$T(s,x,y)=g(s,x)\,y^{\lambda(s)}(\log y)^{\mu}\,\mathrm{e}^{\mathrm{i}\varphi(x,y)}\gamma(s,x,y),$$

where $g\in\mathcal{C}_{\Sigma}^{\mathcal{M}}(X),\ \mu\in\mathbb{N},\ \varphi\in\mathcal{S}(B)$ is prepared as in (3.6) with respect to ψ as in (3.2), $\lambda(s)=\dfrac{\ell s+\eta}{d}$, for some $\ell\in\mathbb{Z},\eta\in\mathbb{C}$ and the same d appearing in (3.2), and the transcendental element $\gamma\in\Gamma_{\Sigma}(B)$ has the form

$$\gamma(s,x,y)=\int_{\mathbb{R}}\chi_D(x,y,t)\,t^{\varrho(s)}(\log t)^{\nu}\,\Phi(s,x,y,t)\,\mathrm{e}^{\sigma\mathrm{i}t}\mathrm{d}t,$$

where D is as in Notation 5.1, $\sigma \in \{+,-\}$, $\nu \in \mathbb{N}$, $\varrho(s) = \dfrac{\widetilde{\ell} s + \widetilde{\eta}}{d}$ for some $\widetilde{\ell} \in \mathbb{Z}, \widetilde{\eta} \in \mathbb{C}$ and the same d appearing in (3.2) and Φ is a Ψ-prepared parametric strong function (with Ψ as in (5.3)).

Recall Notation 3.5.

Proposition 5.3 (Preparation)**.** *Let $h \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$. There is a cell decomposition of $\mathbb{R}^{m+1}$ compatible with X such that for each cell A that is open over $\mathbb{R}^m$ (which we may suppose to be of the form* (3.9)*), $h \circ \Pi_A$ is a finite sum of prepared generators on B_A.*

Proof. Write h as a sum of generators of the form (3.8) and apply [CCRS23, Proposition 4.7] simultaneously to all the parametric power-constructible data appearing in the generators: this produces a cell decomposition of $X \times \mathbb{R}^2$ compatible with X, and on each cell D which is open over $X \times \mathbb{R}$, a prepared form of the data with respect to the last variable t, where the coefficient functions are parametric power-constructible functions depending on the variables $(x,y) \in X \times \mathbb{R}$. Now apply [CCRS23, Proposition 4.7] again simultaneously to all the coefficient functions, in order to prepare them with respect to the variable y on suitable cells $B \subseteq X \times \mathbb{R}$, thus refining the cell decomposition. This gives the wanted result, up to trivial manipulations to adjust the definition of ψ and Ψ (see [CCMRS18, pp. 1268-70] for the details). $\square$

Remark 5.4. The proof of [CCRS23, Proposition 4.7] (and indeed that of all preparation results based on the Subanalytic Preparation Theorem in [LR97]) shows that it is possible to choose the same integer d appearing in Definition 5.2 for all prepared generators on all cells. Thus d is a data of the cell decomposition and not of a single prepared generator on a single cell. We will hence call a *d-cell decomposition* a cell-decomposition with data $d \in \mathbb{N} \setminus \{0\}$ and we will say that a generator is *d-prepared* on one of the cells of the composition. Similar easy manipulations show that if a generator T is d-prepared on a cell of a d-cell decomposition, then T is also d^2-prepared and the decomposition can also be considered as a d^2-cell decomposition.

Our aim is to refine the previous preparation statement so as to write $h \circ \Pi_A$ as a finite sum of generators which are either strongly integrable or monomial in y.

The first remark is that if we consider a cell A such that B_A has bounded y-fibers, then the prepared generators which appear in $h \circ \Pi_A$ are strongly integrable.

Proposition 5.5. *Suppose B as in* (3.1) *has bounded y-fibers and let $T \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$ be a generator which is prepared on B. Then T is strongly integrable.*

Proof. Let

$$G(s,x,y,t) = g(s,x)\, y^{\lambda(s)} (\log y)^{\mu} \chi_D(x,y,t)\, t^{\varrho(s)} (\log t)^{\nu}\, \Phi(s,x,y,t) \in \mathcal{C}_{\Sigma}^{\mathcal{M}}\left(X \times \mathbb{R}^2\right),$$

so that

$$T(s,x,y) = \int_{\mathbb{R}} G(s,x,y,t)\, \mathrm{e}^{\mathrm{i}(\varphi(x,y)+\sigma t)} \mathrm{d}t. \tag{5.4}$$

Since Φ is bounded and extends continuously to $\overline{D}$, for all $(s,x) \in (\Sigma \setminus P) \times X$, the function $y \longmapsto \int_{\mathbb{R}} |G|\, \mathrm{d}t$ extends continuously to the closed and bounded interval $[a(x), b(x)]$ and is hence integrable on this interval. By Tonelli's Theorem, for all $(s,x) \in (\Sigma \setminus P) \times X$, $(y,t) \longmapsto$

$|G(s,x,y,t)| \in L^1(\mathbb{R}^2)$, so that $G \in \mathcal{C}^{\mathcal{M}}_{\Sigma}(X\times\mathbb{R}^2)_{\mathrm{int}}$ and, by Remark 4.3, T is strongly integrable. $\square$

Next, we refine the statement of Proposition 5.3 for the subclass of $\mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X\times\mathbb{R})$ of those functions which are *naive in the last variable* y (see Definition 5.6 below). On cells with unbounded y-fibers, the functions in this subclass have easily readable asymptotics in y (see Section 7) and can be written as finite sums of generators which are either strongly integrable or monomial in y.

Definition 5.6. Let $B \subseteq X\times\mathbb{R}$ be a subanalytic cell which is open over X. A generator $T \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(B)$ as in (3.8) is *naive in* y if the transcendental element γ does not depend on y. Hence

$$T = \gamma g \mathrm{e}^{\mathrm{i}\varphi}$$

with $\gamma \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X)$, $g \in \mathcal{C}^{\mathcal{M}}_{\Sigma}(B)$ and $\varphi \in \mathcal{S}(B)$.

Proposition 5.7. *Let $h \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X\times\mathbb{R})$ be a finite sum of generators which are naive in y. Then Proposition 5.3 holds for h with the additional property that the prepared generators on B_A are either monomial in y or strongly integrable.*

Proof. Apply Proposition 5.3 to h: the proof shows that this produces a cell decomposition and, for each cell A, a presentation of $h\circ\Pi_A$ as a finite sum of prepared generators which are themselves naive in y. By Proposition 5.5, on cells with bounded y-fibers the generators are strongly integrable. Hence we may concentrate on a cell $B = B_A$ of the form (3.1) with unbounded y-fibers and on a generator T which is prepared on B and naive in y. Thus T has the form

$$T(s,x,y) = f(s,x)\, y^{\lambda(s)} (\log y)^{\mu}\, \mathrm{e}^{\mathrm{i}\varphi(x,y)}\Phi(s,x,y), \tag{5.5}$$

where $f \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X)$, $\lambda(s), \mu$ are as in Definition 5.2 and the prepared forms (with respect to ψ as in (3.2)) of $\varphi \in \mathcal{S}(B)$ and $\Phi \in \mathcal{A}_{\Sigma}(B)$ are as in (3.6) and (3.5), respectively. Up to partitioning X into subanalytic cells, we may suppose that $|\varphi_0|$ is either bounded from above or bounded away from zero. If $|\varphi_0|$ is bounded and $\omega < 0$ then $\mathrm{e}^{\mathrm{i}\varphi}$ is a complex-valued ψ-prepared subanalytic strong function. If $\omega \geq 0$ then write

$$\varphi(x,y) = Q(x,y) + \varphi_{>}(x,y),$$

where $Q \in \mathcal{S}(X)\left[y^{\frac{1}{d}}\right]$ is a polynomial in the variable $y^{\frac{1}{d}}$ with coefficients subanalytic functions of x and

$$\varphi_{>}(x,y) = \varphi_0(x)\,(a(x))^{\frac{\omega}{d}} \left(\frac{a(x)}{y}\right)^{\frac{1}{d}} \sum_{k\geq 0} b_{k+\omega+1}(x) \left(\frac{a(x)}{y}\right)^{\frac{k}{d}}.$$

If $|\varphi_0|$ is bounded from above then clearly $\mathrm{e}^{\mathrm{i}\varphi_{>}}$ is a complex-valued ψ-prepared subanalytic strong function. If $|\varphi_0|$ is bounded away from zero, then we write

$$\varphi_{>}(x,y) = (a(x))^{\frac{\omega}{d}} \left(\frac{a(x)\,\varphi_0(x)^d}{y}\right)^{\frac{1}{d}} \sum_{k\geq 0} b_{k+\omega+1}(x)\,\varphi_0(x)^{-k} \left(\frac{a(x)\,\varphi_0(x)^d}{y}\right)^{\frac{k}{d}}.$$

At the price of creating a new cell with bounded y-fibers (which can be dealt with by Proposition 5.5), we may suppose that $y > a(x)\,\varphi_0(x)^d$ on B, so that, up to modifying the definition of ψ, it is clear that $\varphi_>$ is bounded on B and hence $\mathrm{e}^{\mathrm{i}\varphi_>}$ is a complex-valued ψ-prepared subanalytic strong function (we deal in the same way with the case when $|\varphi_0|$ is bounded away from zero and $\omega < 0$).

It follows that $\mathrm{e}^{\mathrm{i}\varphi_>}$ can be absorbed into Φ and hence

$$T(s,x,y) = y^{\lambda(s)} (\log y)^{\mu}\, \mathrm{e}^{\mathrm{i}Q(x,y)} \sum_{k} f_k(s,x)\, y^{-\frac{k}{d}},$$

where $f_k(s,x) = f(s,x)\,\xi_k(s,x)\,a(x)^{\frac{k}{d}} \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X)$, in the notation of (3.5). Since Σ has bounded width, there exists $k_0 \in \mathbb{N}$ such that, setting $\lambda_k(s) = \lambda(s) - \frac{k}{d}$, there exists $s \in \Sigma$ such that $\Re(\lambda_k(s)) \geq -1$ if and only if $k \leq k_0$. It follows that

$$T(s,x,y) = \sum_{k \leq k_0} f_k(s,x)\, y^{\lambda_k(s)} (\log y)^{\mu}\, \mathrm{e}^{\mathrm{i}Q(x,y)} + R(s,x,y),$$

where the terms of the sum are monomial in y and R is strongly integrable. □

In order to extend Proposition 5.7 to generators which are not naive in y we need some preparatory work.

Lemma 5.8. *Let T be a prepared generator (as in Definition 5.2) on a cell B with unbounded y-fibers. Suppose that, either*

(1) D has unbounded t-fibers and $\forall s \in \Sigma,\ \Re(\lambda(s)) < -1$, or
(2) D has bounded t-fibers and $\forall s \in \Sigma,\ \Re(\lambda(s)) + \frac{\Delta}{d} < -1$ and $\Re(\varrho(s)) \leq 0$.

Then T is strongly integrable.

Proof. Suppose that T has no poles outside some closed discrete set $P \subseteq \mathbb{C}$. If we write T as in (5.4), then it suffices to prove that $G \in \mathcal{C}_{\Sigma}^{\mathcal{M}}(X \times \mathbb{R}^2)_{\mathrm{int}}$. By Tonelli's Theorem, it is enough to prove that for all $(s,x) \in (\Sigma \setminus P) \ni X$, the function

$$f_{(s,x)} : y \longmapsto \int_{\mathbb{R}} |G(s,x,y,t)|\, \mathrm{d}t$$

is in $L^1(\mathbb{R})$.

(1) Let $\widetilde{b} = +\infty$. Since $\forall s \in \Sigma,\ \Re(\varrho(s)) < -1$, there is a positive constant M such that

$$\int_1^{+\infty} \left| t^{\varrho(s)} (\log t)^{\nu}\, \Phi(s,x,y,t) \right| \mathrm{d}t \leq M.$$

Therefore,

$$\left| f_{(s,x)}(y) \right| \leq M\, |g(s,x)|\, \chi_B(x,y)\, y^{\lambda(s)} (\log y)^{\mu}$$

and, as $\forall s \in \Sigma,\ \Re(\lambda(s)) < -1$, we have that $f_{(s,x)} \in L^1(\mathbb{R})$.

(2) Let now $\widetilde{b} < +\infty$. Since $\forall s \in \Sigma,\ \Re(\varrho(s)) \leq 0$ and Φ is bounded, there is a positive constant $M(x)$ such that

$$\int_{\widetilde{a}(x,y)}^{\widetilde{b}(x,y)} \left| t^{\varrho(s)} (\log t)^{\nu}\, \Phi(s,x,y,t) \right| \mathrm{d}t \leq M(x)\, y^{\frac{\Delta}{d}} (\log y)^{\nu}.$$

Therefore,

$$\left|f_{(s,x)}(y)\right| \leq M(x)\,|g(s,x)|\,\chi_B(x,y)\,y^{\lambda(s)+\frac{\Delta}{d}}(\log y)^{\mu+\nu}$$

and, as $\forall s \in \Sigma,\ \Re(\lambda(s)) + \frac{\Delta}{d} < -1$, we have that $f_{(s,x)} \in L^1(\mathbb{R})$.

□

Lemma 5.9. *Let T be a prepared generator (as in Definition 5.2) on a cell B with unbounded y-fibers and let $k_0 \in \mathbb{N}$. Then T can be rewritten as a finite sum of prepared generators which are either naive in y or such that for all $s \in \Sigma,\ \Re(\varrho(s)) < -k_0$.*

Proof. Suppose that T has no poles outside some closed discrete set $P \subseteq \mathbb{C}$. Write T on B as

$$g(s,x)\,y^{\lambda(s)}(\log y)^{\mu}\,\mathrm{e}^{\mathrm{i}\varphi(x,y)}\gamma(s,x,y),$$

where

$$\gamma(s,x,y) = \int_{\widetilde{a}(x,y)}^{\widetilde{b}(x,y)} t^{\varrho(s)}(\log t)^{\nu}\,\Phi(s,x,y,t)\,\mathrm{e}^{\sigma \mathrm{i}t}\mathrm{d}t.$$

Since B has unbounded y-fibers, Φ has the following Ψ-prepared nested form (see [CCRS23, Remark 3.7]) with respect to the last three components of Ψ:

$$\Phi(s,x,y,t) = F \circ (s,x,\Xi(x,y,t)),$$

where

$$F(s,x,Y_0,Y_1,Y_2) = \sum_{k,m,n} \xi_{k,m,n}(s,x)\,Y_0^k Y_1^m Y_2^n \in \mathcal{A}_{\Sigma}(x)\,[\![Y_0,Y_1,Y_2]\!]$$

is strongly convergent and

$$\Xi(x,y,t) = \left(\left(\frac{a(x)}{y}\right)^{\frac{1}{d}}, \left(\frac{a_0(x)\,y^{\frac{\alpha}{d}}}{t}\right)^{\frac{1}{d}}, \left(\frac{t}{b_0(x)\,y^{\frac{\beta}{d}}}\right)^{\frac{1}{d}}\right)$$

(the variable Y_2 and the last component of Ξ are missing if D has unbounded t-fibers).

Fix $(s,x,y) \in (\Sigma \setminus P) \times B$ and apply integration by parts to the transcendental element γ, where we integrate $\mathrm{e}^{\mathrm{i}\sigma t}$ and derivate $f(t) := t^{\varrho(s)}(\log t)^{\nu}\,\Phi(s,x,y,t)$. For this, write

$$f'(t) = t^{-1}\left[\varrho(s)\,t^{\varrho(s)}(\log t)^{\nu}\,\Phi + \nu t^{\varrho(s)}(\log t)^{\nu-1}\,\Phi + t^{\varrho(s)}(\log t)^{\nu}\,\widetilde{\Phi}\right],$$

where

$$\widetilde{\Phi}(s,x,y,t) = \widetilde{F} \circ (s,x,\Xi(x,y,t)) \text{ with } \widetilde{F} = -\frac{1}{d}Y_1\frac{\partial F}{\partial Y_1} + \frac{1}{d}Y_2\frac{\partial F}{\partial Y_2}.$$

In particular, $\widetilde{F}$ is strongly convergent and $\widetilde{\Phi}$ is a Ψ-prepared parametric strong function. Notice that each of the terms of $f'(t)\,\frac{\mathrm{e}^{\mathrm{i}\sigma t}}{\mathrm{i}\sigma}$ gives rise to a prepared generator such that the exponent of t in the transcendental element is $\varrho(s) - 1$. The other terms produced by integration by parts are of the form

$$-\,\sigma\mathrm{i}\widetilde{c}(x,y)^{\varrho(s)}(\log \widetilde{c}(x,y))^{\nu}\,\mathrm{e}^{\sigma\mathrm{i}\widetilde{c}(x,y)}\Phi(s,x,y,\widetilde{c}(x,y)), \tag{5.6}$$

where $\widetilde{c}$ is either $\widetilde{a}$ or $\widetilde{b}$ (or the whole term is replaced by zero, if $\widetilde{b}=+\infty$, since then in this case for all $s\in\Sigma,\ \Re(\varrho(s))<-1<0$). Now, if $\widetilde{c}=\widetilde{b}$, then

$$\Xi\left(x,y,\widetilde{b}(x,y)\right)=\left(\left(\frac{a(x)}{y}\right)^{\frac{1}{d}},\left(\frac{a_0(x)\,y^{\frac{\alpha}{d}}}{b_0(x)\,y^{\frac{\beta}{d}}u_b(x,y)}\right)^{\frac{1}{d}},(u_b(x,y))^{\frac{1}{d}}\right)$$

and

$$\frac{a_0(x)\,y^{\frac{\alpha}{d}}}{b_0(x)\,y^{\frac{\beta}{d}}u_b(x,y)}=\left[\frac{a_0(x)}{b_0(x)\,a(x)^{\frac{\beta-\alpha}{d}}}\right](u_b(x,y))^{-1}\left(\frac{a(x)}{y}\right)^{\frac{\beta-\alpha}{d}}.$$

The term between square brackets is bounded on B, because all the other terms are, so we can add it to the list of functions $c(x)$ in the definition of ψ (see 3.2). The unit u_b is ψ-prepared hence $\Phi\left(s,x,y,\widetilde{b}(x,y)\right)$ is a ψ-prepared parametric strong function. A similar calculation shows that so is $\Phi(s,x,y,\widetilde{a}(x,y))$. Thus the terms 5.6 are generators which are naive in y and prepared with respect to ψ.

We iterate the process to further reduce the exponent of t: since Σ has bounded width, $\sup_{s\in\Sigma}(\Re(\varrho(s)))\in\mathbb{R}$. Let $M:=\left\lfloor\sup_{s\in\Sigma}(\Re(\varrho(s)))\right\rfloor$. By integrating by parts $M+k_0+1$ times, we can rewrite T as a finite sum of generators which are either naive in y and prepared with respect to ψ, or such that the exponent of t in the transcendental element γ is $\varrho(s)-M-k_0-1$, whose real part is $<-k_0$. □

Proposition 5.10. *Let T be a prepared generator (as in Definition 5.2) on a cell B with unbounded y-fibers and suppose that T has no poles outside some closed discrete set $P\subseteq\mathbb{C}$. Then there exist a closed discrete set $P'\subseteq\mathbb{C}$ containing P and such that $P'\setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice, a finite partition of B into subcells and, on each subcell B' which is open over X, finitely many generators T_i which are either naive in y or strongly integrable on B', such that for all $(s,x,y)\in(\Sigma\setminus P')\times B'$, $T(s,x,y)=\sum T_i(s,x,y,)$.*

Proof. Recall Notation 5.1. Note that $0\le\alpha\le\beta$. There are three cases:

(1) $\alpha=\beta=0$
(2) $\alpha>0$
(3) $\alpha=0,\beta>0$

Define $C:=\sup_{s\in\Sigma}(\Re(\lambda(s)))$.

(1) The case $\beta=0$ also includes the case $\widetilde{b}(x,y)=b_0(x)=+\infty$. We claim that, at the price of creating a new cell with bounded y-fibers (which can be handled using Proposition 5.5), we may suppose that for all $(x,y)\in B$
   - $\widetilde{a}(x,y)\le a_0(x)$ and $b_0(x)\le\widetilde{b}(x,y)$;
   - $|\widetilde{a}(x,y)-a_0(x)|\le 1$ and $\left|\widetilde{b}(x,y)-b_0(x)\right|\le 1$

   The proof of the claim can be found in [CCMRS18, p.1277] and only uses basic o-minimal properties of subanalytic sets and functions.
   Therefore, we may write the transcendental element γ as the sum of three integrals, with integration bounds, respectively, $(\widetilde{a}(x,y),a_0(x))$, $(a_0(x),b_0(x))$ and $\left(b_0(x),\widetilde{b}(x,y)\right)$

(if $\widetilde{b}=+\infty$, then the second integral has $+\infty$ as upper integration bound and the third integral is missing). The integral with bounds $(\widetilde{a}(x,y), a_0(x))$ can be written as

$$\mathrm{e}^{\mathrm{i}\sigma a_0(x)}\int_{a_0(x)}^{a_0(x)u_a(x,y)} t^{\varrho(s)}(\log t)^{\nu}\,\Phi(s,x,y,t)\,\mathrm{e}^{\sigma\mathrm{i}(t-a_0(x))}\mathrm{d}t, \tag{5.7}$$

where, thanks to the claim, $\mathrm{e}^{\sigma\mathrm{i}(t-a_0(x))}$ is a complex-valued subanalytic function on B. Hence the integrand is in $\mathcal{C}^{\mathcal{M}}_{\Sigma}(B)$ and we can invoke Theorem 2.18 to obtain that the term containing (5.7) can be written, outside some closed discrete set P' containing P, as a generator of $\mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X\times\mathbb{R})$ which is naive in y. The integral with bounds $\left(b_0(x), \widetilde{b}(x,y)\right)$, if present, is handled similarly. For the integral with bounds $(a_0(x), b_0(x))$, notice that now the variable y only appears in the parametric strong function Φ, which we can write in nested form with respect to the last component of ψ as

$$\sum_{k\geq 0}\widetilde{\xi}_k(s,x,t)\left(\frac{a(x)}{y}\right)^{\frac{k}{d}}, \tag{5.8}$$

for some $\widetilde{\xi}_k\in\mathcal{A}_{\Sigma}(E)$, where $E=\{(x,t):\ x\in X,\ a_0(x)<t<b_0(x)\}$. Let $k_0=\lceil d(C+1)\rceil+1$, so that for all $s\in\Sigma$, $\Re(\lambda(s))-\frac{k_0}{d}<-1$, and let

$$\Phi_{>k_0}=\Phi-\sum_{k\leq k_0}\widetilde{\xi}_k(s,x,t)\left(\frac{a(x)}{y}\right)^{\frac{k}{d}}.$$

Setting

$$f_k(s,x)=g(s,x)(a(x))^{\frac{k}{d}}\int_{a_0(x)}^{b_0(x)} t^{\varrho(s)}(\log t)^{\nu}\,\widetilde{\xi}_k(s,x,t)\,\mathrm{e}^{\sigma\mathrm{i}t}\mathrm{d}t\in\mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X),$$

write the term

$$g(s,x)\,y^{\lambda(s)}(\log y)^{\mu}\,\mathrm{e}^{\mathrm{i}\varphi(x,y)}\int_{a_0(x)}^{b_0(x)} t^{\varrho(s)}(\log t)^{\nu}\,\mathrm{e}^{\sigma\mathrm{i}t}\Phi(s,x,y,t)\,\mathrm{d}t$$

as the following sum of generators which are naive in y

$$\sum_{k\leq k_0} f_k(s,x)\,y^{\lambda(s)-\frac{k}{d}}(\log y)^{\mu}\,\mathrm{e}^{\mathrm{i}\varphi(x,y)},$$

plus the term

$$g(s,x)(a(x))^{\frac{k_0}{d}}\,y^{\lambda(s)-\frac{k_0}{d}}(\log y)^{\mu}\,\mathrm{e}^{\mathrm{i}\varphi(x,y)}\int_{a_0(x)}^{b_0(x)} t^{\varrho(s)}(\log t)^{\nu}\,\Phi_{>k_0}(s,x,y,t)\,\mathrm{e}^{\sigma\mathrm{i}t}\mathrm{d}t, \tag{5.9}$$

which is strongly integrable on B by definition of k_0 and since $\Phi_{>k_0}$ is bounded.

(2) Let $N_0 = \lceil \frac{dC+\Delta+d}{\alpha} \rceil + 1$. By Lemma 5.9, we may suppose that for all $s \in \Sigma$, $\Re(\varrho(s)) < -N_0$. Write

$$t^{\varrho(s)} = t^{\varrho(s)+N_0} \left(a_0(x) y^{\frac{\alpha}{d}}\right)^{-N_0} \left[\left(\frac{a_0(x) y^{\frac{\alpha}{d}}}{t}\right)^{\frac{1}{d}}\right]^{dN_0}.$$

The rightmost term in the above formula can be absorbed into Φ and the central term can be factored out of the integral defining the transcendental element γ. By the choice of N_0, for all $s \in \Sigma$,

$$\Re(\lambda(s)) - \frac{N_0\alpha}{d} + \frac{\Delta}{d} < -1, \tag{5.10}$$

so by Lemma 5.8 (either of the two conditions, depending on the nature of the t-fibers of the cell D) T is strongly integrable on B.

(3) Let $N_0 = \left\lceil \frac{dC+\Delta+d}{\beta} \right\rceil + 1$ and $k_0 = d(N_0 - 1)$. By Lemma 5.9, we may suppose that for all $s \in \Sigma$, $\Re(\varrho(s)) < -N_0$. This implies in particular that for all $s \in \Sigma$,

$$\Re(\varrho(s)) + \frac{k_0}{d} < -1, \tag{5.11}$$

$$\Re(\varrho(s)) + \frac{k_0+1}{d} \leq 0, \tag{5.12}$$

$$\Re(\lambda(s)) - \frac{\beta(k_0+1)}{d^2} + \frac{\Delta}{d} < -1. \tag{5.13}$$

First, we split $\Phi(s,x,y,t)$ into the sum of two series, by separating the positive and the negative powers of t:

$$\Phi = \sum_{k>0} \xi_k^{<}(s,x,y) \left(\frac{a_0(x)}{t}\right)^{\frac{k}{d}} + \sum_{k\geq 0} \xi_k^{>}(s,x,y) \left(\frac{t}{b_0(x) y^{\frac{\beta}{d}}}\right)^{\frac{k}{d}}.$$

Next, write

$$\gamma_{\leq k_0}(s,x,y) = \int_{a_0(x)u_a(x,y)}^{b_0(x)y^{\frac{\beta}{d}}u_b(x,y)} t^{\varrho(s)} (\log t)^{\nu}\, \Phi_{\leq k_0}(s,x,y,t)\, \mathrm{e}^{\sigma \mathrm{i} t} \mathrm{d}t,$$

$$\gamma_{>k_0}(s,x,y) = \int_{a_0(x)u_a(x,y)}^{b_0(x)y^{\frac{\beta}{d}}u_b(x,y)} t^{\varrho(s)} (\log t)^{\nu} \left(\frac{t}{b_0(x) y^{\frac{\beta}{d}}}\right)^{\frac{k_0+1}{d}} \Phi_{>k_0}(s,x,y,t)\, \mathrm{e}^{\sigma \mathrm{i} t} \mathrm{d}t,$$

where

$$\Phi_{\leq k_0} = \sum_{k>0} \xi_k^{<}(s,x,y) \left(\frac{a_0(x)}{t}\right)^{\frac{k}{d}} + \sum_{k=0}^{k_0} \xi_k^{>}(s,x,y) \left(\frac{t}{b_0(x) y^{\frac{\beta}{d}}}\right)^{\frac{k}{d}},$$

$$\Phi_{>k_0} = \sum_{k\geq 0} \xi_{k+k_0+1}^{>}(s,x,y) \left(\frac{t}{b_0(x) y^{\frac{\beta}{d}}}\right)^{\frac{k}{d}}.$$

By (5.11) and linearity, we may write $\gamma_{\leq k_0}$ as the sum of two integrals with upper integration bound equal to $+\infty$ and the lower integration bounds equal to, respectively, $a_0(x)\,u_a(x,y)$ and $b_0(x)\,y^{\frac{\beta}{d}}u_b(x,y)$. The first integral falls within the scope of the first part of this proof, whereas the second integral falls within the scope of the second part of this proof.

It remains to consider

$$\begin{aligned}T_{>k_0}(s,x,y) &:= g(s,x)\,y^{\lambda(s)}(\log y)^{\mu}\,\mathrm{e}^{\mathrm{i}\varphi(x,y)}\gamma_{>k_0}(s,x,y)\\ &= g(s,x)\,y^{\lambda(s)-\frac{\beta(k_0+1)}{d^2}}(\log y)^{\mu}\,\mathrm{e}^{\mathrm{i}\varphi(x,y)}\,(b_0(x))^{-\frac{\beta(k_0+1)}{d^2}}\cdot\\ &\quad\cdot\int_{a_0(x)u_a(x,y)}^{b_0(x)y^{\frac{\beta}{d}}u_b(x,y)} t^{\varrho(s)+\frac{k_0+1}{d}}(\log t)^{\nu}\,\Phi_{>k_0}(s,x,y,t)\,\mathrm{e}^{\sigma\mathrm{i}t}\mathrm{d}t.\end{aligned} \tag{5.14}$$

By (5.12) and (5.13), $T_{\geq k_0}$ satisfies the second condition in Lemma 5.8.

□

6. Interpolation and stability under integration

In this section we finish the proof of Theorem 2.9. For this, it suffices to consider the 1-dimensional case, Theorem 6.5 below (the general n-dimensional case follows from Fubini's Theorem, see the end of Section 4.2), the proof of which requires an analysis of the integration locus.

With this in mind, we adapt [CCRS23, Definition 6.1] to the current setting.

Definition 6.1. Given $N,d\in\mathbb{N}\setminus\{0\}$ and $\{(\ell_i,r_i):\ 1\leq i\leq N\}\subseteq\mathbb{R}^2$, define

$$\begin{aligned}\Xi_{i,0,-} &= \emptyset,\\ \Xi_{i,0,\circ} &= \{s\in\Sigma:\ \ell_i\Re(s)+r_i+d<0\}\quad(1\leq i\leq N),\\ \Xi_{i,j,-} &= \{s\in\Sigma:\ \ell_i\Re(s)+r_i+d=j-1\}\quad(1\leq i\leq N,\ j\in\mathbb{N}\setminus\{0\}),\\ \Xi_{i,j,\circ} &= \{s\in\Sigma:\ j-1<\ell_i\Re(s)+r_i+d<j\}\quad(1\leq i\leq N,\ j\in\mathbb{N}\setminus\{0\}).\end{aligned}$$

The collection

$$\mathcal{G}=\{\Xi_{i,j,\star}:\ 1\leq i\leq N,\ j\in\mathbb{N},\ \star\in\{-,\circ\}\}$$

is called the *grid* of *denominator* d and *data* $(d,\{(\ell_i,r_i):\ 1\leq i\leq N\})$. A $\mathcal{G}$-*cell* is a nonempty subset $S\subseteq\Sigma$ such that

$$\forall\Xi\in\mathcal{G},\ \Xi\cap S=\emptyset \text{ or } S\subseteq\Xi, \text{ and } S=\bigcap\{\Xi\in\mathcal{G}:\ S\subseteq\Xi\}.$$

Finally, given a prepared generator T as in Definition 5.2, we call the tuple

$$\left(d^2,\left\{\left(d\ell+\delta\widetilde{\ell},d\Re(\eta)+\delta\Re(\widetilde{\eta})\right):\ \delta\in\{0,\alpha,\beta\}\right\}\right)$$

the *grid data* of T.

Remarks 6.2.

(1) Since Σ has bounded width, a grid $\mathcal{G}$ induces a finite partition $\mathcal{R}(\mathcal{G})$ of Σ into $\mathcal{G}$-cells, and each $\mathcal{G}$-cell is either an open vertical substrip of Σ or a vertical line.

(2) A prepared generator T generates a grid of data the grid data of T. In this case, if T is monomial in y then on each $\mathcal{G}$-cell S the real part of the exponent of y is either always < -1 or always ≥ -1.

Notation 6.3. Given a subanalytic set $X \subseteq \mathbb{R}^m$ and functions $Q_1, Q_2 \in \mathcal{S}(X)\left[y^{\frac{1}{d}}\right]$ which are polynomials in $y^{\frac{1}{d}}$ with coefficients subanalytic functions of x, it is clearly possible to partition X into finitely many subanalytic cells such that for each cell X', either for all $x \in X'$, $Q_1(x, \cdot)$ and $Q_2(x, \cdot)$ define the same polynomial function, or for all $x \in X'$, $Q_1(x, \cdot)$ and $Q_2(x, \cdot)$ define different polynomial functions. In this case, we will say, respectively, that $Q_1 = Q_2$ on X' or $Q_1 \neq Q_2$ on X'.

Proposition 6.4 (Splitting)**.** *Let $h \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$ be without poles outside some closed discrete set $P \subseteq \mathbb{C}$. There are a closed discrete set $P' \subseteq \mathbb{C}$ containing P and such that $P' \setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice, $d \in \mathbb{N} \setminus \{0\}$, finite sets $J_{\mathrm{int}}, J_{\mathrm{mon}} \subseteq \mathbb{N}$ and a d-cell decomposition (see Remark 5.4) of $\mathbb{R}^{m+1}$ compatible with X such that for each cell A that is open over $\mathbb{R}^m$ (which we may suppose to be of the form* (3.9)*),*

$$h \circ \Pi_A = \sum_{j \in J_{\mathrm{int}}} T_j + \sum_{j \in J_{\mathrm{mon}}} T_j, \tag{6.1}$$

where each T_j is a d-prepared generator without poles outside P' (see Definition 5.2 and Remark 5.4). Moreover, using the notation in Definition 4.2,

(1) For every $j \in J_{\mathrm{int}}$, T_j is strongly integrable on B_A and if B_A has unbounded y-fibers, then, in the notation of Definition 5.2, for all $s \in \Sigma$, $\Re(\lambda(s)) < -1$.

(2) For every $j \in J_{\mathrm{mon}}$, T_j is monomial in y, with monomial data $(d, \ell_j, \eta_j, \mu_j, Q_j)$, where:

(a) for all $x \in X$, $Q_j(x, 0) = 0$ and for all $i, j \in J_{\mathrm{mon}}$, either $Q_i = Q_j$ on X or $Q_i \neq Q_j$ on X (see Notation 6.3);

(b) the tuples $(\ell_j, \eta_j, \mu_j, Q_j) \in \mathbb{Z} \times \mathbb{C} \times \mathbb{N} \times \mathcal{S}(X)\left[y^{\frac{1}{d}}\right]$ $(j \in J_{\mathrm{mon}})$ are pairwise distinct;

(c) there is a grid $\mathcal{G}$ such that for all $\mathcal{G}$-cell S, for all $j \in J_{\mathrm{mon}}$, either $\Re\left(\frac{\ell_j s + \eta_j}{d}\right) < -1$ for all $s \in S$, or $\Re\left(\frac{\ell_j s + \eta_j}{d}\right) \geq -1$ for all $s \in S$.

Proof. Apply Proposition 5.3 to h. This produces d and a d-cell decomposition of $\mathbb{R}^{m+1}$ such that on each cell A open over X, $h \circ \Pi_A$ is a finite sum of prepared generators T. Collect the grid data of all the prepared generators and generate the corresponding grid $\mathcal{G}$ with denominator d^2. For each cell A, apply Propositions 5.10 and 5.7 to each prepared generator T on B_A. This produces a refinement of the d-cell decomposition and rewrites T on each cell as a finite sum of prepared generators T' which are either strongly integrable (and satisfying condition (1)) or monomial in y. Up to absorbing $\mathrm{e}^{\mathrm{i}Q_j(x,0)}$ into $f_j(s,x)$ and up to partitioning X into subanalytic cells, we may suppose that item (2.a) in the statement of the proposition is satisfied. Summing like terms we may also suppose that item (2.b) is satisfied. Revisiting the proofs of Propositions 5.10 and 5.7, which are based on integration by parts of the transcendental elements and series expansion of parametric strong functions on cells with unbounded y-fibers, we see that if the exponents of y and t in the original prepared generator T are $\lambda(s)$ and $\varrho(s)$ respectively, then the exponents of y in the newly created monomial generators T' have the form $\lambda(s) - \frac{k}{d} + \frac{\delta}{d}(\varrho(s) - k')$, for some $k, k' \in \mathbb{N}$ and $\delta \in \{0, \alpha, \beta\}$.

In particular, the grid generated by the grid data of the new monomial generators does not create any new cell. By Remark 5.4 we may rename d^2 as d and adapt accordingly the definitions of $\ell_j, \eta_j, \alpha_j, \beta_j$, so that, by Remark 6.2 (2), item (2.c) in the statement of the proposition is also satisfied. $\square$

Theorem 6.5 (Interpolation and integration locus)**.** *Let $h \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X \times \mathbb{R})$ be without poles outside some closed discrete set $P \subseteq \mathbb{C}$. There are a closed discrete set $P' \subseteq \mathbb{C}$ containing P and such that $P' \backslash P$ is contained in a finitely generated $\mathbb{Z}$-lattice and a function $H \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X)$ without poles outside P' such that*

$$\forall (s,x) \in \mathrm{Int}\left(h; (\Sigma \setminus P') \times X\right), \quad \int_{\mathbb{R}} h(s,x,y)\,\mathrm{d}y = H(s,x).$$

Moreover, there exists a grid $\mathcal{G}$ such that

$$\mathrm{Int}\left(h; (\Sigma \setminus P') \times X\right) = \bigcup_{S \in \mathcal{R}(\mathcal{G})} \left\{ (s,x) :\ s \in S \setminus P',\ \bigwedge_{j \in J_S} f_j(s,x) = 0 \right\}, \tag{6.2}$$

for a suitable finite set J_S and suitable $f_j \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X)$ without poles outside P'.

Proof. Apply Proposition 6.4 to h: this produces a closed discrete set $P' \subseteq \mathbb{C}$ containing P and such that $P' \setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice, $d \in \mathbb{N} \setminus \{0\}$, finite sets $J_{\mathrm{int}}, J_{\mathrm{mon}} \subseteq \mathbb{N}$, a grid $\mathcal{G}$ and a d-cell decomposition, such that the conclusion of the proposition holds. By linearity of the integral, it suffices to prove the statement of the theorem for $h \restriction A$, where A is a cell of the decomposition which is open over X. Recall Notation 3.5 and note that

$$\frac{\partial \Pi_A}{\partial y}(x,y) = \sigma_A \tau_A y^{\tau_A - 1}.$$

Thus, up to multiplying each T_j in (6.1) by $\frac{\partial \Pi_A}{\partial y}(x,y)$, we may write that for all $(s,x) \in \mathrm{Int}\left(h \restriction A; (\Sigma \setminus P') \times X\right)$,

$$\int_{A_x} h(s,x,y)\,\mathrm{d}y = \int_{a_A(x)}^{b_A(x)} \left(\sum_{j \in J_{\mathrm{int}}} T_j(s,x,y) + \sum_{j \in J_{\mathrm{mon}}} T_j(s,x,y) \right) \mathrm{d}y.$$

If B_A has bounded y-fibers, then we are done by Proposition 5.5 and Corollary 4.6.

If B_A has unbounded y-fibers, then for all $j \in J_{\mathrm{mon}}$, T_j has the form $f_j(s,x)\, y^{\lambda_j(s)} (\log y)^{\mu_j} \mathrm{e}^{\mathrm{i}Q_j(x,y)}$, with $\lambda_j(s) = \frac{\ell_j s + \eta_j}{d}$, and for all $\mathcal{G}$-cell S there is a set $J_S \subseteq J_{\mathrm{mon}}$ such that for all $j \in J_S$, $\mathrm{Int}\left(T_j \chi_{B_A}; (S \setminus P') \times X\right) = \{(s,x) \in (S \setminus P') \times X :\ f_j(s,x) = 0\}$ whereas for all $j \in J_{\mathrm{mon}} \setminus J_S$, $\mathrm{Int}\left(T_j \chi_{B_A}; (S \setminus P') \times X\right) = (S \setminus P') \times X$. Thus, the set

$$E := \bigcap_{j \in J_{\mathrm{int}} \cup J_{\mathrm{mon}}} \mathrm{Int}\left(T_j \chi_{B_A}; (\Sigma \setminus P') \times X\right)$$

is of the form of the right hand side of (6.2) and, applying either Corollary 4.6 or Proposition 4.8 to $T_j \chi_{B_A}$ and possibly enlarging P', we find $H \in \mathcal{C}^{\mathcal{M},\mathcal{F}}_{\Sigma}(X)$ without poles outside P' which interpolates the integral of $h \restriction A$ for all $(s,x) \in E$.

Note that $E \subseteq \mathrm{Int}\left(h \restriction A; (\Sigma \setminus P') \times X\right)$. It remains to show that, up to possibly enlarging P', the set E coincides with $\mathrm{Int}\left(h \restriction A; (\Sigma \setminus P') \times X\right)$. Let

$$P_A = P' \cup \{s \in \mathbb{C} :\ \exists i,j \in J_{\mathrm{mon}} \text{ such that } i \neq j,\ \lambda_i(s) = \lambda_j(s),\ \mu_i = \mu_j,\ Q_i = Q_j\}.$$

By item (2.b) in Proposition 6.4, if $s \in \Sigma$ is such that $\lambda_i(s) = \lambda_j(s)$ for some $i \neq j$ such that $\mu_i = \mu_j$, $Q_i = Q_j$, then necessarily $\ell_i \neq \ell_j$, so $P_A \setminus P'$ is finite.

By definition of P_A, if $s \in \Sigma \setminus P_A$ and $i, j \in J_{\text{mon}}$ are such that $i \neq j$, $\mu_i = \mu_j$ and $\Re(\lambda_i(s)) = \Re(\lambda_j(s))$, then $Q_i = Q_j \Longrightarrow \Im(\lambda_i(s)) \neq \Im(\lambda_j(s))$.

Let $(s_0, x_0) \in \operatorname{Int}(h \restriction A; (\Sigma \setminus P_A) \times X)$ and let S be the $\mathcal{G}$-cell to which s_0 belongs. Define

$$\rho_j = \Re(\lambda_j(s_0)), \ \sigma_j = \Im(\lambda_j(s_0)), \ p_j(y) = Q_j(x_0, y) \in \mathbb{R}\left[y^{\frac{1}{d}}\right].$$

Let (r_0, ν_0) be the lexicographic maximum of the set $\{(\rho_j, \mu_j) : j \in J_S\}$ and let $J_0 = \{j \in J_S : (\rho_j, \mu_j) = (r_0, \nu_0)\}$. Then

$$\sum_{j \in J_S} T_j(s_0, x_0, y) = y^{r_0} (\log y)^{\nu_0} \sum_{j \in J_0} f_j(s_0, x_0)\, y^{\mathrm{i}\sigma_j} \mathrm{e}^{\mathrm{i}p_j(y)} + \sum_{j \in J_S \setminus J_0} f_j(s_0, x_0)\, y^{\rho_j + \mathrm{i}\sigma_j} (\log y)^{\mu_j} \mathrm{e}^{\mathrm{i}p_j(y)}.$$

Since $(s_0, x_0) \in \operatorname{Int}(h \restriction A; (\Sigma \setminus P_A) \times X)$, it follows from [CCRS23, Proposition 3.4 (1)] that $\bigwedge_{j \in J_0} f_j(s_0, x_0) = 0$. By repeating this procedure with the index set $J_S \setminus J_0$, we end up proving that $(s_0, x_0) \in E$. □

Remark 6.6. Given $h \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$ and a strip $\Sigma' \supseteq \Sigma$, apply Proposition 6.4 to h and consider the extension h' of h to Σ'. The proof shows that Proposition 6.4 applies to h' with different generators T'_j but with the same $d, \mathcal{G}$ and P', by integrating by parts some of the transcendental elements appearing in the strongly integrable generators T_j. For the same reason, Theorem 6.5 applies to h' with a different H but the same $P', \mathcal{G}$.

7. Asymptotic expansions and limits

7.1. **Asymptotic expansions.** In this section we study the behaviour of a function h, in $\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$, and, in $\mathcal{C}^{\mathbb{C},\mathcal{F}}(X \times \mathbb{R})$, seen as a function of the last variable y with parameters ($s \in \Sigma$ and) $x \in X$. We are interested in "the germ at $+\infty$ in y" of h, hence we will work in restriction to cells of the form (3.1) with unbounded y-fibers. As we are only interested in the behaviour at $+\infty$ in y, we will often replace the cell B by some smaller cell B', still of base X and with unbounded y-fibers, but whose lower boundary function is some analytic subanalytic function a' which satisfies that for all $x \in X$, $a(x) \leq a'(x)$. As X serves as a space of parameters, we will also often partition X into finitely many subanalytic cells and suppose, as we did in the previous sections, that X itself is one of the cells of the partition. Finally, if $h \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$ has no poles outside some closed discrete set $P \subseteq \mathbb{C}$, as Σ also serves as a space of parameters, we will often replace P by some bigger closed discrete set $P' \subseteq \mathbb{C}$ such that $P' \setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice.

Summing up, the sentence "if B is a cell of base X with unbounded y-fibers and $h \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$ has no poles outside some closed discrete set $P \subseteq \mathbb{C}$, then, *up to partitioning* X, *shrinking* B *and enlarging* P, Property (*) holds for h" will be used as a shorthand for the following: *there are a finite partition of X into subanalytic cells X' and a closed discrete set $P' \subseteq \mathbb{C}$ such that $P' \setminus P$ is contained in a finitely generated $\mathbb{Z}$-lattice, and for each cell X' there is a cell $B' \subseteq B$ of base X' and with unbounded y-fibers such that Property (*) holds for $h \restriction (\Sigma \setminus P') \times B'$.*

Our first result concerns the class $\mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(X \times \mathbb{R})$.

Theorem 7.1. *Let B be as in (3.1) with unbounded y-fibers and $h \in \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(B)$ be without poles outside some closed discrete set $P \subseteq \mathbb{C}$. Up to partitioning X, shrinking B and enlarging P, there is a sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_{\Sigma}^{\mathcal{M},\mathcal{F}}(B)$ of generators which are monomial in y such that:*

(1) For all $N \in \mathbb{N}$ there are $j_N \in \mathbb{N}$ and a function $C_N : (\Sigma \setminus P) \times X \longrightarrow (0, +\infty)$ such that

$$\forall (s, x, y) \in (\Sigma \setminus P) \times B, \ \left| h(s, x, y) - \sum_{j \leq j_N} T_j(s, x, y) \right| \leq C_N(s, x)\, y^{-N}.$$

(2) If h is a finite sum of generators which are naive in y then we can choose the sequence $(T_n)_{n\in\mathbb{N}}$ so that the series $\sum_{j\in\mathbb{N}} T_j$ converges absolutely to h.

Proof. We first prove the two statements for a function h which is a finite sum of generators which are naive in y: write $h = \sum_{i\in I} T_i$, where I is a finite index set and T_i has the form (5.5). Argueing as in the proof of Proposition 5.7 and using Remark 3.6, up to partitioning X and shrinking B, we may suppose that

$$T_i(s, x, y) = f_i(s, x)\, y^{\lambda_i(s)} (\log y)^{\mu_i}\, \mathrm{e}^{\mathrm{i}Q_i(x,y)} \Phi_i(s, x, y),$$

where the $Q_i \in \mathcal{S}(X)\left[y^{\frac{1}{d}}\right]$ satisfy items (2.a) and (2.b) of Proposition 6.4 and Φ_i is as in (3.5). Using the (absolutely convergent) series expansion of Φ_i, we write

$$\begin{aligned} h(s, x, y) &= \sum_{i\in I} f_i(s, x) (\log y)^{\mu_i}\, \mathrm{e}^{\mathrm{i}Q_i(x,y)} \sum_{k\in\mathbb{N}} \xi_{i,k}(s, x) (a(x))^{\frac{k}{d}}\, y^{\lambda_i(s)-\frac{k}{d}} \\ &= \sum_{i\in I, k\in\mathbb{N}} f_{i,k}(s, x)\, y^{\lambda_{i,k}(s)} (\log y)^{\mu_i}\, \mathrm{e}^{\mathrm{i}Q_i(x,y)}, \end{aligned} \tag{7.1}$$

where $f_{i,k}(s, x) = f_i(s, x)\, \xi_{i,k}(s, x) (a(x))^{\frac{k}{d}}$ and $\lambda_{i,k}(s) = \lambda_i(s) - \frac{k}{d}$. This proves the second statement of the theorem for h.

Fix $N \in \mathbb{N}$, let $\mu := \max_{i\in I} \mu_i$, $K := \sup_{i\in I, s\in\Sigma} |\Re(\lambda_i(s))|$ and choose $k_0 \in \mathbb{N}$ such that $k_0 \geq d(K + N + 1)$. Let

$$\begin{aligned} h_{\geq k_0}(s, x, y) &:= h(s, x, y) - \sum_{i\in I} f_i(s, x) (\log y)^{\mu_i}\, \mathrm{e}^{\mathrm{i}Q_i(x,y)} \sum_{k<k_0} \xi_{i,k}(s, x) (a(x))^{\frac{k}{d}}\, y^{\lambda_i(s)-\frac{k}{d}} \\ &= \sum_{i\in I} f_i(s, x)\, y^{\lambda_i(s)-\frac{k_0}{d}} (\log y)^{\mu_i}\, \mathrm{e}^{\mathrm{i}Q_i(x,y)} \sum_{k\geq 0} \xi_{i,k+k_0}(s, x) (a(x))^{\frac{k}{d}}\, y^{-\frac{k}{d}}. \end{aligned}$$

Setting $C_N(s, x) = \left(\frac{(\log a(x))^{\mu}}{a(x)} + \frac{1}{\mathrm{e}}\right) \sum_{i\in I} |f_i(s, x)| \sum_{k\geq 0} |\xi_{i,k+k_0}(s, x)|$, by the choice of k_0 we have

$$|h_{\geq k_0}(s, x, y)| \leq C_N(s, x)\, y^{-N},$$

which proves the first statement of the theorem for h.

Suppose now that h is not a finite sum of generators which are naive in y. Apply Proposition 5.3 and Remark 3.6 to h: up to shrinking B, this writes h as a finite sum of prepared generators as in Definition 5.2. Let T be one such generator: for our aim it is enough to show that, given $N \in \mathbb{N}$, we can rewrite T as a finite sum of generators which are either naive in y or such that we can control their asymptotics by y^{-N}. For this, we revisit the proof of Proposition 5.10 and argue according to the nature of the integration bounds in the transcendental element of T.

Recall Notation 5.1.

If $\alpha=\beta=0$, then, up to partitioning X, shrinking B and enlarging P, we may rewrite T as a a finite sum of generators which are naive in y plus a term of the form (5.9), where we can expand $\Phi_{>k_0}$ as an absolutely convergent series in the variable y as in (5.8). Permuting integral and summation, we obtain that this last term can be written as an absolutely convergent series of the form (7.1). Hence we can apply the first part of the proof to this last term.

If $\alpha>0$, then chose $\ell_0\in\mathbb{N}$ such that for all $s\in\Sigma$, $\Re\left(\lambda\left(s\right)\right)+\frac{\alpha}{d}\left(\Re\left(\varrho\left(s\right)\right)-\ell_0+1\right)+\frac{\Delta}{d}<-\left(N+1\right)$. If we integrate by parts as in Lemma 5.9 ℓ_0 times, then we create finitely many terms which are naive in y and an integral rest of the form

$$R\left(s,x,y\right)=g\left(s,x\right)y^{\lambda(s)}\left(\log y\right)^{\mu}\mathrm{e}^{\mathrm{i}\varphi(x,y)}\int_{\widetilde{a}(x,y)}^{\widetilde{b}(x,y)}t^{\varrho(s)-\ell_0}\left(\log t\right)^{\nu}\Phi\left(s,x,y,t\right)\mathrm{e}^{\sigma\mathrm{i}t}\mathrm{d}t, \tag{7.2}$$

where (5.10) is satisfied.

If $\widetilde{b}=+\infty$, then

$$\begin{aligned}\left|R\left(s,x,y\right)\right|&\leq\widetilde{C}_N\left(s,x\right)y^{\Re(\lambda(s))+\frac{\alpha}{d}(\Re(\varrho(s))-\ell_0+1)}\left(\log y\right)^{\mu+\nu}\\&\leq C_N\left(s,x\right)y^{-N},\end{aligned}$$

for suitable positive functions $\widetilde{C}_N,C_N$.

If $\widetilde{b}<+\infty$, then

$$\begin{aligned}\left|R\left(s,x,y\right)\right|&\leq\widetilde{C}_N\left(s,x\right)y^{\Re(\lambda(s))+\frac{\alpha}{d}(\Re(\varrho(s))-\ell_0)+\frac{\Delta}{d}}\left(\log y\right)^{\mu+\nu}\\&\leq C_N\left(s,x\right)y^{-N},\end{aligned}$$

for suitable positive functions $\widetilde{C}_N,C_N$.

If $\alpha=0$ and $\beta>0$ then choose $k_0\in\mathbb{N}$ such that for all $s\in\Sigma$, $\Re\left(\lambda\left(s\right)\right)-\frac{\beta}{d}\left(\frac{k_0+1}{d}\right)+\frac{\Delta}{d}<-\left(N+1\right)$ and $\ell_0\in\mathbb{N}$ such that $\ell_0>\Re\left(\varrho\left(s\right)\right)+\frac{k_0}{d}+1$. Then (5.11) and (5.12) are satisfied if we replace $\varrho\left(s\right)$ by $\varrho\left(s\right)-\ell_0$. If we integrate by parts ℓ_0 times, then we create finitely many terms which are naive in y and an integral rest of the form (7.2). Proceeding as in the third part of the proof of Proposition 5.10, we are left to deal with a term $R_{>k_0}$ of the form (5.14) which satisfies

$$\begin{aligned}\left|R_{>k_0}\left(s,x,y\right)\right|&\leq\widetilde{C}_N\left(s,x\right)y^{\Re(\lambda(s))-\frac{\beta}{d}\left(\frac{k_0+1}{d}\right)+\frac{\Delta}{d}}\left(\log y\right)^{\mu+\nu}\\&\leq C_N\left(s,x\right)y^{-N},\end{aligned}$$

for suitable positive functions $\widetilde{C}_N,C_N$. □

Our next goal is to concentrate on the subclass $\mathcal{C}^{\mathbb{C},\mathcal{F}}\left(X\times\mathbb{R}\right)$ and deduce from Theorem 7.1 a more precise result on the asymptotic behaviour of h in y, in the sense of [CCMRS18, Definition 7.1] but uniformly in the variables $x\in X$.

First, we restate and improve Theorem 7.1 for functions in the class $\mathcal{C}^{\mathbb{C},\mathcal{F}}\left(X\times\mathbb{R}\right)$.

Definition 7.2. Let $\mathscr{E} \subseteq \mathcal{C}^{\mathbb{C},\mathcal{F}}(X \times (0,+\infty))$ be the $\mathbb{C}$-vector space of all functions of the form

$$E(x,y) = \sum_{j\in J} f_j(x)\, \mathrm{e}^{\mathrm{i}\zeta_j(x,y)},$$

where J is a finite index set, $f_j \in \mathcal{C}^{\mathbb{C},\mathcal{F}}(X)$, $\zeta_j(x,y) = \sigma_j \log y + Q_j(x,y)$ with $\sigma_j \in \mathbb{R}$, $Q_j \in \mathcal{S}(X)\left[y^{\frac{1}{d}}\right]$. We require moreover that for all $j \in J$, for all $x \in X$, $Q_j(x,0) = 0$, for all $i,j \in J$, either $Q_i = Q_j$ on X or $Q_i \neq Q_j$ on X and if $i \neq j$ then for all $x \in X$, the functions $y \longmapsto \zeta_i(x,y)$ and $y \longmapsto \zeta_j(x,y)$ are distinct.

Remark 7.3. By [CCRS23, Proposition 3.4 (2)], if $E \in \mathscr{E} \setminus \{0\}$ then for all $x \in X$, either $y \longmapsto E(x,y)$ is identically zero or there exist $\varepsilon(x) > 0$ and a sequence $(y_n)_{n\in\mathbb{N}}$ such that $\lim_{n\longrightarrow+\infty} y_n = +\infty$ and for all for all $n \in \mathbb{N}$, $|E(x,y_n)| > \varepsilon(x)$.

Definition 7.4. A function $h \in \mathcal{C}^{\mathbb{C},\mathcal{F}}(X \times \mathbb{R})$ has a *power-log asymptotic expansion with coefficients in* $\mathscr{E}$ if there are a collection $\{E_n :\ n \in \mathbb{N}\} \subseteq \mathscr{E}$, a sequence $(r_n, \nu_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \times \mathbb{N}$ which is strictly decreasing with respect to the lexicographic order, a cell B as in (3.1) with unbounded y-fibers and for all $N \in \mathbb{N}$, a function $C_N : X \longrightarrow (0,+\infty)$ such that, for all $(x,y) \in B$,

$$\left| h(x,y) - \sum_{n<N} E_n(x,y)\, y^{r_n} (\log y)^{\nu_n} \right| \leq C_N(x)\, y^{r_N} (\log y)^{\nu_N}. \tag{7.3}$$

If moreover the series $\sum_{n\in\mathbb{N}} E_n(x,y)\, y^{r_n} (\log y)^{\nu_n}$ converges absolutely to h, then we say that h has a *convergent* power-log asymptotic expansion with coefficients in $\mathscr{E}$.

Note that the sequence of real functions $(g_n(y))_{n\in\mathbb{N}} = (y^{r_n} (\log y)^{\nu_n})_{n\in\mathbb{N}}$ forms an *asymptotic scale* at $+\infty$ in the sense that, for all $n \in \mathbb{N}$, $\lim_{y\longrightarrow+\infty} \frac{g_{n+1}(y)}{g_n(y)} = 0$.

Recall the definition of the system $\mathcal{C}^{\mathbb{C}}$ of power-constructible functions and that of the system $\mathcal{C}^{\mathbb{C},\mathrm{i}\mathcal{S}}$, given in Section 2.1.

Definition 7.5. Let $\mathcal{C}^{\mathbb{C},\mathcal{F}}_{\text{naive}}(X \times \mathbb{R})$ be the additive group generated by the generators which are naive in y, i.e. of the form

$$\gamma g \mathrm{e}^{\mathrm{i}\varphi} \quad \left(\gamma \in \mathcal{C}^{\mathbb{C},\mathcal{F}}(X),\ g \in \mathcal{C}^{\mathbb{C}}(X \times \mathbb{R}),\ \varphi \in \mathcal{S}(X \times \mathbb{R})\right).$$

Note that $\mathcal{C}^{\mathbb{C},\mathcal{F}}_{\text{naive}}(X \times \mathbb{R})$ is a $\mathbb{C}$-algebra and

$$\mathcal{C}^{\mathbb{C}}(X \times \mathbb{R}) \subseteq \mathcal{C}^{\mathbb{C},\mathrm{i}\mathcal{S}}(X \times \mathbb{R}) \subseteq \mathcal{C}^{\mathbb{C},\mathcal{F}}_{\text{naive}}(X \times \mathbb{R}) \subseteq \mathcal{C}^{\mathbb{C},\mathcal{F}}(X \times \mathbb{R}). \tag{7.4}$$

Theorem 7.6. *Every $h \in \mathcal{C}^{\mathbb{C},\mathcal{F}}(X \times \mathbb{R})$ has, up to partitioning X, a power-log asymptotic expansion with coefficients in $\mathscr{E}$. If moreover $h \in \mathcal{C}^{\mathbb{C},\mathcal{F}}_{\text{naive}}(X \times \mathbb{R})$, then such an asymptotic expansion is convergent.*

Proof. Suppose first that $h \in \mathcal{C}^{\mathbb{C},\mathcal{F}}_{\text{naive}}(X \times \mathbb{R})$, so that, up to partitioning X and on some cell B with unbounded y-fibers, h can be written as in (7.1), where the functions $f_{i,k}$ only depend on the variables x and $\lambda_{i,k} = \lambda_i - \frac{k}{d} \in \mathbb{C}$. Let $\rho_{i,k} = \Re(\lambda_i) - \frac{k}{d}$, $\sigma_i = \Im(\lambda_i)$ and define

$\zeta_i(x,y)=\sigma_i\log y+Q_i(x,y)$. Hence we can write h as the sum of the absolutely convergent series of functions

$$\sum_{(i,k)\in I\times\mathbb{N}} f_{i,k}(x)\, y^{\rho_{i,k}}(\log y)^{\mu_i}\,\mathrm{e}^{\mathrm{i}\zeta_i(x,y)}.$$

The set $\{\rho_{i,k}:\ i\in I, k\in\mathbb{N}\}$ is contained in a finitely generated $\mathbb{Z}$-lattice and, since I is finite, so is the set $\{\mu_i:\ i\in I\}$. Hence the set

$$J=\{(r,\nu):\ \exists\,(i,k)\in I\times\mathbb{N}\text{ s.t. }(\rho_{i,k},\mu_i)=(r,\nu)\}$$

is countable and, for $(r,\nu)\in J$, the set $J_{(r,\nu)}=\{(i,k)\in I\times\mathbb{N}:\ \rho_{i,k}=r,\ \mu_i=\nu\}$ is finite. Fix a bijection

$$\mathbb{N}\ni n\longmapsto(r_n,\nu_n)\in J$$

which is decreasing with respect to the lexicographic order and define

$$E_n(x,y)=\sum_{(i,k)\in J_{(r_n,\nu_n)}} f_{i,k}(x)\,\mathrm{e}^{\mathrm{i}\zeta_i(x,y)}.$$

These are the coefficients of a convergent power-log asymptotic expansion of h in the asymptotic scale $\{y^{r_n}(\log y)^{\nu_n}:\ n\in\mathbb{N}\}$.

Suppose now that $h\notin\mathcal{C}^{\mathbb{C},\mathcal{F}}_{\text{naive}}(X\times\mathbb{R})$. Revisiting the proof of Theorem 7.1, given $N\in\mathbb{N}$, we may write h as a finite sum of generators which are either naive in y (and hence have a convergent power-log asymptotic expansion in some common asymptotic scale $\left(y^{r_n}(\log y)^{\nu_n}\right)_{n\in\mathbb{N}}$ with coefficients $\{E_n:\ n\in\mathbb{N}\}\subseteq\mathscr{E}$) or whose module is bounded $C_N(x)\,y^{\lfloor r_N\rfloor-1}$, where C_N is some positive function in $\mathcal{C}^{\mathbb{C},\mathcal{F}}(X)$. In particular, h has a (not necessarily convergent) power-log asymptotic expansion as in (7.3). $\square$

Remark 7.7. Argueing as in [CCMRS18, Lemma 7.2] and using Remark 7.3, one sees that if h has a power-log asymptotic expansion in a certain power-log asymptotic scale and with coefficients in $\mathscr{E}$, then its coefficients are uniquely determined. Note that the proof of Theorem 7.6 shows that the power-log asymptotic scales $\left(y^{r_n}(\log y)^{\nu_n}\right)_{n\in\mathbb{N}}$ appearing in the asymptotic expansions of functions in $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ have the property that the sequence $(r_n,\nu_n)_{n\in\mathbb{N}}$ has the same order type as ω. In particular, the union of two such asymptotic scales is again an asymptotic scale of the same type, so a function in $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ cannot have two different asymptotic expansions in two different power-log asymptotic scales.

Corollary 7.8. *The systems $\mathcal{C}^{\mathbb{C},\mathrm{i}\mathcal{S}}$ and $\mathcal{C}^{\mathcal{M},\mathrm{i}\mathcal{S}}$ are not stable under parametric integration.*

Proof. We give two examples of functions which are in $\mathcal{C}^{\mathbb{C},\mathcal{F}}(\mathbb{R})$ but not in $\mathcal{C}^{\mathcal{M},\mathrm{i}\mathcal{S}}(\mathbb{R})$.

The function $f: y\longmapsto\mathrm{e}^{-|y|}$ belongs to $\mathcal{C}^{\mathbb{C},\mathcal{F}}(\mathbb{R})$, since it can be obtained as a parametric integral of a function in $\mathcal{C}^{\mathbb{C},\mathrm{i}\mathcal{S}}(\mathbb{R}^2)$ (it is the inverse Fourier transform of the semialgebraic function $t\longmapsto\frac{2}{1+4\pi^2t^2}$, see for example [GW99]). If f were in $\mathcal{C}^{\mathcal{M},\mathrm{i}\mathcal{S}}(\mathbb{R})$ then it would also be in $\mathcal{C}^{\mathbb{C},\mathrm{i}\mathcal{S}}(\mathbb{R})$ and by Theorem 7.6 f would have a convergent power-log asymptotic expansion with coefficients in $\mathscr{E}$. Now, argueing as in [CCMRS18, Example 7.4] and using Remark 7.3, one sees that no exponentially flat function can have such a convergent power-log asymptotic expansion.

Now consider the function

$$\mathrm{Si}\,(y) = \begin{cases} \int_0^y \frac{\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{-\mathrm{i}t}}{2\mathrm{i}t}\mathrm{d}t & y > 0 \\ 0 & y \le 0 \end{cases}$$

which is obtained as a parametric integral of a function in $\mathcal{C}^{\mathbb{C},\mathrm{i}\mathcal{S}}\left(\mathbb{R}^2\right)$. It is well-known that Si has a divergent power-log asymptotic expansion with coefficients in $\mathcal{E}$ (see [AS65] and [CCMRS18, Example 7.5]. By Theorem 7.6 and Remark 7.7, Si $\notin \mathcal{C}^{\mathcal{M},\mathrm{i}\mathcal{S}}\left(\mathbb{R}\right)$. □

Remark 7.9. Let $X \subseteq \mathbb{R}^m$ be a subanalytic open set and $K \subseteq X$ be a compact subanalytic subset. It is possible to construct a C^∞ function $\eta \in \mathcal{C}^{\mathbb{C},\mathcal{F}}(X)$ such that $\eta\,(X) \subseteq [0,1]$ and $\eta \equiv 1$ on a neighbourhood of K in X (in particular $\mathcal{C}^{\mathbb{C},\mathcal{F}}(X)$ contains smooth functions with compact support). One way to do this is to consider the function

$$\nu : x \mapsto f(1 - \|x\|^2),$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^m$ and

$$f : t \mapsto \begin{cases} \mathrm{e}^{-\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t \le 0 \end{cases}.$$

Note that, considering the first example in the proof of Corollary 7.8 and using the stability under right-composition with subanalytic functions observed in Remark 2.8, we obtain that $\nu \in \mathcal{C}^{\mathbb{C},\mathcal{F}}(X)$. We can then define η as the convolution of ν with the subanalytic characteristic function of a sufficiently small tubular neighbourhood of K in X (see for instance [Hör03, Theorem 1.4.1]), and thus obtain that $\nu \in \mathcal{C}^{\mathbb{C},\mathcal{F}}(X)$ by Theorem 2.9.

We have at our disposal several results concerning the asymptotics at infinity of integral transforms, and in particular of Fourier and Mellin transforms, of functions with support in $[0,+\infty)$ having an asymptotic expansions at the origin in the scale $\{x^\alpha \log^\beta : \alpha, \beta \in \mathbb{R}\}$ (see for instance [BH86, Won89, WL78]). In this situation, the integral transforms have an asymptotic expansion at $+\infty$ in the same power-log scale. On the other hand, to our knowledge very little known beyond this scale, in particular with respect to asymptotic scales detecting exponentially small terms (see [Lom00]), a question that is relevant to the class $\mathcal{C}^{\exp}$ of [CCMRS18] and to our class $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ by Remark 7.9, but which seems to require new tools.

7.2. **Pointwise limits.** In this section we prove the stability of the class $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ under pointwise limits.

Notation 7.10. For $X \subseteq \mathbb{R}^m$ and $h\colon X \times \mathbb{R} \to \mathbb{C}$, let

$$\mathrm{Lim}\,(h, X) := \{x \in X : \lim_{y\to+\infty} h\,(x,y) \text{ exists}\}.$$

Theorem 7.11. *Let $h \in \mathcal{C}^{\mathbb{C},\mathcal{F}}\left(X \times \mathbb{R}\right)$. There exist $f, g \in \mathcal{C}^{\mathbb{C},\mathcal{F}}\left(X\right)$ such that*

$$\mathrm{Lim}\,(h, X) = \{x \in X : f\,(x) = 0\}$$

and such that for all $x \in \mathrm{Lim}\,(h,X)$,

$$\lim_{y\to+\infty} h\,(x,y) = g\,(x)\,.$$

Proof. Apply Proposition 6.4 to h and concentrate on a cell A with unbounded y-fibers (so that, by Remark 3.6, $A = B_A$ and Π_A is the identity map). By condition (1), the prepared generators T_j which are strongly integrable tend indeed to a limit and this limit is zero. Hence we may suppose that $h = \sum_{i\in I} T_i$ for some finite index set I, where each T_i is a monomial generator of the form

$$T_i(x,y) = f_i(x)\, y^{\lambda_i} (\log y)^{\mu_i} \,\mathrm{e}^{\mathrm{i}Q_i(x,y)},$$

where $\lambda_i \in \mathbb{C}$ with $\Re(\lambda_i) \geq 0$. Write the finite set

$$J = \{(r,\nu) \in [0,+\infty) \times \mathbb{N} :\ \exists i \in I \text{ s.t. } \Re(\lambda_i) = r,\ \mu_i = \nu\}$$

as

$$J = \{(r_0,\nu_0), \ldots, (r_N,\nu_N)\}$$

for some $N \in \mathbb{N}$, and suppose that $(r_0,\nu_0) > \ldots > (r_N,\nu_N)$ with respect to the lexicographic order. For $j = 0,\ldots,N$, define

$$J_j = \{i \in I :\ \Re(\lambda_i) = r_j,\ \mu_i = \nu_j\}.$$

Writing $\zeta_i(x,y) = \Im(\lambda_i)\log y + Q_i(x,y)$ and $E_j(x,y) = \sum_{i\in J_j} f_i(x)\,\mathrm{e}^{\mathrm{i}\zeta_i(x,y)}$, we obtain that

$$h(x,y) = \sum_{j\leq N} E_j(x,y)\, y^{r_j} (\log y)^{\nu_j}.$$

Let $x \in \mathrm{Lim}(h,X)$. Suppose that there exists $i \in J_0$ such that $f_i(x) = 0$. Then, by Condition (2.a) of Proposition 6.4 and by [CCRS23, Proposition 3.4(2)] we have necessarily that $r_0 = \nu_0 = 0$. Hence we may suppose that $N = 0$ and $h(x,y) = \sum_{i\in J_0} f_i(x)\,\mathrm{e}^{\mathrm{i}\zeta_i(x,y)}$. If there exists $i \in J_0$ such that either $\Im(\lambda_i) \neq 0$ or $Q_i(x,y) \neq 0$, then by [CCRS23, Proposition 3.4(3)] we obtain that $f_i(x) = 0$. Notice that there is at most one index $i_0 \in J_0$ such that $\zeta_{i_0}(x,y) = 0$. To conclude, we define and

$$f(x) = \sum_{i\in\widetilde{I}} |f_i(x)|^2.$$

As the class $\mathcal{C}^{\mathbb{C},\mathcal{F}}$ is clearly stable under complex conjugation, f belongs to $\mathcal{C}^{\mathbb{C},\mathcal{F}}$. Finally, define $g(x) = f_{i_0}(x)$, if there exists a (necessarily unique) index $i_0 \in I$ such that $\Re(\lambda_{i_0}) = \mu_{i_0} = 0$ and $\zeta_{i_0}(x,y) = 0$, and $g = 0$ otherwise. $\square$

8. The Fourier-Plancherel transform and L^p-limits

We deal here with the question of parametric families of functions of $\mathcal{C}^{\mathbb{C},\mathcal{F}}$, to provide non-compensation arguments in this framework, useful for L^p-completeness and the L^2-Fourier transform, also known as the Plancherel transform, or the Fourier-Plancherel transform. In [CCMRS18, Section 8], this is treated in the case of the system $\mathcal{C}^{\exp}$, which we generalize to our setting of $\mathcal{C}^{\mathbb{C},\mathcal{F}}$.

We recall from [CCMRS18] what it means for a *family* of functions to be *continuously uniformly distributed modulo 1*, which extends notions from [Wey16, KN74].

Let X be a nonempty subset of $\mathbb{R}^m$, $N \in \mathbb{N}\setminus\{0\}$ and $\rho = (\rho_1,\ldots,\rho_N) : X \times [0,+\infty) \to \mathbb{R}^N$ be a map. If $I_1,\ldots,I_N \subseteq \mathbb{R}$ are bounded intervals with nonempty interior, we denote by I the box $\prod_{j=1}^N I_j$ and, for $T \geq 0$ and $x \in X$, we let

$$W^x_{\rho,I,T} := \{t \in [0,T] : \{\rho(x,t)\} \in I\},$$

where $\{\rho(x,t)\}$ denotes the vector of fractional parts $(\{\rho_1(x,t)\},\dots,\{\rho_N(x,t)\})$ of the components of ρ, that is to say for $x \in \mathbb{R}$, $x = x - \lfloor x \rfloor$.

Definition 8.1. With this notation, we say that the map ρ is *continuously uniformly distributed modulo* 1 *on* X (abbreviated as *c.u.d. mod* 1 *on* X) if for every box $I \subseteq [0,1)^N$,

$$\lim_{T\to+\infty} \sup_{x\in X} \frac{\mathrm{vol}_1\left(W^x_{\rho,I,T}\right)}{T} = \mathrm{vol}_N(I).$$

We will use the c.u.d. mod 1 property in Lemma 8.5. In our context we have to deal with sums of complex exponential functions, with phase of type $\varphi(x,y) = \sigma \log y + p(x,y)$, where p is a polynomial in y, or more exactly in $y^{\frac{1}{d}}$, for some positive integer d, and with coefficients some functions of the variable x. We cannot directly use the c.u.d. mod 1 property for those phases, since $\log y$ is not a c.u.d. mod 1 function (although φ turns out to be c.u.d. mod 1 when p is not constant). To overcome this technical difficulty, we compose φ with $(x,y) = (x,\mathrm{e}^t)$ to obtain a phase of type $\phi(x,t) = \sigma t + p(x,\mathrm{e}^t)$. Now we can use the c.u.d. mod 1 property, the change of variables $y = \mathrm{e}^t$ being harmless in view of the conclusion of Lemma 8.5.

Proposition 8.2. *Let $\ell, p \in \mathbb{N}$ and X a compact subset of $\mathbb{R}^m$. Consider a map $\rho = (\phi_1,\cdots,\phi_\ell,\rho_1,\dots,\rho_p) : X\times[0,+\infty) \to \mathbb{R}^{\ell+p}$, where for each $i \in \{1,\dots,\ell\}$, for each $j \in \{1,\dots,p\}$,*

$$\phi_i(x,t) = g_i(x)\,\mathrm{e}^{\frac{\delta_i}{d}t}, \quad \rho_j(x,t) = \sigma_j t$$

for some continuous (nonzero) functions $g_i : X \to \mathbb{R}$, positive integers d and δ_i, and for σ_j real numbers. Assume that for each $x \in X$, the functions $t \mapsto \phi_1(x,t),\dots,t\mapsto \phi_\ell(x,t), t\mapsto \rho_1(x,t),\dots,t\mapsto\rho_p(x,t)$ are linearly independent over $\mathbb{Q}$. Then ρ is c.u.d. mod 1 on X.

Before proving the proposition, we make a remark.

Remark 8.3. In the notation of Proposition 8.2, let $\delta = \max\{\delta_1,\dots,\delta_\ell\}$, and for each $k \in \{1,\dots,\delta\}$, let $I_k = \{i \in \{1,\dots,\ell\} : \delta_i = k\}$. The assumption that $t \mapsto \phi_1(x,t),\dots,t\mapsto \phi_\ell(x,t), t\mapsto\rho_1(x,t),\dots,t\mapsto\rho_p(x,t)$ are linearly independent over $\mathbb{Q}$ for each $x \in X$ is equivalent to saying that for each $k \in \{1,\dots,\delta\}$ and $x \in X$, the family of real numbers $(g_i(x))_{i\in I_k}$ is linearly independent over $\mathbb{Q}$, and that the family of real numbers $(\sigma_1,\dots,\sigma_m)$ is linearly independent over $\mathbb{Q}$.

Proof of Proposition 8.2. We may assume that $\ell \geq 1$, since if $\ell = 0$ and the family of linear maps $(t\mapsto\sigma_1 t,\dots,t\mapsto\sigma_p t)$ is linearly independent over $\mathbb{Q}$, then the map ρ is well-known to be c.u.d. mod 1 (see [KN74, Exercise 9.27]).

Assuming $\ell \geq 1$, the proof consists in satisfying the version in families of the criterion (8.1) (see [KN74, Theorem 9.9] for the basic case, and [CCMRS18, Proposition 8.7] for the version in families): for any $h = (\alpha_1,\dots,\alpha_\ell,\beta_1,\dots,\beta_p) \in \mathbb{Z}^{\ell+p}$, $h \neq 0$,

$$\lim_{T\to+\infty} \frac{1}{T}\int_1^T \mathrm{e}^{2\pi\mathrm{i}\langle h,\rho(x,t)\rangle}\,\mathrm{d}t = 0, \tag{8.1}$$

uniformly in $x \in X$. We prove in fact that for some $T_0 \geq 1$, $J(T) = \displaystyle\int_{T_0}^T \mathrm{e}^{2\pi\mathrm{i}\langle h,\rho(x,t)\rangle}\,\mathrm{d}t$ is bounded from above by a constant not depending on $x \in X$. To do this, we follow the proof of [CCRS23, Proposition 3.4]: we fix $h \in \mathbb{Z}^{\ell+p}$, define, in the notation of Remark 8.3, $G(x) = \sum_{i\in I_\delta} \alpha_i g_i(x)$, $G_k(x) = \sum_{j\in I_k} \alpha_i g_i(x)$, for $k \in \{1, \ldots, \delta-1\}$, and $\sigma = \sum_{j=1}^m \beta_j \sigma_j$, and we write

$$H(x,t) = \frac{\langle h, \rho(x,t)\rangle}{G(x)} = \mathrm{e}^{\frac{\delta}{d}t} + \frac{G_{\delta-1}(x)}{G(x)}\mathrm{e}^{\frac{\delta-1}{d}t} + \cdots + \frac{G_1(x)}{G(x)}\mathrm{e}^{\frac{t}{d}} + \frac{\sigma t}{G(x)}. \tag{8.2}$$

For simplicity we assume that $I_k \neq \emptyset$, for $k = 1, \ldots, \delta$, which is harmless. Note that the continuous functions $G_1, \ldots, G_{\delta-1}$ are bounded from above on X. By Remark 8.3 the function G has no zero in X, since for each $x \in X$, the components of ρ are linearly independent over $\mathbb{Q}$, and therefore, again by continuity on X, $|G|$ is bounded below by a constant $C > 0$ on X. It follows that we can fix T_0 sufficiently large so that, for each $x \in X$, $t \mapsto H(x,t)$ and $t \mapsto \dfrac{\partial H}{\partial t}(x,t)$ are strictly increasing (to $+\infty$) on $[T_0, +\infty)$, and we can assume that for all $x \in X$, $\dfrac{\partial H}{\partial t}(x, T_0) \geq 1$.

Denoting for each $x \in X$, $t = V(x,u)$ the inverse of $u = H(x,t)$, we perform the change of variables $u = H(x,t)$ in $J(T)$ to obtain

$$J(T) = \int_{T_0}^{T} \mathrm{e}^{2\pi\mathrm{i}G(x)H(x,t)}\,\mathrm{d}t = \int_{H(x,T_0)}^{H(x,T)} \frac{\mathrm{e}^{2\pi\mathrm{i}G(x)u}}{\frac{\partial H}{\partial t}(x, V(x,u))}\,\mathrm{d}u.$$

Now, since $u \mapsto \dfrac{1}{\frac{\partial H}{\partial t}(x, V(x,u))}$ is monotonically decreasing on $[H(x,T_0), +\infty)$, by the Second Mean Value Theorem for integrals applied to the real part of $J(T)$, we have

$$\Re(J(T)) = \frac{1}{\frac{\partial H}{\partial t}(x, T_0)} \int_{H(x,T_0)}^{\tau} \cos(2\pi G(x)u)\,\mathrm{d}u,$$

for some $\tau \in (H(x,T_0), H(x,T)]$. Since $u \mapsto \cos(2\pi G(x)u)$ has an antiderivative with period $\frac{1}{|G(x)|}$, and since $\frac{1}{|G(x)|} \leq \frac{1}{C}$, the integral on the right side may be replaced with an integral over an interval of length at most $\frac{1}{C}$. From the fact that $\dfrac{\partial H}{\partial t}(x,T_0) \geq 1$, for all $x \in X$, it follows that the real part of $J(T)$ is uniformly bounded from above with respect to $x \in X$, and so is the imaginary part of $J(T)$ by the same computation. $\square$

We now introduce some notation for Lemma 8.4. Consider a cell

$$A = \{(x,t) : x \in A_0, t > a(x)\},$$

where A_0 is connected and open in $\mathbb{R}^m$. Let $f : A \to \mathbb{C}$ be defined by

$$f(x,t) = \sum_{j=1}^{n} f_j(x)\,\mathrm{e}^{\mathrm{i}(\sigma_j t + p_j(x,\mathrm{e}^t))},$$

where $\sigma_1, \ldots, \sigma_n$ are real numbers, $(f_1, \ldots, f_n)$ is a family of (nonzero) analytic functions in $\mathcal{C}^{\mathbb{C},\mathcal{F}}(A_0)$, $p_1(x,T), \ldots, p_n(x,T)$ are polynomials (in $T^{\frac{1}{d}}$, for some positive integer d) of $\mathcal{S}(A_0)\left[T^{\frac{1}{d}}\right]$, with analytic coefficients in $\mathcal{S}(A_0)$, and $p_j(x,0) = 0$ for all $j \in \{1, \ldots, n\}$ and all $x \in A_0$. We furthermore assume that for $j \neq j'$ in $\{1, \ldots, n\}$, $\sigma_j t + p_j(x,t) \neq \sigma_{j'} t + p_{j'}(x,t)$ (as functions).

Lemma 8.4. *In above notation, we may express f on A as*

$$f\left(x,t\right)=F\left(x,\rho\left(x,t\right)\right)$$

where $\rho=\left(\phi_{1},\cdots,\phi_{\ell},\rho_{1},\ldots,\rho_{p}\right)$ for some $\ell,p\in\mathbb{N}$, and where for each $i\in\{1,\ldots,\ell\}$ and for each $j\in\{1,\ldots,p\}$,

$$\phi_{i}(x,t)=g_{i}\left(x\right)e^{\frac{\delta_{i}}{d}t},\quad\rho_{j}\left(x,t\right)=\sigma_{j}t$$

for some analytic functions g_i in $\mathcal{S}\left(A_0\right)$, $\delta_i\in\mathbb{N}$, $\sigma_j\in\mathbb{R}$, and where $F\left(x,z_1,\ldots,z_{\ell+p}\right)$ is a Laurent polynomial in the variables $\mathrm{e}^{2\pi\mathrm{i}z_1},\ldots,\mathrm{e}^{2\pi\mathrm{i}z_{\ell+p}}$ with analytic coefficients in $\mathcal{C}^{\mathbb{C},\mathcal{F}}\left(A_0\right)$. If $n=1$ and if $\sigma_1=0$, $p_1=0$, then $\ell+p=0$ and $F\left(x\right)=f_1\left(x\right)$. Otherwise we have $\ell+p>0$, and

1. *there exists a set $A_0'\subseteq A_0$ such that $\mathrm{vol}_m\left(A_0\setminus A_0'\right)=0$ and for every $x\in A_0'$, $z\mapsto F\left(x,z\right)$ is nonconstant,*
2. *for every open set $\Omega\subseteq A_0$ and every real number $\lambda<\mathrm{vol}_m\left(\Omega\right)$, there exists a real number T_0 and a compact set $K\subseteq\Omega\cap A_0'$ such that $K\times[T_0,+\infty)\subseteq A$, $\lambda\leq\mathrm{vol}_m\left(K\right)\leq\mathrm{vol}_m(\Omega)$, and $\rho\restriction K\times[T_0,+\infty)$ is c.u.d. mod 1 on K.*

Proof. The case $n=1$, $\sigma_1=0$ and $p_1=0$ being trivial, we may assume that $\sigma_1\neq 0$ or $p_1\neq 0$. For each $j\in\{1,\ldots,n\}$, we write

$$\sigma_{j}t+p_{j}\left(x,\mathrm{e}^{t}\right)=\sigma_{j}t+\sum_{k=1}^{D}h_{j,k}\left(x\right)\mathrm{e}^{\frac{k}{d}t}$$

with $D\in\mathbb{N}$ and $h_{j,k}\in\mathcal{S}\left(A_0\right)$. For each $k\in\{1,\ldots,D\}$, fix $I_k\subseteq\{1,\ldots,n\}$ such that $\left(h_{i,k}\right)_{i\in I_k}$ is a basis of the $\mathbb{Q}$-vector space generated by the family $\left(h_{j,k}\right)_{j\in\{1,\ldots,n\}}$ (as functions of x), and fix $Q\subseteq\{1,\ldots,n\}$ such that $\left(\sigma_q\right)_{q\in Q}$ is a basis of the $\mathbb{Q}$-vector space generated by the family $\left(\sigma_j\right)_{j\in\{1,\ldots,n\}}$. We then set

$$I=\{(i,k):k\in\{1,\ldots,D\},i\in I_k\}\subseteq\{1,\ldots,D\}\times\{1,\ldots,n\}.$$

We fix a positive integer η such that for each $(j,k)\in\{1,\ldots,n\}\times\{1,\ldots,D\}$,

$$h_{j,k}=\sum_{i\in I_k}\frac{\alpha_{j;i,k}}{\eta}h_{i,k},\ \ \sigma_j=\sum_{q\in Q}\frac{\beta_{j;q}}{\eta}\sigma_q$$

for unique tuples $\left(\alpha_{j;i,k}\right)_{i\in I_k}$ and $\left(\beta_{j;q}\right)_{q\in L}$ of elements of $\mathbb{Z}$. With this notation we have

$$\begin{aligned}f\left(x,t\right)&=\sum_{j=1}^{n}f_{j}\left(x\right)\mathrm{e}^{\mathrm{i}\sigma_{j}t+\mathrm{i}\sum_{k=1}^{d}h_{j,k}(x)\mathrm{e}^{\frac{k}{d}t}}\\&=\sum_{j=1}^{n}f_{j}\left(x\right)\mathrm{e}^{\mathrm{i}\sum_{q\in Q}\frac{\beta_{j;q}}{\eta}\sigma_{q}t+\mathrm{i}\sum_{k=1}^{d}\sum_{i\in I_k}\frac{\alpha_{j;i,k}}{\eta}h_{i,k}(x)\mathrm{e}^{\frac{k}{d}t}}\\&=\sum_{j=1}^{n}f_{j}\left(x\right)\prod_{q\in Q}\left(\mathrm{e}^{2\pi\mathrm{i}\rho_{q}(t)}\right)^{\beta_{j;q}}\prod_{(i,k)\in I}\left(\mathrm{e}^{2\pi\mathrm{i}\phi_{i,k}(x,t)}\right)^{\alpha_{j;i,k}}=F\left(x,\left(\phi_{i,k}\left(x,t\right)\right)_{(i,k)\in I},\left(\rho_{q}\right)_{q\in Q}\right)\end{aligned}$$

where for each $(i,k)\in I$, $\phi_{i,k}(x,t)=\frac{h_{i,k}(x)}{2\pi\eta}\mathrm{e}^{\frac{k}{d}t}$, for each $q\in Q$, $\rho_q(t)=\frac{\sigma_q}{2\pi\eta}t$, and

$$F(x,(z_{i,k})_{(i,k)\in I},(z_q)_{q\in Q})=\sum_{j=1}^{n}f_j(x)\prod_{q\in Q}\left(\mathrm{e}^{2\pi\mathrm{i}z_q}\right)^{\beta_{j;q}}\prod_{(i,k)\in I}\left(\mathrm{e}^{2\pi\mathrm{i}z_{i,k}}\right)^{\alpha_{j;i,k}}.$$

For each $j\in\{1,\dots n\}$, f_j is a nonzero analytic function on the connected and open set A_0, so the set

$$U:=\{x\in A_0: f_j(x)\neq 0 \text{ for all } j\in\{1,\dots,n\}\}$$

satisfies $\mathrm{vol}_m(A_0\setminus U)=0$. Denote by $\mathcal{F}$ the Laurent polynomial associated to F

$$\mathcal{F}(x,(Z_{i,k})_{(i,k)\in I},(Z_q)_{q\in Q})=\sum_{j=1}^{n}f_j(x)\prod_{q\in Q}Z_q^{\beta_{j;q}}\prod_{(i,k)\in I}Z_{i,k}^{\alpha_{j;i,k}}.$$

Note that $F(x,z)=F(x,(z_{i,k})_{(i,k)\in I},(z_q)_{q\in Q})=\mathcal{F}(x,(\mathrm{e}^{2\pi\mathrm{i}z_{i,k}})_{(i,k)\in I},(\mathrm{e}^{2\pi\mathrm{i}z_q})_{q\in Q})$.

Since we assumed $\sigma_1\neq 0$ or $p_1\neq 0$, we can always suppose $\sigma_1\in(\sigma_q)_{q\in Q}$ or, for some k, $h_{1,k}\in(h_{i,k})_{i\in I_k}$, respectively. Thus $\mathcal{F}$ certainly contains a term of the form $f_1(x)Z_1$ or $f_1(x)Z_{1,k}$. Moreover, since for $j\neq j'$ in $\{1,\dots,n\}$, $\sigma_j t+p_j(x,t)\neq\sigma_{j'}t+p_{j'}(x,t)$ (as functions), the monomial terms in the above expression of $\mathcal{F}$ cannot cancel out. It follows that for each $x\in U$, $\mathcal{F}$ is not constant as a Laurent polynomial, and in particular, for each $x\in U$, not constant on the real torus $(S^1)^{|Q|+|I|}$. As a consequence, for each $x\in U$, the trigonometric polynomial $z\mapsto F(x,z)$ is not constant.

Observe that since $(h_{i,k})_{i\in I_k}$ is independent over $\mathbb{Q}$ (as functions of x), for each $k\in\{1,\dots,D\}$ and nonzero tuple $c=(c_i)\in\mathbb{Z}^{|I_k|}$, $\sum_{i\in I_k}c_ih_{i,k}$ is a nonzero analytic function on A_0, so the set $\left\{x\in U:\sum_{i\in I_k}c_ih_{i,k}(x)=0\right\}$ cannot have positive measure, and the set

$$A_0':=U\setminus\left(\bigcup_{k=1}^{D}\bigcup_{c\in\mathbb{Z}^{|I_k|}\setminus\{0\}}\left\{x\in U:\sum_{i\in I_k}c_ih_{i,k}(x)=0\right\}\right)$$

satisfies $\mathrm{vol}_m(A_0\setminus A_0')=0$ as well. This gives (1), for this set $A_0'\subset U$.

The set A_0' is defined such that for each $k\in\{1,\dots,D\}$, for each $x\in A_0'$, the family of numbers $(h_{i,k}(x))_{(i,k)\in I}$ is linearly independent over $\mathbb{Q}$. By Remark 8.3, for each $x\in A_0'$ the family of functions $(t\mapsto\phi_{i,k}(x,t))_{(i,k)\in I}$ is also linearly independent over $\mathbb{Q}$. On the other hand the family of functions $(t\mapsto\rho_q(t))_{q\in Q}$ is linearly independent over $\mathbb{Q}$, since so is the family of real numbers $(\sigma_q)_{q\in Q}$. In particular, for each $x\in A_0'$, the family of functions $t\mapsto\rho(x,t)=\left((\phi_{i,k}(x,t))_{(i,k)\in I},(\rho_q(t))_{q\in Q}\right)$ is linearly independent over $\mathbb{Q}$.

Given an open set $\Omega\subseteq A_0$ and a positive real number λ with $\lambda<\mathrm{vol}_m(\Omega)=\mathrm{vol}_m(\Omega\cap A_0')$, the inner regularity of the Lebesgue measure shows that we may fix a compact set $K\subseteq\Omega\cap A_0'$ with $\mathrm{vol}_m(K)\geq\lambda$. Since K is compact and $a(x)$ is continuous, we may fix T_0 sufficiently large so that $K\times[T_0,+\infty)\subseteq A$. Proposition 8.2 then shows that the restriction of ρ to $K\times[T_0,+\infty)$ is c.u.d. mod 1 on K, which completes the proof of (2). $\square$

Recall that

$$f(x,y)=\sum_{j=1}^{n}f_j(x)\,y^{\mathrm{i}\sigma_j}\mathrm{e}^{\mathrm{i}p_j(x,y)},$$

where $\sigma_1, \ldots, \sigma_n$ are real numbers, $(f_1, \ldots, f_n)$ is a family of (nonzero) analytic functions in $\mathcal{C}^{\mathbb{C},\mathcal{F}}(A_0)$, $p_1(x,T), \ldots, p_n(x,T)$ are polynomials (in $T^{\frac{1}{d}}$, for some positive integer d) of $\mathcal{S}(A_0)\left[T^{\frac{1}{d}}\right]$, with analytic coefficients in $\mathcal{S}(A_0)$, and $p_j(x,0) = 0$ for all $j \in \{1, \ldots, n\}$ and $x \in A_0$. Furthermore we assume that for $j \neq j'$ in $\{1, \ldots, n\}$, $\sigma_j + p_j(x,y) \neq \sigma_{j'} + p_{j'}(x,y)$ (as functions).

Lemma 8.5. *In the notation above, there exist $\varepsilon > 0$, $\Delta > 0$, a strictly increasing sequence $(y_j)_{j\in\mathbb{N}}$ in $\mathbb{R}$ diverging to $+\infty$, a compact set $K \subset A_0$, and a sequence $(X_j)_{j\in\mathbb{N}}$ of Lebesgue measurable subsets of K, with, for all $j \in \mathbb{N}$, $\mathrm{vol}_m(X_j) \geq \Delta$, $X_{2j+1} \subseteq X_{2j}$, and such that , for all $x_0 \in X_{2j}$, $x_1 \in X_{2j+1}$,*

$$|f(x_0, y_{2j})| \geq \varepsilon \text{ and } |f(x_0, y_{2j}) - f(x_1, y_{2j+1})| \geq \varepsilon.$$

Proof. Let $\tilde{f}(x,t) := f(x, \mathrm{e}^t)$ for any (x,t) such that $(x, \mathrm{e}^t) \in A$. Then we can apply Lemma 8.4 to $\tilde{f}$, so that the hypothesis of [CCMRS18, Lemma 8.10] is satisfied by $\tilde{f}$. It immediately follows that the conclusions of our lemma are satisfied by $\tilde{f}$, for a sequence of real numbers $(t_j)_{j\in\mathbb{N}}$ diverging to $+\infty$. It now suffices to set $y_j = \mathrm{e}^{t_j}$ to conclude the proof of the lemma. □

Definition 8.6. Let $X \subseteq \mathbb{R}^m$ and $f \colon X \times \mathbb{R} \to \mathbb{C}$ be Lebesgue measurable, and $p \in [1, +\infty]$. For each $y \in \mathbb{R}$, define $f_y : X \to \mathbb{C}$ by $f_y(x) = f(x,y)$ for all $x \in X$. We say that the family of functions $(f_y)_{y\in\mathbb{R}}$ is *Cauchy in $L^p(X)$ as $y \to +\infty$* if for each $y \in \mathbb{R}$, $f_y \in L^p(X)$ and for all $\varepsilon > 0$ there exists $y_0 \in \mathbb{R}$ such that

$$\|f_y - f_{y'}\|_p < \varepsilon \quad \text{for all } y, y' \geq y_0.$$

Theorem 8.7. *Let $p \in [1, +\infty]$ and $f \in \mathcal{C}^{\mathbb{C},\mathcal{F}}(X \times \mathbb{R})$, for some subanalytic set $X \subseteq \mathbb{R}^m$, and suppose that $(f_y)_{y\in\mathbb{R}}$ is Cauchy in $L^p(X)$ as $y \to +\infty$. Then there exist $g \in \mathcal{C}^{\mathbb{C},\mathcal{F}} \cap L^p(X)$ and a subanalytic set $X_0 \subseteq X$ such that $\mathrm{vol}_m(X \setminus X_0) = 0$,*

$$\lim_{y\to+\infty} \|f_y - g\|_p = 0,$$

and

$$\lim_{y\to+\infty} f(x,y) = g(x) \quad \text{for all } x \in X_0.$$

Proof. Writing f as a sum of generators as in Theorem 7.11, we proceed as in the proof of [CCMRS18, Proposition 8.2], using Lemma 8.5 instead of [CCMRS18, Lemma 8.10]. □

As a direct consequence of Proposition 8.7 (see for instance the proof of [CCMRS18, Theorem 8.3]) we obtain the following result.

Theorem 8.8. *Let $\widetilde{\mathscr{F}}$ be the Fourier-Plancherel extension of the Fourier transform to $L^2(\mathbb{R}^n)$. Then, the image of $\mathcal{C}^{\mathbb{C},\mathcal{F}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ under $\widetilde{\mathscr{F}}$ is $\mathcal{C}^{\mathbb{C},\mathcal{F}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. transform*

Stability under parametric Plancherel-Fourier transforms is formulated and shown similarly.

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Univ. Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France, and, KU Leuven, Department of Mathematics, B-3001 Leuven, Belgium
E-mail address: `Raf.Cluckers@univ-lille.fr`
URL: `http://rcluckers.perso.math.cnrs.fr/`

Université Savoie Mont Blanc, LAMA, CNRS UMR 5127, F-73000 Chambéry, France
E-mail address: `georges.comte@univ-smb.fr`
URL: `http://gcomte.perso.math.cnrs.fr/`

Institut de Mathématiques de Jussieu – Paris Rive Gauche, Université Paris Cité and Sorbonne Université, CNRS, IMJ-PRG, F-75013 Paris, France
E-mail address: `tamara.servi@imj-prg.fr`
URL: `http://www.logique.jussieu.fr/~servi/index.html`