Subsumptions of Algebraic Rewrite Rules

Thierry Boy de la Tour

▶ To cite this version:

Thierry Boy de la Tour. Subsumptions of Algebraic Rewrite Rules. 2023. hal-04087851v1

HAL Id: hal-04087851

https://hal.univ-grenoble-alpes.fr/hal-04087851v1

Preprint submitted on 3 May 2023 (v1), last revised 27 Sep 2023 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Subsumptions of Algebraic Rewrite Rules

Thierry Boy de la Tour
Univ. Grenoble Alpes, CNRS, Grenoble INP, LIG,
38000 Grenoble, France
`thierry.boy-de-la-tour at imag.fr`

May 3, 2023

Abstract

What does it mean for an algebraic rewrite rule to subsume another rule (that may then be called a subrule)? We view subsumptions as rule morphisms such that the simultaneous application of a rule and a subrule (i.e. the application of a morphism) yields the same result as a single application of the rule. Simultaneous applications of whole subcategories of rules are obtained by Global Coherent Transformations and illustrated on graphs in the DPO approach. Other approaches are possible since these transformations are formulated in an abstract Rewriting Environment, and such environments exist for DPO, SqPO, PBPO rules.

1 Introduction

In Global Transformations [16] rules in the form of pairs (L, R) of graphs (or objects in a category $\mathcal{C}$) are applied simultaneously to an input graph (as in L-systems [10] and cellular automata [9]). Such rules are related by pairs of $\mathcal{C}$-morphisms to form a category. These morphisms come from representing possible overlaps of rules as subrules whose applications are induced by the overlapping applications of rules, therefore establishing a link between these. By computing a colimit of a diagram involving the morphisms between occurrences of right-hand sides, Global Transformations offer the possibility to merge items (vertices or edges) in these occurrences of right-hand sides.

This form of rules has the advantage of simplicity, first because the notion of subrule can be identified with morphisms in $\mathcal{C}^2$, and second because the input graph is completely removed. Indeed, when all occurrences of L have been found in the input graph G, the output graph H is produced solely from the corresponding occurrences of R, thus effectively removing G. In particular, if no L has any match in G then H is the empty graph.

More standard approaches to rewriting use rules for replacing matched parts of the input by new parts. These substitutions are performed by first removing the matched part and then adding the new part, which is performed by a pushout. But since there is no general algebraic way of deleting parts of a $\mathcal{C}$-object, several approaches have been devised, from DPO [7] to PBPO [4] rules. These rules always have an interface K with $\mathcal{C}$-morphisms to L and R, but can be more complicated. Hence the necessity of a general notion of morphism between rules.

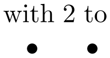
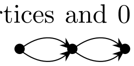
In Section 3 an intuitive analysis of rule subsumptions on a simple example with DPO-rules leads to morphisms between the corresponding direct transformations. The intuition is made clear in Section 4 where Global Coherent Transformations are defined and illustrated on the running example with DPO rules. This definition is carried out in a *Rewriting Environment* that provides the relevant categories of rules, direct transformations and partial transformations. This derives from the Parallel Coherent Transformations defined in [2] (only for a variant of DPO rules), where sets or rules can be applied simultaneously on an input object. One important difference with Global Transformations is that overlapping rules may conflict if one rule deletes an item that another rule preserves. Only non conflicting (so called *coherent*) matchings can be applied simultaneously. The notion of Parallel Coherence from [2] is therefore adapted in order to embrace rule morphisms.

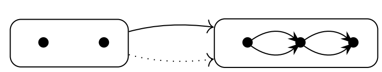
Section 5 is devoted to the analysis of Rewriting Environments, and yields natural definitions of environments for the SqPO and PBPO approaches. Future work and open questions are found in Section 6.

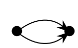
2 Notations

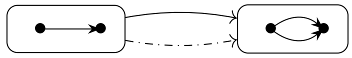
Embeddings are injective functors (as in [11]), all other notions are compatible with [15]. We also use *meets* and *sums* of functors, see [13].

For any category $\mathcal{C}$, we write $G \in \mathcal{C}$ to indicate that G is a $\mathcal{C}$-object, and $\mathcal{C}^\bullet$ is the discrete category on $\mathcal{C}$-objects. Then G also denotes the functor from the terminal category $\mathbf{1}$ to $\mathcal{C}^\bullet$ that maps the object of $\mathbf{1}$ to G. The *slice* category $\mathcal{C} \setminus G$ has as objects $\mathcal{C}$-morphisms of codomain G, and as morphisms $h : f \to g$ $\mathcal{C}$-morphisms such that $g \circ h = f$. The *coslice* category $G \setminus \mathcal{C}$ has as objects $\mathcal{C}$-morphisms of domain G, and as morphisms $h : f \to g$ $\mathcal{C}$-morphisms such that $h \circ f = g$.

We will use the standard notion of graphs with multiple directed edges. The initial graph is denoted $\varnothing$. In the running example we will use graphs with 2 to 3 vertices and 0 to 4 edges denoted directly by their drawings, as in 
and  . We will only use monomorphisms between these graphs, and only those that map the leftmost (resp. rightmost) vertex of the domain graph to the leftmost (resp. rightmost) vertex of the codomain graph, represented by plain arrows, and those that swap these vertices, represented as dotted arrows. For example we consider only two possible morphisms:



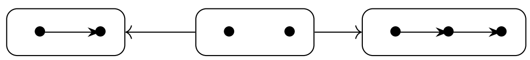
The two morphisms from to  will be distinguished similarly:



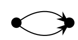
3 Subrules in DPO Graph Transformations

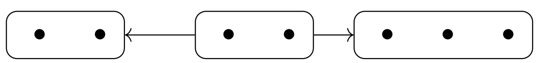
The notion of a rule ρ being a subrule of a rule ρ', or more generally of a morphism $\sigma : \rho \to \rho'$, covers the idea that ρ represents a part (specified by σ) of what ρ' achieves, and therefore that any application of ρ' entails and subsumes a particular application (obtained through σ) of ρ. In [16] the morphisms σ are pairs of unrelated graph morphisms. But in the DPO approach the left and right-hand sides of a rule are related by morphisms from an interface graph K, hence the graph morphisms in σ should also be related.

Example 3.1. In the running example we transform every directed edge in a graph into a pair of consecutive edges. This can be expressed as the following rule



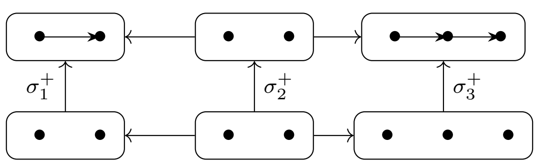
(ρ')

We do not wish to transform loops in this way, hence we adopt the DPO approach restricted to monic matches. We also wish to create only one middle vertex for parallel edges, so that the input graph $G =$  in our running example shall be transformed into $H =$  . In order to merge the two vertices created by the two simultaneous applications of ρ' on G we need to link them through the application of a common subrule on their overlap. Consider the rule



(ρ)

The right hand side expresses the fact that the middle vertex is created depending on the overlap $\bullet \quad \bullet$ and not on the edges of G. Thus we need to link the middle vertices from ρ and ρ' right-hand sides through a morphism $\sigma^+ : \rho \to \rho'$, given as three $\mathcal{C}$-morphisms:



(σ^+)

The two square diagrams commute, and we easily understand that this is necessary for ρ to be a subrule of ρ'. But commutation would also hold if the interface graph of ρ were $\varnothing$, and then ρ would remove the overlap $\bullet \quad \bullet$. This would conflict with ρ' that preserves the overlap. We need the two rules to behave similarly on the overlap, which means that the interface of the subrule ρ is determined by the way the interface of ρ' intersects the overlap. This can be expressed by stating that the left square should be a pullback.

Definition 3.2 (categories $\mathscr{R}_{\mathrm{DPO}}$, $\mathscr{R}_{\mathrm{DPOm}}$)**.** Let **sp** be the category generated by the graph

$$\mathrm{L} \xleftarrow{\ \mathrm{l}\ } \mathrm{K} \xrightarrow{\ \mathrm{r}\ } \mathrm{R}$$

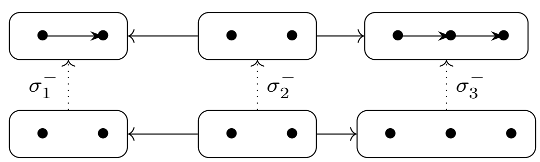
For any category $\mathcal{C}$, let $\mathscr{R}_{\mathrm{DPO}}$ be the category whose objects are the functors $\rho : \mathbf{sp} \to \mathcal{C}$ such that $\rho\mathrm{l}$ is monic, and morphisms $\sigma : \rho \to \rho'$ are triples $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of $\mathcal{C}$-morphisms such that

$$\begin{tikzcd}
\rho\mathrm{L} \arrow[d, "\sigma_1"'] & \rho\mathrm{K} \arrow[l, tail, "\rho\mathrm{l}"'] \arrow[r, "\rho\mathrm{r}"] \arrow[d, "\sigma_2"] \arrow[dl, phantom, "\llcorner", very near start] & \rho\mathrm{R} \arrow[d, "\sigma_3"] \\
\rho'\mathrm{L} & \rho'\mathrm{K} \arrow[l, tail, "\rho'\mathrm{l}"] \arrow[r, "\rho'\mathrm{r}"'] & \rho'\mathrm{R}
\end{tikzcd}$$

commutes in $\mathcal{C}$ and the left square is a pullback. The obvious composition $\sigma' \circ \sigma$ is given by $(\sigma'_1 \circ \sigma_1,\, \sigma'_2 \circ \sigma_2,\, \sigma'_3 \circ \sigma_3)$, and the obvious identity is $1_\rho = (1_{\rho\mathrm{L}},\, 1_{\rho\mathrm{K}},\, 1_{\rho\mathrm{R}})$.

Let $\mathscr{R}_{\mathrm{DPOm}}$ be the subcategory of $\mathscr{R}_{\mathrm{DPO}}$ with all rules and morphisms σ such that σ_1 and σ_2 are monics.

Example 3.3. We consider two morphisms of rules, σ^+ above and $\sigma^- : \rho \to \rho'$ that swaps the left and right vertices:



(σ^-)

There are two obvious matchings m_1 and m_2 of ρ' in G, and they induce two matchings of ρ in G, say $m^+ = m_1 \circ \sigma_1^+ = m_2 \circ \sigma_1^+$ and $m^- = m_1 \circ \sigma_1^- = m_2 \circ \sigma_1^-$.

The existence of DPO-transformations that correspond to these matchings requires the existence of pushout complements and is subject to the gluing condition (see [8, 6]). We say that an item (edge or vertex) of G is *marked for removal* by $m : \rho\mathrm{L} \to G$ if it has a preimage by the matching m that has none by $\rho\mathrm{l}$ (those are the items that have no preimage in the pushout complement, see [3]). Then the *gluing condition* states that (1) items marked for removal have only one preimage by the matching m, and (2) if a vertex adjacent to an edge is marked for removal, then so is this edge. We first see that this condition is inherited (backward) along the morphisms of $\mathscr{R}_{\mathrm{DPOm}}$.

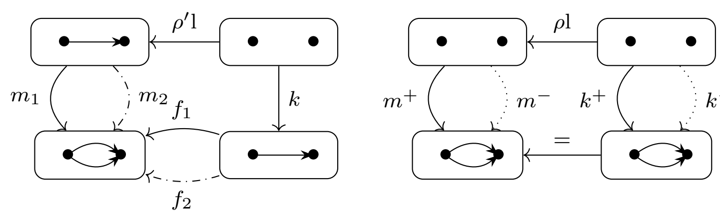
Proposition 3.4. *If $\mathcal{C}$ is the category of graphs, $\sigma : \rho \to \rho'$ is a morphism in $\mathscr{R}_{\mathrm{DPO}}$ such that σ_1 is monic and $m : \rho'\mathrm{L} \to G$ is a matching that satisfies the gluing condition then so does the matching $m \circ \sigma_1 : \rho\mathrm{L} \to G$.*

Proof. We start with condition (1). Let x, x' be items in $\rho\mathrm{L}$ such that $m \circ \sigma_1(x) = m \circ \sigma_1(x')$ and, say, x has no preimage by $\rho\mathrm{l}$. If $\sigma_1(x)$ had a preimage y by $\rho'\mathrm{l}$ then x and y would have a common preimage in the pullback $\rho\mathrm{K}$, a contradiction.

Hence $\sigma_1(x)$ has no preimage by $\rho'\mathrm{l}$ so that $m(\sigma_1(x))$ is marked for removal by m, hence $\sigma_1(x) = \sigma_1(x')$ by the gluing condition on m, hence $x = x'$.

Condition (2). Let v be a vertex that has no preimage by $\rho\mathrm{l}$ and is adjacent to an edge e in $\rho\mathrm{L}$, then as above $\sigma_1(v)$ has no preimage by $\rho'\mathrm{l}$. If e had a preimage e' by $\rho\mathrm{l}$ then $\rho'\mathrm{l} \circ \sigma_2(e') = \sigma_1 \circ \rho\mathrm{l}(e') = \sigma_1(e)$, i.e., $\sigma_1(e)$ would have a preimage by $\rho'\mathrm{l}$ in contradiction with the gluing condition on m. Hence $m\circ\sigma_1(e)$ is marked for removal by $m \circ \sigma_1$. □

Example 3.5. We see that m_1 and m_2 satisfy the gluing condition, hence they have a pushout complement by $\rho'\mathrm{l}$ and so do m^+ and m^- by $\rho\mathrm{l}$. The four pushouts $(\rho'\mathrm{l}, m_1, k, f_1)$, $(\rho'\mathrm{l}, m_2, k, f_2)$, $(\rho\mathrm{l}, m^+, k^+, 1_G)$, $(\rho\mathrm{l}, m^-, k^-, 1_G)$ are depicted below.



The following result reveals the relationship induced by morphisms $\sigma : \rho \to \rho'$ on these pushouts and hence on the corresponding direct DPO transformations.

Proposition 3.6. *If $\mathcal{C}$ is the category of graphs, $\sigma : \rho \to \rho'$ is a morphism in $\mathscr{R}_{\mathrm{DPO}}$, $m : \rho'\mathrm{L} \to G$ and $m \circ \sigma_1 : \rho\mathrm{L} \to G$ have pushout complements as below, then there is a unique graph morphism d such that*

$$\begin{tikzcd}
 & \rho\mathrm{L} \arrow[dl, "\sigma_1"'] \arrow[dd] & & \rho\mathrm{K} \arrow[ll, tail, "\rho\mathrm{l}"'] \arrow[dl, "\sigma_2"] \arrow[dd, "k"] \\
\rho'\mathrm{L} \arrow[dd, "m"'] & & \rho'\mathrm{K} \arrow[ll, tail, "\rho'\mathrm{l}"] \arrow[dd, "k'"'] & \\
 & G \arrow[dl, dash, "="'] & & D \arrow[ll, tail, "f"'] \arrow[dl, dashed, tail, "d"] \\
G & & D' \arrow[ll, tail, "f'"] &
\end{tikzcd}$$

commutes.

Proof. The front and back faces are pushouts. For all item x in D', $f'(x)$ is not marked for removal by m and we show that is also the case by $m \circ \sigma_1$. Suppose otherwise, then $f'(x)$ has a preimage y by $m \circ \sigma_1$ that has no preimage by $\rho\mathrm{l}$. However, $\sigma_1(y)$ has a preimage y' by $\rho'\mathrm{l}$, and since the top face is a pullback there should be a common preimage of y and y' in $\rho\mathrm{K}$, a contradiction. Thus we let $d(x)$ be the unique preimage of $f'(x)$ by f, so that $f \circ d = f'$, and since

f, f' are monomorphisms then so is d, and it is obviously unique. Then we see that

$$f \circ k = m \circ \sigma_1 \circ \rho \mathrm{l} = m \circ \rho' \mathrm{l} \circ \sigma_2 = f' \circ k' \circ \sigma_2 = f \circ d \circ k' \circ \sigma_2$$

hence the right face of the cube commutes. $\square$

The existence of d means that all items marked for removal by $m \circ \sigma_1$, i.e., removed by the subrule ρ, are also removed by ρ'.

Definition 3.7 (categories $\mathscr{D}_{\mathrm{DPO}}$, $\mathscr{D}_{\mathrm{DPOm}}$, functors $\mathsf{R}_{\mathrm{DPO}}$, $\mathsf{R}_{\mathrm{DPOm}}$)**.** Let $\mathbf{dt}$ be the category generated by the graph

$$\begin{tikzcd}
\mathrm{L} \arrow[d, "\mathrm{m}"'] & \mathrm{K} \arrow[l, "\mathrm{l}"'] \arrow[r, "\mathrm{r}"] \arrow[d, "\mathrm{k}"] & \mathrm{R} \arrow[d, "\mathrm{n}"] \\
\mathrm{G} & \mathrm{D} \arrow[l, "\mathrm{f}"] \arrow[r, "\mathrm{g}"'] & \mathrm{H}
\end{tikzcd}$$

with relations $\mathrm{m} \circ \mathrm{l} = \mathrm{f} \circ \mathrm{k}$ and $\mathrm{n} \circ \mathrm{r} = \mathrm{g} \circ \mathrm{k}$. A *direct DPO-transformation* in $\mathcal{C}$ is a functor $\delta : \mathbf{dt} \to \mathcal{C}$ such that $\delta \mathrm{l}$ is monic and the two squares $(\delta \mathrm{l}, \delta \mathrm{m}, \delta \mathrm{k}, \delta \mathrm{f})$, $(\delta \mathrm{r}, \delta \mathrm{n}, \delta \mathrm{k}, \delta \mathrm{g})$ are pushouts.

Let $\mathscr{D}_{\mathrm{DPO}}$ be the category whose objects are direct DPO-transformations in a category $\mathcal{C}$ and whose morphisms $\mu : \delta \to \delta'$ are 4-tuples $(\mu_1, \mu_2, \mu_3, \mu_4)$ of $\mathcal{C}$-morphisms such that μ_4 is monic, the following diagram

$$\begin{tikzcd}
 & \delta\mathrm{L} \arrow[dl, "\mu_1"'] \arrow[dd, "\delta\mathrm{m}"] & & \delta\mathrm{K} \arrow[ll, tail, "\delta\mathrm{l}"'] \arrow[rr, "\delta\mathrm{r}"] \arrow[dl, "\mu_2"] \arrow[dd, "\delta\mathrm{k}"] & & \delta\mathrm{R} \arrow[dl, "\mu_3"] \\
\delta'\mathrm{L} \arrow[dd, "\delta'\mathrm{m}"'] & & \delta'\mathrm{K} \arrow[ll, tail, "\delta'\mathrm{l}"'] \arrow[rr, "\delta'\mathrm{r}"] \arrow[dd, "\delta'\mathrm{k}"'] & & \delta'\mathrm{R} & \\
 & \delta\mathrm{G} \arrow[dl, dash, "="'] & & \delta\mathrm{D} \arrow[ll, "\delta\mathrm{f}"'] & & \\
\delta'\mathrm{G} & & \delta'\mathrm{D} \arrow[ll, "\delta'\mathrm{f}"] \arrow[ur, tail, "\mu_4"'] & & &
\end{tikzcd}$$

commutes and the top left square is a pullback, with obvious composition and identities. Let $\mathsf{R}_{\mathrm{DPO}}$ be the obvious functor from $\mathscr{D}_{\mathrm{DPO}}$ to $\mathscr{R}_{\mathrm{DPO}}$. Let $\mathscr{D}_{\mathrm{DPOm}}$ be the full subcategory of $\mathscr{D}_{\mathrm{DPO}}$ whose objects are the direct tranformations δ such that $\delta\mathrm{m}$ is monic, and let $\mathsf{R}_{\mathrm{DPOm}} : \mathscr{D}_{\mathrm{DPOm}} \to \mathscr{R}_{\mathrm{DPOm}}$ be the corresponding restriction of $\mathsf{R}_{\mathrm{DPO}}$.

4 Global Coherent Transformations

The result of a transformation will be computed exclusively from data extracted from direct transformations as follows.

Definition 4.1 (category $\mathcal{C}_{\mathrm{pt}}$, functors In, $\mathsf{P}_{\mathrm{DPOm}}$)**.** Let $\mathbf{pt}$ be the category generated by the graph

$$\begin{tikzcd} \mathrm{G} & \mathrm{D} \arrow[l, "\mathrm{f}"'] & \mathrm{K} \arrow[l, "\mathrm{k}"'] \arrow[r, "\mathrm{r}"] & \mathrm{R} \end{tikzcd}$$

For any category $\mathcal{C}$, the category $\mathcal{C}_{\mathrm{pt}}$ of *partial transformations* has as objects functors $\tau : \mathbf{pt} \to \mathcal{C}$, and as morphisms $\nu : \tau \to \tau'$ triples (ν_1, ν_2, ν_3) such that

$$\begin{tikzcd} \tau\mathrm{G} \arrow[d, dash, "="'] & \tau\mathrm{D} \arrow[l, "\tau\mathrm{f}"'] & \tau\mathrm{K} \arrow[l, "\tau\mathrm{k}"'] \arrow[r, "\tau\mathrm{r}"] \arrow[d, "\nu_2"'] & \tau\mathrm{R} \arrow[d, "\nu_3"] \\ \tau'\mathrm{G} & \tau'\mathrm{D} \arrow[l, "\tau'\mathrm{f}"] \arrow[u, "\nu_1"'] & \tau'\mathrm{K} \arrow[l, "\tau'\mathrm{k}"] \arrow[r, "\tau'\mathrm{r}"'] & \tau'\mathrm{R} \end{tikzcd}$$

commutes in $\mathcal{C}$, with obvious composition and identities.

Let $\mathsf{In} : \mathcal{C}_{\mathrm{pt}} \to \mathcal{C}^{\bullet}$ be the *input functor* defined as $\mathsf{In}\tau = \tau\mathrm{G}$ and $\mathsf{In}\nu = 1_{\tau\mathrm{G}}$. Let $\mathsf{P}_{\mathrm{DPOm}} : \mathscr{D}_{\mathrm{DPOm}} \to \mathcal{C}_{\mathrm{pt}}$ be the obvious functor.

Using inverse images of the functors $\mathsf{P}_{\mathrm{DPOm}}$ and $\mathsf{R}_{\mathrm{DPOm}}$ we can easily focus on the direct transformations of concern (and the morphisms between them), i.e., the transformations *of* a graph *by* a rule system.

Definition 4.2 (Rewriting Environments, rule systems, notations D_δ, $\pi_1\mu \ldots$)**.** For any category $\mathcal{C}$, a *Rewriting Environment* for $\mathcal{C}$ consists of a category $\mathcal{D}$ of *direct transformations*, a category $\mathcal{R}$ of *rules* and two functors

$$\begin{tikzcd} \mathcal{R} & \mathcal{D} \arrow[l, "\mathsf{R}"'] \arrow[r, "\mathsf{P}"] & \mathcal{C}_{\mathrm{pt}} \end{tikzcd}$$

A *rule system* in a Rewriting Environment is a category $\mathcal{S}$ with an embedding $\mathsf{I} : \mathcal{S} \to \mathcal{R}$ (alternately, $\mathcal{S}$ is a subcategory of $\mathcal{R}$ and I is the inclusion functor).

Given a rule system and an *input* object G, we build the categories $\mathcal{D}|_G$, $\mathcal{D}|_G^{\mathcal{S}}$ and functors I_G, $\mathsf{I}_{\mathcal{S}}$, $\mathsf{R}|_G^{\mathcal{S}}$ as meets of previous functors:

$$\begin{tikzcd} \mathcal{S} \arrow[r, tail, "\mathsf{I}"] & \mathcal{R} & & \\ & & & \\ & \mathcal{D} \arrow[uu, "\mathsf{R}"'] \arrow[r, "\mathsf{P}"] & \mathcal{C}_{\mathrm{pt}} \arrow[r, "\mathsf{In}"] & \mathcal{C}^{\bullet} \\ & & & \\ \mathcal{D}|_G^{\mathcal{S}} \arrow[uuuu, "\mathsf{R}|_G^{\mathcal{S}}"] \arrow[r, tail, "\mathsf{I}_{\mathcal{S}}"'] \arrow[ur, phantom, "\urcorner"] & \mathcal{D}|_G \arrow[uu, tail, "\mathsf{I}_G"] \arrow[rr] \arrow[ur, phantom, "\urcorner"] & & \mathbf{1} \arrow[uu, tail, "G"'] \end{tikzcd}$$

For any $\delta \in \mathcal{D}|_G^{\mathcal{S}}$ we write D_δ for $(\mathsf{P}\mathsf{I}_G\mathsf{I}_{\mathcal{S}}\delta)\mathrm{D}$ and similarly f_δ etc. For any $\mu : \delta \to \delta'$ in $\mathcal{D}|_G^{\mathcal{S}}$ we write $\pi_1\mu$ for the first coordinate of $\mathsf{P}\mathsf{I}_G\mathsf{I}_{\mathcal{S}}\mu$ and similarly $\pi_2\mu$, $\pi_3\mu$.

Example 4.3. For $\mathcal{S}$ we take the subcategory $\rho \overset{\sigma^+}{\underset{\sigma^-}{\rightrightarrows}} \rho'$ of $\mathscr{R}_{\mathrm{DPO}}$. To the matchings m_1 and m_2 of ρ' in G correspond two[1] transformations in $\mathscr{D}_{\mathrm{DPOm}}$ denoted δ_1 and δ_2 (their left pushouts are depicted on the left of Example 3.5). To the matchings m^+ and m^- of ρ in G correspond another two transformations denoted δ^+ and δ^- (on the right of Example 3.5). To each $i = 1, 2$ correspond one morphism $\mu_i^+ : \delta^+ \to \delta_i$ such that $\mathsf{R}_{\mathrm{DPOm}}\mu_i^+ = \sigma^+$ and one morphism $\mu_i^- : \delta^- \to \delta_i$ such that $\mathsf{R}_{\mathrm{DPOm}}\mu_i^- = \sigma^-$.

Thus $\mathscr{D}_{\mathrm{DPOm}}|_G^{\mathcal{S}}$ is the subcategory

$$\begin{tikzcd} & \delta_1 & \\ \delta^+ \arrow[ur, "\mu_1^+"] \arrow[dr, "\mu_2^+"'] & & \delta^- \arrow[ul, "\mu_1^-"'] \arrow[dl, "\mu_2^-"] \\ & \delta_2 & \end{tikzcd}$$

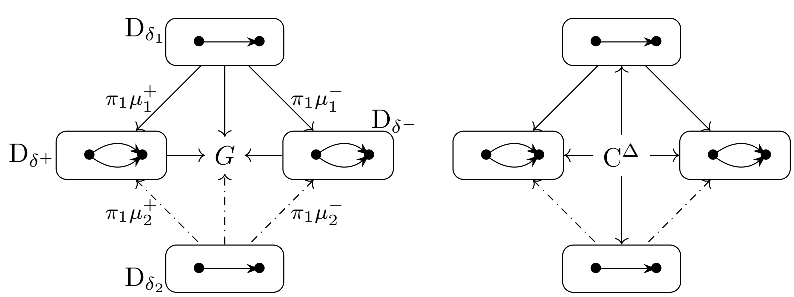
of $\mathscr{D}_{\mathrm{DPOm}}$.

The transformation will use the data accessible through $\mathsf{P} \circ \mathsf{I}_G \circ \mathsf{I}_{\mathcal{S}}$. We first need to build an interface between the input G and the expected output H. In Parallel Coherent Transformation [2] the interface is obtained as a limit of the morphisms $\mathrm{f}_\delta : \mathrm{D}_\delta \to G$ for all δ in a set Δ of direct transformations, hence of a diagram that is a sink to G and thus corresponds to a discrete diagram in $\mathcal{C} \setminus G$. In Global Coherent Transformations the interface (denoted C^Δ below) is obtained similarly, but now Δ is a category and the diagram contains the morphisms $\pi_1\mu : \mathrm{D}_{\delta'} \to \mathrm{D}_\delta$ for all $\mu : \delta \to \delta'$ in Δ.

Definition 4.4 (functor $\mathsf{P}_{\mathrm{G}}^{\Delta}$, limit $f^\Delta : \mathrm{C}^\Delta \to G$, cone γ^Δ)**.** For any subcategory Δ of $\mathcal{D}|_G^{\mathcal{S}}$ let $\mathsf{P}_{\mathrm{G}}^{\Delta} : \Delta^{\mathrm{op}} \to \mathcal{C} \backslash G$ be the contravariant functor that maps every $\delta \in \Delta$ to $\mathrm{f}_\delta : \mathrm{D}_\delta \to G$ and every morphism μ of Δ to $\pi_1\mu$. Let $f^\Delta : \mathrm{C}^\Delta \to G$ be the limit of $\mathsf{P}_{\mathrm{G}}^{\Delta}$ and γ^Δ be the limit cone from f^Δ to $\mathsf{P}_{\mathrm{G}}^{\Delta}$.

Note that if Δ is empty then the limit f^Δ of the empty diagram is the terminal object of $\mathcal{C} \setminus G$, that is 1_G, hence $\mathrm{C}^\Delta = G$.

Example 4.5. Let $\Delta = \mathscr{D}_{\mathrm{DPOm}}|_G^{\mathcal{S}}$. The diagram on the left below corresponds to the functor $\mathsf{P}_{\mathrm{G}}^{\Delta}$ together with the morphisms $\mathrm{f}_{\delta_i^\pm} : \mathrm{D}_{\delta_i^\pm} \to G$ (objects in $\mathcal{C} \setminus G$). The limit of this diagram yields $\mathrm{C}^\Delta = \bullet \quad \bullet$ and the limit cone is represented on the right.



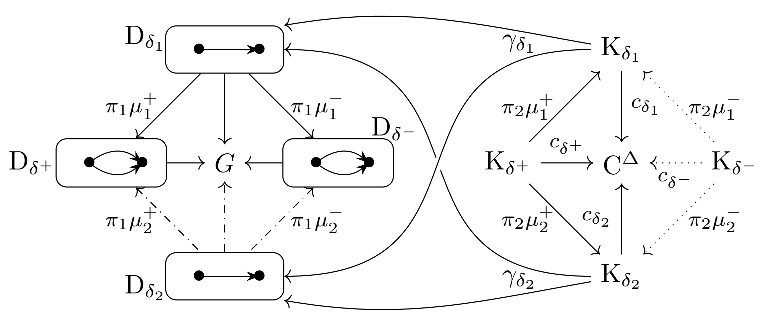
[1] We consider transformations only up to isomorphisms, see Footnote 2.

We next need to check that the transformations in Δ do not conflict with each other, i.e., that for all $\delta \in \Delta$ the image of K_δ in G is not only preserved by δ (in D_δ) but also by all other transformations $\delta' \in \Delta$. This is ensured by finding (natural) cones from these K_δ to the $\mathrm{D}_{\delta'}$, which we shall formulate with $\mathsf{P}^{\Delta}_{\mathrm{G}}$, hence in $\mathcal{C} \setminus G$.

Definition 4.6 (system of cones, morphisms c_δ, global coherence)**.** A *system of cones for* Δ is a set of cones γ_δ from $\mathrm{f}_\delta \circ \mathrm{k}_\delta$ to $\mathsf{P}^{\Delta}_{\mathrm{G}}$ such that $\gamma_\delta \delta = \mathrm{k}_\delta$ for all $\delta \in \Delta$, and $\gamma_\delta = \gamma_{\delta'} \circ \pi_2 \mu$ for all $\mu : \delta \to \delta'$ in Δ. We then let $c_\delta : \mathrm{f}_\delta \circ \mathrm{k}_\delta \to f^{\Delta}$ be the unique morphism in $\mathcal{C} \setminus G$ such that $\gamma_\delta = \gamma^{\Delta} \circ c_\delta$. Δ is *globally coherent* if there exists a system of cones for Δ.

Note that if $\gamma_{\delta'}$ is a cone from $\mathrm{f}_{\delta'} \circ \mathrm{k}_{\delta'}$ to $\mathsf{P}^{\Delta}_{\mathrm{G}}$ then $\gamma_{\delta'} \circ \pi_2 \mu$ is a cone from $\mathrm{f}_\delta \circ \mathrm{k}_\delta$ to $\mathsf{P}^{\Delta}_{\mathrm{G}}$, hence global coherence means that we should find cones for the direct transformations (say δ_1 and δ_2) from the top rules, with the constraint that they should be compatible on common subtransformations $\delta_1 \leftarrow \delta \to \delta_2$. If $\mathcal{S}$ and therefore Δ are discrete, this amounts to parallel coherence, see [2].

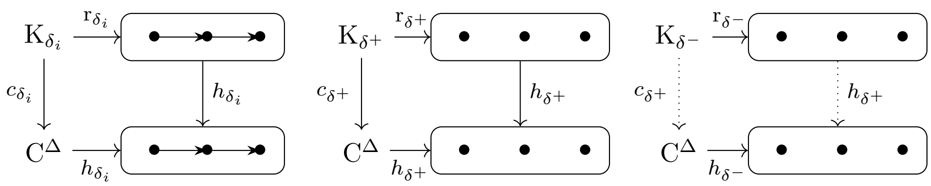
Example 4.7. On our example the four graphs $\mathrm{K}_{\delta_i^\pm}$ are equal to $\bullet \quad \bullet$. It is easy to build the four cones from the four morphisms from K_{δ_i} to D_{δ_i} depicted below, by composing them with the $\pi_1 \mu_i^\pm$ on the left and the $\pi_2 \mu_i^\pm$ on the right. On the right are also depicted the morphisms $c_{\delta_i^\pm}$.



Definition 4.8 (morphisms $h_\delta : \mathrm{C}^\Delta \to H_\delta$)**.** If Δ is globally coherent for all $\delta \in \Delta$ then c_δ can be viewed as a $\mathcal{C}$-morphism $c_\delta : \mathrm{K}_\delta \to \mathrm{C}^\Delta$, and we consider the following pushout in $\mathcal{C}$.

$$\begin{tikzcd} \mathrm{K}_\delta \arrow[r, "\mathrm{r}_\delta"] \arrow[d, "c_\delta"'] & \mathrm{R}_\delta \arrow[d, "n_\delta"] \\ \mathrm{C}^\Delta \arrow[r, "h_\delta"'] & H_\delta \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Example 4.9. On our example we get:



We now turn h into a functor.

Proposition 4.10. *For every $\mu : \delta \to \delta'$ in Δ there exists a unique h_μ such that*

$$\begin{tikzcd}
 & H_\delta \arrow[d, dashed, "h_\mu"] & R_\delta \arrow[l, "n_\delta"'] \arrow[d, "\pi_3\mu"] \\
C^\Delta \arrow[ur, "h_\delta"] \arrow[r, "h_{\delta'}"'] & H_{\delta'} & R_{\delta'} \arrow[l, "n_{\delta'}"]
\end{tikzcd}$$

commutes.

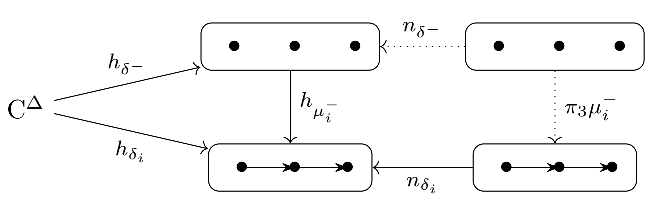
Proof. Since $\gamma^\Delta \circ c_\delta = \gamma_\delta = \gamma_{\delta'} \circ \pi_2\mu = \gamma^\Delta \circ c_{\delta'} \circ \pi_2\mu$ then by the unicity of c_δ the left face of the following cube commutes.

$$\begin{tikzcd}
 & K_\delta \arrow[rr, "r_\delta"] \arrow[dl, "\pi_2\mu"'] \arrow[dd, "c_\delta" near start] & & R_\delta \arrow[dl, "\pi_3\mu"'] \arrow[dd, "n_\delta"] \\
K_{\delta'} \arrow[rr, "r_{\delta'}" near start, crossing over] \arrow[dd, "c_{\delta'}"'] & & R_{\delta'} & \\
 & C^\Delta \arrow[rr, "h_\delta" near start] \arrow[dl, equal] & & H_\delta \arrow[dl, dashed, "h_\mu"] \\
C^\Delta \arrow[rr, "h_{\delta'}"'] & & H_{\delta'} \arrow[from=uu, "n_{\delta'}" near end, crossing over] &
\end{tikzcd}$$

Since the top and front faces also commute then $n_{\delta'} \circ \pi_3\mu \circ r_\delta = h_{\delta'} \circ c_\delta$, and since the back face is a pushout we get the result. $\square$

Corollary 4.11. *By unicity we get $h_{\mu' \circ \mu} = h_{\mu'} \circ h_\mu$.*

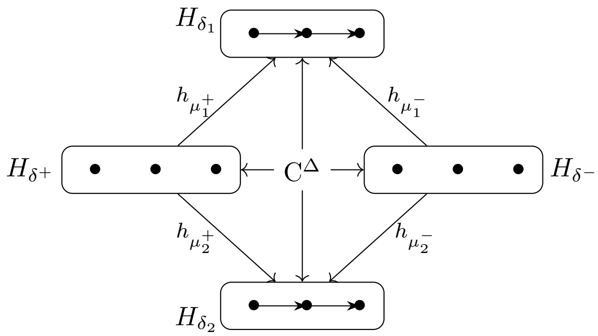
Example 4.12. For instance the morphisms $\mu_i^- : \delta^- \to \delta_i$ yield the morphisms $h_{\mu_i^-}$ depicted below.

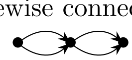


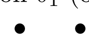
Definition 4.13 (functor $\mathsf{P}_{\mathrm{H}}^{\Delta}$, colimit $h^{\Delta} : \mathrm{C}^{\Delta} \to H^{\Delta}$)**.** If Δ is globally coherent let $\mathsf{P}_{\mathrm{H}}^{\Delta} : \Delta \to \mathrm{C}^{\Delta} \setminus \mathcal{C}$ be the functor defined by $\mathsf{P}_{\mathrm{H}}^{\Delta}\delta = h_{\delta}$ (interpreted as an object of $\mathrm{C}^{\Delta} \setminus \mathcal{C}$) and $\mathsf{P}_{\mathrm{H}}^{\Delta}\mu = h_{\mu}$ for all $\mu : \delta \to \delta'$ in Δ. Let $h^{\Delta} : \mathrm{C}^{\Delta} \to H^{\Delta}$ be the colimit[2] of $\mathsf{P}_{\mathrm{H}}^{\Delta}$, then the $\mathcal{C}$-span $G \xleftarrow{f^{\Delta}} \mathrm{C}^{\Delta} \xrightarrow{h^{\Delta}} H^{\Delta}$ is a *Global Coherent Transformation by* Δ.

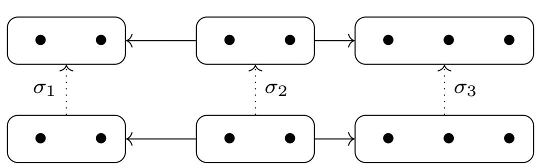
If Δ is empty then the colimit h^{Δ} of the empty diagram is the initial object of $\mathrm{C}^{\Delta} \setminus \mathcal{C}$, that is $1_{\mathrm{C}^{\Delta}}$, hence $H^{\Delta} = \mathrm{C}^{\Delta} = G$. Generally, the functor $\mathsf{P}_{\mathrm{H}}^{\Delta}$ depends on the choice of cones γ_{δ} for $\delta \in \Delta$, hence h^{Δ} is not determined by Δ.

Example 4.14. The functor $\mathsf{P}_{\mathrm{H}}^{\Delta}$ applied to Δ yields the following diagram



The leftmost vertices of these five graphs are connected as images or preimages of each other, and similarly for the five right vertices, and the four middle vertices. The four edges are not likewise connected, hence the colimit of this diagram is the expected result $H =$  . We therefore see that the two middle vertices created in δ_1 and δ_2 are merged by their common subtransformation δ^+ (or δ^-), but also that the two middle vertices created in δ^+ and δ^- are merged by their common upper transformation δ_1 (or δ_2).

If we apply $\mathcal{S}$ to the graph $G' =$  then rule ρ' does not apply to G' and hence the two matchings of ρ in G' apply independently, thus adding two vertices to G'. We can merge them by adding to $\mathcal{S}$ the following rule morphism $\sigma : \rho \to \rho$ that swaps the left and right vertices:



We have $\sigma^2 = 1_{\rho}$ hence σ is an automorphism of ρ. Adding σ to $\mathcal{S}$ means that ρ will be applied modulo automorphisms; this generalizes to the algebraic

[2] Global Coherent Transformations are obtained as limits and colimits of diagrams whose index category is Δ, hence are not affected by isomorphisms in Δ, which can therefore be replaced by its skeleton.

context the notion of Parallel Rewriting Modulo Automorphism devised in an algorithmic approach in [1].

Since $\sigma^+ \circ \sigma = \sigma^-$ and $\sigma^- \circ \sigma = \sigma^+$, the new rule system is

$$\mathcal{S}' = \begin{tikzcd} \rho \arrow[loop left, "\sigma"] \arrow[r, bend left, "\sigma^+"] \arrow[r, bend right, "\sigma^-"'] & \rho' \end{tikzcd}$$

If we apply $\mathcal{S}'$ to G, we add two new morphisms in $\mathscr{D}_{\mathrm{DPOm}}|_G^{\mathcal{S}}$, i.e,

$$\Delta' = \mathscr{D}_{\mathrm{DPOm}}|_G^{\mathcal{S}'} = \begin{tikzcd} & \delta_1 & \\ \delta^+ \arrow[ur, "\mu_1^+"] \arrow[dr, "\mu_2^+"'] \arrow[rr, bend left] & & \delta^- \arrow[ul, "\mu_1^-"'] \arrow[dl, "\mu_2^-"] \arrow[ll, bend left] \\ & \delta_2 & \end{tikzcd}$$

It is easy to see that the Global Coherent Transformation by Δ' is the same as above with Δ.

We finally prove that, appart from this mechanism of sharing common transformations, isolated transformations always subsume their subtransformations, so that morphisms in $\mathcal{R}$ are rule subsumptions as intended.

Proposition 4.15. *If Δ is restricted to $\mu : \delta \to \delta'$ and Δ' to δ' then Δ and Δ' are globally coherent and $H^\Delta \simeq H^{\Delta'}$.*

Proof. $\mathsf{P}_{\mathrm{G}}^{\Delta}$ yields the left diagram below, whose limit f^Δ is $\mathrm{f}_{\delta'}$, clearly the same as $f^{\Delta'}$, so that $\mathrm{C}^\Delta \simeq \mathrm{C}^{\Delta'} \simeq \mathrm{D}_{\delta'}$. There is obviously a unique system of cones for Δ'. There is an obvious system of cones for Δ, with $\gamma_{\delta'}$ from $\mathrm{f}_{\delta'} \circ \mathrm{k}_{\delta'}$ to $\mathsf{P}_{\mathrm{G}}^{\Delta}$ (middle diagram) such that $\gamma_{\delta'}\delta = \pi_1\mu \circ \mathrm{k}_{\delta'}$ (by naturality of $\gamma_{\delta'}$), and $\gamma_\delta = \gamma_{\delta'} \circ \pi_2\mu$ is a suitable cone from $\mathrm{f}_\delta \circ \mathrm{k}_\delta$ to $\mathsf{P}_{\mathrm{G}}^{\Delta}$ (since $\pi_1\mu \circ \mathrm{k}_{\delta'} \circ \pi_2\mu = \mathrm{k}_\delta$), hence this system of cones is unique and Δ, Δ' are globally coherent. $\mathsf{P}_{\mathrm{H}}^{\Delta}$ yields the diagram on the right, whose colimit is $h_{\delta'}$, hence $H^\Delta \simeq H_{\delta'} \simeq H^{\Delta'}$. Note that this is the pushout of $\mathrm{r}_{\delta'}$ and $c_{\delta'} = \mathrm{k}_{\delta'}$.

$$\begin{tikzcd} & \mathrm{D}_\delta \arrow[dl, "\mathrm{f}_\delta"'] \\ G & \\ & \mathrm{D}_{\delta'} \arrow[ul, "\mathrm{f}_{\delta'}"] \arrow[uu, "\pi_1\mu"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{D}_\delta \arrow[dl, "\mathrm{f}_\delta"'] & \mathrm{K}_\delta \arrow[l, "\mathrm{k}_\delta"'] \arrow[dd, "\pi_2\mu"] \\ G & & \\ & \mathrm{D}_{\delta'} \arrow[ul, "\mathrm{f}_{\delta'}"] \arrow[uu, "\pi_1\mu"'] & \mathrm{K}_{\delta'} \arrow[l, "\mathrm{k}_{\delta'}"] \end{tikzcd} \qquad \begin{tikzcd} & H_\delta \arrow[dd, "h_\mu"] \\ \mathrm{C}^\Delta \arrow[ur, "h_\delta"] \arrow[dr, "h_{\delta'}"'] & \\ & H_{\delta'} \end{tikzcd}$$

$\square$

5 Rewriting Environments and Their Properties

Definitions 3.2, 3.7 and 4.1 provide two Rewriting Environments that we may call DPO and DPOm. By Proposition 3.6 it is obvious that $\mathsf{R}_{\mathrm{DPO}}$ and $\mathsf{R}_{\mathrm{DPOm}}$ are faithful when $\mathcal{C}$ is the category of graphs. This is easily seen to generalize to all adhesive categories [12]. In fact, we can generalize Proposition 3.6 as follows:

Proposition 5.1. *If $\mathcal{C}$ is adhesive, $\delta, \delta' \in \mathscr{D}_{\mathrm{DPO}}$ and $\sigma : \mathsf{R}_{\mathrm{DPO}}\delta \to \mathsf{R}_{\mathrm{DPO}}\delta'$ such that $\delta\mathrm{m} = \delta'\mathrm{m} \circ \sigma_1$ then there exists a unique $\mu : \delta \to \delta'$ such that $\mathsf{R}_{\mathrm{DPO}}\mu = \sigma$.*

Proof. Let $G = \mathsf{InP}\delta = \mathsf{InP}\delta'$, we consider the following diagram

$$\begin{tikzcd}
 & \delta\mathrm{L} \arrow[dl, "\sigma_1"'] \arrow[dd, "\delta\mathrm{m}"] & & \delta\mathrm{K} \arrow[ll, tail, "\delta\mathrm{l}"'] \arrow[dl, "\sigma_2"] \arrow[dd, "\delta\mathrm{k}"] \arrow[dddr, dashed, bend left, "z"] & \\
\delta'\mathrm{L} \arrow[dd, "\delta'\mathrm{m}"'] & & \delta'\mathrm{K} \arrow[ll, tail, "\delta'\mathrm{l}"'] \arrow[dd, "\delta'\mathrm{k}"] & & \\
 & \delta\mathrm{G} \arrow[dl, equal] & & \delta\mathrm{D} \arrow[ll, tail, "\delta\mathrm{f}"'] & \\
\delta'\mathrm{G} & & \delta'\mathrm{D} \arrow[ll, tail, "\delta'\mathrm{f}"] & & P \arrow[ll, tail, "x"] \arrow[ul, tail, "y"]
\end{tikzcd}$$

where the bottom face is a pullback. By [12, Lemma 4.2] monics are stable under pushouts hence $\delta\mathrm{f}$ and $\delta'\mathrm{f}$ are monics and therefore also x and y. By the commuting properties we have $\delta\mathrm{f} \circ \delta\mathrm{k} = \delta'\mathrm{f} \circ \delta'\mathrm{k} \circ \sigma_2$, hence there exists a unique z such that $y \circ z = \delta\mathrm{k}$ and $x \circ z = \delta'\mathrm{k} \circ \sigma_2$.

The front face is a pushout along the monic $\delta'\mathrm{l}$, hence it is a pullback [12, Lemma 4.3], as is the top face, hence by composition the square formed by $\delta\mathrm{l}$, $\delta\mathrm{m}$, $\delta'\mathrm{f}$, $\delta'\mathrm{k} \circ \sigma_2$ is also a pullback.

The back face is a pushout along the monic $\delta\mathrm{l}$, hence it is a VK-square and bottom face of the commuting cube below

$$\begin{tikzcd}
 & \delta\mathrm{K} \arrow[rr, "z"] \arrow[dl, "1"'] \arrow[dd, "1"] & & P \arrow[dl, tail, "x"'] \arrow[dd, tail, "y"] \\
\delta\mathrm{K} \arrow[rr, "\delta'\mathrm{k} \circ \sigma_2"] \arrow[dd, tail, "\delta\mathrm{l}"'] & & \delta'\mathrm{D} \arrow[dd, tail, "\delta'\mathrm{f}"] & \\
 & \delta\mathrm{K} \arrow[rr, "\delta\mathrm{k}"] \arrow[dl, tail, "\delta\mathrm{l}"'] & & \delta\mathrm{D} \arrow[dl, tail, "\delta\mathrm{f}"] \\
\delta\mathrm{L} \arrow[rr, "\delta\mathrm{m}"'] & & G &
\end{tikzcd}$$

Its front and right faces are pullbacks. Since $\delta\mathrm{l}$ is monic then its left face is a pullback, and since y is monic its back face is also a pullback. Hence its top face is a pushout, and since isomorphisms are preserved by pushouts, x is an isomorphism.

Let $d = y \circ x^{-1}$, we see that $\delta\mathrm{f} \circ d = \delta'\mathrm{f}$ and $d \circ \delta'\mathrm{k} \circ \sigma_2 = y \circ z = \delta\mathrm{k}$, so that $\mu = (\sigma_1, \sigma_2, \sigma_3, d)$ is a morphism from δ to δ' in $\mathscr{D}_{\mathrm{DPO}}$ such that $\mathsf{R}_{\mathrm{DPO}}\mu = \sigma$. Its unicity is obvious. $\square$

A property that one might reasonably expect is that when a rule applies and yields a direct transformation then its subrules also apply and yield subtransformations. We express this by means of the following notion.

Definition 5.2 (left-full)**.** A functor $\mathsf{F} : \mathcal{A} \to \mathcal{B}$ is *left-full* if for all $a' \in \mathcal{A}$, all $b \in \mathcal{B}$ and all $g : b \to Fa'$, there exist $a \in \mathcal{A}$ and $f : a \to a'$ such that $Ff = g$.

It is obvious that left-fullness is closed by composition.

Lemma 5.3. I_G *is a left-full embedding.*

Proof. The functor $G : \mathbf{1} \to \mathcal{C}$ is an embedding hence so is I_G. For all $\delta' \in \mathcal{D}|_G$, $\delta \in \mathcal{D}$ and $\mu : \delta \to \mathsf{I}_G\delta'$ we have $\mathsf{InP}\delta = \mathsf{I}_G\mathsf{InP}\delta' = G$ hence $\mathsf{InP}\mu = 1_G$. Since G and 1_G also have preimages by functor G there must be preimages $\delta_1 \in \mathcal{D}|_G$ and $\mu_1 : \delta_1 \to \delta'$ in $\mathcal{D}|_G$ such that $\mathsf{I}_G\mu_1 = \mu$, hence I_G is left-full. $\square$

Proposition 5.4. *If* R *is left-full (resp. faithful) then so is* $\mathsf{R}|_G^{\mathcal{S}}$ *for every rule system* $\mathcal{S}$ *and every* $G \in \mathcal{C}$*.*

Proof. For all $\delta' \in \mathcal{D}|_G^{\mathcal{S}}$, $\rho \in \mathcal{S}$ and $\sigma : \rho \to \rho'$, where $\rho' = \mathsf{R}|_G^{\mathcal{S}}\delta'$, we have $\mathsf{I}\rho' = \mathsf{R}\mathsf{I}_G\mathsf{I}_{\mathcal{S}}\delta'$ and $\mathsf{I}\sigma : \mathsf{I}\rho \to \mathsf{I}\rho'$ in $\mathcal{R}$, and since by Lemma 5.3 $\mathsf{R} \circ \mathsf{I}_G$ is left-full then there exists $\delta_1 \in \mathcal{D}|_G$ and $\mu_1 : \delta_1 \to \mathsf{I}_{\mathcal{S}}\delta'$ such that $\mathsf{R}\mathsf{I}_G\mu_1 = \mathsf{I}\sigma$. Thus $\mathsf{I}\rho$ and $\mathsf{I}\sigma$ have preimages by I and $\mathsf{R} \circ \mathsf{I}_G$, hence they must have preimages $\delta \in \mathcal{D}|_G^{\mathcal{S}}$ and $\mu : \delta \to \delta'$ such that $\mathsf{I}_G\mu = \mu_1$ and $\mathsf{R}|_G^{\mathcal{S}}\mu = \sigma$.

If R is faithful, since I_G is faithful then so is $\mathsf{R} \circ \mathsf{I}_G$, and hence so is $\mathsf{R}|_G^{\mathcal{S}}$. $\square$

Hence when R is left-full and faithful every morphism $\sigma : \rho \to \rho'$ in $\mathcal{S}$ is reflected by a morphism in $\mathcal{D}|_G^{\mathcal{S}}$ whenever ρ' is reflected by a direct transformation δ' (i.e., whenever ρ' applies to G), and this morphism is uniquely determined by σ and δ'. According to Proposition 3.4 it is obvious that $\mathsf{R}_{\mathrm{DPOm}}$ is left-full (when $\mathcal{C}$ is the category of graphs). It is easy to see that $\mathsf{R}_{\mathrm{DPO}}$ is not left-full (since σ_1 may not be monic).

We next consider the case of Sesqui-Pushouts [5]. It is based on the notion of final pullback complement that allows not only to remove parts of the input G but also to make copies of parts of G (when $\rho\mathrm{l}$ below is not monic).

Definition 5.5. For any category $\mathcal{C}$, let $\mathscr{R}_{\mathrm{SqPO}}$ be the category whose objects are the functors $\rho : \mathbf{sp} \to \mathcal{C}$ and morphisms $\sigma : \rho \to \rho'$ are triples $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of $\mathcal{C}$-morphisms such that

$$\begin{tikzcd}
\rho\mathrm{L} \arrow[d, "\sigma_1"'] & \rho\mathrm{K} \arrow[l, "\rho\mathrm{l}"'] \arrow[r, "\rho\mathrm{r}"] \arrow[d, "\sigma_2"] \arrow[dl, phantom, very near start, "\llcorner"] & \rho\mathrm{R} \arrow[d, "\sigma_3"] \\
\rho'\mathrm{L} & \rho'\mathrm{K} \arrow[l, "\rho'\mathrm{l}"] \arrow[r, "\rho'\mathrm{r}"'] & \rho'\mathrm{R}
\end{tikzcd}$$

commutes in $\mathcal{C}$ and the left square is a pullback, with obvious composition and identities. Let $\mathscr{R}_{\mathrm{SqPOm}}$ be the subcategory with morphisms σ such that σ_1 and σ_2 are monics.

A *direct SqPO-transformation* in $\mathcal{C}$ is a functor $\delta : \mathbf{dt} \to \mathcal{C}$ such that $\delta\mathrm{f}$,$\delta\mathrm{k}$ is a final pullback complement of $\delta\mathrm{m}$, $\delta\mathrm{l}$, and $(\delta\mathrm{r}, \delta\mathrm{n}, \delta\mathrm{k}, \delta\mathrm{g})$ is a pushout.

Proposition 5.6. *For every direct SqPO-transformations δ, δ' with corresponding SqPO-rules ρ, ρ', every $\sigma : \rho \to \rho'$ in $\mathscr{R}_{\mathrm{SqPO}}$ such that $\delta\mathrm{m} = \delta'\mathrm{m} \circ \sigma_1$, there exists a unique $\mathcal{C}$-morphism d such that*

$$\begin{tikzcd}
 & \delta\mathrm{L} \arrow[dl, "\sigma_1"'] \arrow[dd, "\delta\mathrm{m}"] & & \delta\mathrm{K} \arrow[ll, "\delta\mathrm{l}"'] \arrow[dl, "\sigma_2"] \arrow[dd, "\delta\mathrm{k}"] \\
\delta'\mathrm{L} \arrow[dd, "\delta'\mathrm{m}"'] & & \delta'\mathrm{K} \arrow[ll, "\delta'\mathrm{l}"'] \arrow[dd, "\delta'\mathrm{k}"'] & \\
 & \delta\mathrm{G} \arrow[dl, equal] & & \delta\mathrm{D} \arrow[ll, "\delta\mathrm{f}"'] \\
\delta'\mathrm{G} & & \delta'\mathrm{D} \arrow[ll, "\delta'\mathrm{f}"] \arrow[ur, dashed, "d"'] &
\end{tikzcd}$$

commutes.

Here the existence of d means not only that ρ removes at least as much as its subrule ρ', but also that it makes at least as many copies of the items of G. It is then easy to define the category $\mathscr{D}_{\mathrm{SqPO}}$ of direct SqPO-transformations, the category $\mathscr{D}_{\mathrm{SqPOm}}$ of direct SqPO-transformations with monic matches and faithful functors $\mathsf{R}_{\mathrm{SqPO}} : \mathscr{D}_{\mathrm{SqPO}} \to \mathscr{R}_{\mathrm{SqPO}}$ and $\mathsf{R}_{\mathrm{SqPOm}} : \mathscr{D}_{\mathrm{SqPOm}} \to \mathscr{R}_{\mathrm{SqPOm}}$, as in Definition 3.7. We leave this to the reader. We then see that

Proposition 5.7. *In the category of graphs $\mathsf{R}_{\mathrm{SqPOm}}$ is left-full.*

Proof. For all $\delta' \in \mathscr{D}_{\mathrm{SqPOm}}$ and $\sigma : \rho \to \mathsf{R}_{\mathrm{SqPOm}}\delta'$ in $\mathscr{R}_{\mathrm{SqPOm}}$, the matching $\delta'\mathrm{m} \circ \sigma_1 : \rho\mathrm{L} \to \delta'\mathrm{G}$ is monic hence by [5, Construction 6] $\delta'\mathrm{m} \circ \sigma_1, \rho\mathrm{l}$ has a final pullback complement, hence there is a $\delta \in \mathscr{D}_{\mathrm{SqPOm}}$ such that $\delta\mathrm{m} = \delta'\mathrm{m} \circ \sigma_1$ and $\mathsf{R}_{\mathrm{SqPOm}}\delta = \rho$, and by Proposition 5.6 there is a (unique) $\mu : \delta \to \delta'$ in $\mathscr{D}_{\mathrm{SqPOm}}$ such that $\mathsf{R}_{\mathrm{SqPOm}}\mu = \sigma$. □

We now consider the case of PBPO-rules [4], that also enables copies of parts of G but with better control of the way they are linked together and to the rest of G. The drawback is that matchings of the left-hand side of a rule into G should be completed with a co-match form G to a given type of the left-hand side.

Definition 5.8 (category $\mathscr{D}_{\mathrm{PBPO}}$, direct PBPO-transformations)**.** Let **pb** be the category generated by the graph

$$\begin{tikzcd}
\mathrm{L} \arrow[d, "\mathrm{t_L}"'] & \mathrm{K} \arrow[l, "\mathrm{l}"'] \arrow[r, "\mathrm{r}"] \arrow[d, "\mathrm{t_K}"] & \mathrm{R} \arrow[d, "\mathrm{t_R}"] \\
\mathrm{T_L} & \mathrm{T_K} \arrow[l, "\mathrm{u}"] \arrow[r, "\mathrm{v}"'] & \mathrm{T_R}
\end{tikzcd}$$

with relations $t_L \circ l = u \circ t_K$ and $t_R \circ r = v \circ t_K$. A *PBPO-rule* in $\mathcal{C}$ is a functor $\rho : \mathbf{pb} \to \mathcal{C}$. A morphism $\sigma : \rho \to \rho'$ is a 5-tuple $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$ of $\mathcal{C}$-morphisms such that

$$
\begin{tikzcd}
 & \rho L \arrow[dl, "\sigma_1"'] \arrow[dd, "\rho t_L"] & & \rho K \arrow[ll, "\rho l"'] \arrow[rr, "\rho r"] \arrow[dl, "\sigma_2"'] \arrow[dd, "\rho t_K"] & & \rho R \arrow[dl, "\sigma_3"] \\
\rho' L \arrow[dd, "\rho' t_L"'] & & \rho' K \arrow[ll, "\rho' l"'] \arrow[rr, "\rho' r"] \arrow[dd, "\rho' t_K"'] & & \rho' R & \\
 & \rho T_L & & \rho T_K \arrow[ll, "\rho u"] & & \\
\rho' T_L \arrow[ur, "\sigma_4"] & & \rho' T_K \arrow[ll, "\rho' u"] \arrow[ur, "\sigma_5"'] & & &
\end{tikzcd}
$$

commutes. Let $\mathscr{D}_{\mathrm{PBPO}}$ be the category of morphisms of PBPO-rules on $\mathcal{C}$, with the obvious composition and identities.

Let **pbt** be the category generated by

$$
\begin{tikzcd}
L \arrow[d, "m"] \arrow[dd, bend right, "t_L"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] \arrow[dd, bend right, "t_K"'] & R \arrow[d, "n"] \arrow[dd, bend right, "t_R"'] \\
G \arrow[d, "t_G"] & D \arrow[l, "f"] \arrow[r, "g"'] \arrow[d, "t_D"] & H \arrow[d, "t_H"] \\
T_L & T_K \arrow[l, "u"] \arrow[r, "v"'] & T_R
\end{tikzcd}
$$

with all commuting relations, a *direct PBPO-transformation* in $\mathcal{C}$ is a functor $\delta : \mathbf{pbt} \to \mathcal{C}$ such that $(\delta f, \delta t_G, \delta t_D, \delta u)$ is a pullback and $(\delta r, \delta n, \delta k, \delta g)$ is a pushout.

To every direct PBPO-transformation obviously corresponds a PBPO-rule and a partial transformation.

Proposition 5.9. *For every direct PBPO-transformations δ, δ' with corresponding PBPO-rules ρ, ρ', every $\sigma : \rho \to \rho'$ in $\mathscr{D}_{\mathrm{PBPO}}$ such that $\delta m = \delta' m \circ \sigma_1$ and $\delta t_G = \sigma_4 \circ \delta' t_G$, there exists a unique $\mathcal{C}$-morphism d such that*

$$
\begin{tikzcd}
 & \delta\mathrm{L} \arrow[dl, "\sigma_1"'] \arrow[dd, "\delta\mathrm{m}"] & & \delta\mathrm{K} \arrow[ll, "\delta\mathrm{l}"'] \arrow[dl, "\sigma_2"'] \arrow[dd, "\delta\mathrm{k}"] \\
\delta'\mathrm{L} \arrow[dd, "\delta'\mathrm{m}"'] & & \delta'\mathrm{K} \arrow[ll, "\delta'\mathrm{l}"'] \arrow[dd, "\delta'\mathrm{k}"] & \\
 & \delta\mathrm{G} \arrow[dd, "\delta\mathrm{t_G}"] & & \delta\mathrm{D} \arrow[ll, "\delta\mathrm{f}"'] \arrow[dd, "\delta\mathrm{t_D}"] \\
\delta'\mathrm{G} \arrow[ur, dash, "="] \arrow[dd, "\delta'\mathrm{t_G}"'] & & \delta'\mathrm{D} \arrow[ll, "\delta'\mathrm{f}"] \arrow[ur, dashed, "d"'] \arrow[dd, "\delta'\mathrm{t_D}"] & \\
 & \delta\mathrm{T_L} & & \delta\mathrm{T_K} \arrow[ll, "\delta\mathrm{u}"] \\
\delta'\mathrm{T_L} \arrow[ur, "\sigma_4"] & & \delta'\mathrm{T_K} \arrow[ll, "\delta'\mathrm{u}"] \arrow[ur, "\sigma_5"'] &
\end{tikzcd}
$$

commutes.

Proof. By hypothesis the two front, back, left faces commute, as well as the top and bottom faces. Thus

$$\delta\mathrm{u} \circ \sigma_5 \circ \delta'\mathrm{t_D} = \sigma_4 \circ \delta'\mathrm{u} \circ \delta'\mathrm{t_D} = \sigma_4 \circ \delta'\mathrm{t_G} \circ \delta'\mathrm{f} = \delta\mathrm{t_G} \circ \delta'\mathrm{f},$$

and since $\delta\mathrm{D}$ is a pullback then there exists a unique d such that the right and top face of the bottom cube commute. This also means that $(\delta\mathrm{D}, \delta\mathrm{f}, \delta\mathrm{t_D})$ is a mono-source, and since

$$\delta\mathrm{f} \circ d \circ \delta'\mathrm{k} \circ \sigma_2 = \delta'\mathrm{f} \circ \delta'\mathrm{k} \circ \sigma_2 = \delta'\mathrm{m} \circ \delta'\mathrm{l} \circ \sigma_2 = \delta'\mathrm{m} \circ \circ \sigma_1 \circ \delta\mathrm{l} = \delta\mathrm{m} \circ \delta\mathrm{l} = \delta\mathrm{f} \circ \delta\mathrm{k},$$

$$\delta\mathrm{t_D} \circ d \circ \delta'\mathrm{k} \circ \sigma_2 = \sigma_5 \circ \delta'\mathrm{t_D} \circ \delta'\mathrm{k} \circ \sigma_2 = \sigma_5 \circ \delta'\mathrm{t_K} \circ \sigma_2 = \delta\mathrm{t_K} = \delta\mathrm{t_D} \circ \delta\mathrm{k}$$

then $d \circ \delta'\mathrm{k} \circ \sigma_2 = \delta\mathrm{k}$. □

We leave it to the reader to define a Rewriting Environment for PBPO-rules and transformations, with a left-full faithful functor $\mathsf{R}_{\mathrm{PBPO}} : \mathscr{D}_{\mathrm{PBPO}} \to \mathscr{R}_{\mathrm{PBPO}}$ (provided $\mathcal{C}$ has pushouts and pullbacks).

We finally observe that if $\mathcal{R}_1 \xleftarrow{\mathsf{R}_1} \mathcal{D}_1 \xrightarrow{\mathsf{P}_1} \mathcal{C}_{\mathrm{pt}}$ and $\mathcal{R}_2 \xleftarrow{\mathsf{R}_2} \mathcal{D}_2 \xrightarrow{\mathsf{P}_2} \mathcal{C}_{\mathrm{pt}}$ are Rewriting Environments for $\mathcal{C}$ then so is $\mathcal{R}_1 + \mathcal{R}_2 \xleftarrow{\mathsf{R}_1 + \mathsf{R}_2} \mathcal{D}_1 + \mathcal{D}_2 \xrightarrow{[\mathsf{P}_1, \mathsf{P}_2]} \mathcal{C}_{\mathrm{pt}}$. This means that it is possible to mix rules of different approaches to transform a graph, though of course rules of distinct approaches cannot subsume each other.

6 Conclusion and Future Work

The general notion of a rule subsumption is given through Rewriting Environments, where abstract categories of rules and direct transformations are related to a specific category of partial transformations $\mathcal{C}_{\mathrm{pt}}$. The Global Coherent Transformation is built from partial transformations in a way pertaining both to Parallel Coherent Transformation, by the use of limits on interfaces, and to Global Transformations, by applying categories of rules.

We have provided Rewriting Environments for the most common approaches to algebraic rewriting, except the Single Pushout [14]. This will be done in a forthcoming paper, where we will see that the interface and right-hand side provided in a partial transformation are not necessarily extracted from the applied rule. We also intend to show that Global Transformations can be obtained by a suitable environment (except when Δ is empty).

It may seem strange that, through $\mathcal{C}_{\mathrm{pt}}$, rules are not assumed to have left-hand sides and direct transformations are not assumed to use matchings. The notion of Rewriting Environment is as simple as required to define Global Coherent Transformations, but does not guarantee some properties that the user might reasonably expect. In particular it does not prevent the categories $\mathcal{R}$ and $\mathcal{D}$ from being discrete (which is correct only if no subsumption is possible). In the future we will enhance Rewriting Environments with a notion of matchings in order to better understand their structure.

We also need to further analyze the properties of the DPO Rewriting Environment: when $\mathcal{C}$ is an adhesive category it is an open question whether $\mathsf{R}_{\mathrm{DPOm}}$ is left-full.

The present framework also brings some insight into the notion of parallel rewriting modulo automorphisms [1], that factors out the automorphism group of the rules. With Global Coherent Transformations we have the possibility to factor out subgroups of these automorphism groups. This needs further investigations.

References

[1] T. Boy de la Tour & R. Echahed (2020): *Parallel Rewriting of Attributed Graphs.* *Theoretical Computer Science* 848, pp. 106–132, doi:10.1016/j.tcs.2020.09.025.

[2] T. Boy de la Tour & R. Echahed (2021): *Parallel Coherent Graph Transformations.* In: *Recent Trends in Algebraic Development Techniques, 25th International Workshop, WADT 2020, Revised Selected Papers*, *LNCS* 12669, Springer, pp. 75–97.

[3] Thierry Boy de la Tour (2023): *Algebraic Monograph Transformations*, doi:10.48550/ARXIV.2303.01137.

[4] Andrea Corradini, Dominique Duval, Rachid Echahed, Frédéric Prost & Leila Ribeiro (2019): *The PBPO graph transformation approach.* *J. Log. Algebr. Meth. Program.* 103, pp. 213–231.

[5] Andrea Corradini, Tobias Heindel, Frank Hermann & Barbara König (2006): *Sesqui-Pushout Rewriting.* In: *ICGT 2006*, *LNCS* 4178, Springer, pp. 30–45.

[6] Andrea Corradini, Ugo Montanari, Francesca Rossi, Hartmut Ehrig, Reiko Heckel & Michael Löwe (1997): *Algebraic Approaches to Graph Transfor-*

mation - Part I: Basic Concepts and Double Pushout Approach. In Grzegorz Rozenberg, editor: *Handbook of Graph Grammars and Computing by Graph Transformations, Volume 1: Foundations*, World Scientific, pp. 163–246, doi:10.1142/9789812384720_0003.

[7] Hartmut Ehrig (1979): *Introduction to the algebraic theory of graph grammars (a survey).* In Volker Claus, Hartmut Ehrig & Grzegorz Rozenberg, editors: *Graph-Grammars and Their Application to Computer Science and Biology*, Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 1–69.

[8] Hartmut Ehrig, Karsten Ehrig, Ulrike Prange & Gabriele Taentzer (2006): *Fundamentals of Algebraic Graph Transformation.* Monographs in Theoretical Computer Science. An EATCS Series, Springer, doi:10.1007/3-540-31188-2.

[9] Alexandre Fernandez, Luidnel Maignan & Antoine Spicher (2021): *Cellular Automata and Kan Extensions.* In Alonso Castillo-Ramirez, Pierre Guillon & Kévin Perrot, editors: *27th IFIP WG 1.5 International Workshop on Cellular Automata and Discrete Complex Systems, AUTOMATA 2021*, *OASIcs* 90, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, pp. 7:1–7:12, doi:10.4230/OASIcs.AUTOMATA.2021.7.

[10] Alexandre Fernandez, Luidnel Maignan & Antoine Spicher (2022): *Non-Determinism in Lindenmayer Systems and Global Transformations.* In Stefan Szeider, Robert Ganian & Alexandra Silva, editors: *47th International Symposium on Mathematical Foundations of Computer Science, MFCS 2022*, *LIPIcs* 241, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, pp. 49:1–49:13, doi:10.4230/LIPIcs.MFCS.2022.49.

[11] H. Herrlich & G. E. Strecker (1973): *Category Theory.* Allyn and Bacon.

[12] Stephen Lack & Pawel Sobocinski (2005): *Adhesive and quasiadhesive categories.* *Informatique Théorique et Applications* 39(3), pp. 511–545.

[13] F. W. Lawvere (1963): *Functional Semantics of Algebraic Theories.* Ph.D. thesis, Columbia University.

[14] Michael Löwe (1993): *Algebraic Approach to Single-Pushout Graph Transformation.* *Theoretical Computer Science* 109, pp. 181–224.

[15] Saunders Mac Lane (1997): *Categories for the Working Mathematician*, 2nd edition. *Graduate Texts in Mathematics* 5, Springer-Verlag, New York. (1st ed., 1971).

[16] Luidnel Maignan & Antoine Spicher (2015): *Global Graph Transformations.* In Detlef Plump, editor: *Proceedings of the 6th International Workshop on Graph Computation Models*, *CEUR Workshop Proceedings* 1403, CEUR-WS.org, pp. 34–49. Available at `http://ceur-ws.org/Vol-1403/paper4.pdf`.