Mean-variance dynamic portfolio allocation with transaction costs: a Wiener chaos expansion approach

Areski Cousin, Jérôme Lelong, Tom Picard

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Areski Cousin*   J. Lelong†   T. Picard‡

May 2, 2023

Abstract

This paper studies the multi-period mean-variance portfolio allocation problem with transaction costs. Many methods have been proposed these last years to challenge the famous uni-period Markowitz strategy. But these methods cannot integrate transaction costs or become computationally heavy and hardly applicable. In this paper, we try to tackle this allocation problem by proposing an innovative approach which relies on representing the set of admissible portfolios by a finite dimensional Wiener chaos expansion. This numerical method is able to find an optimal strategy for the allocation problem subject to transaction costs. To complete the study, the link between optimal portfolios submitted to transaction costs and the underlying risk aversion is investigated. Then a competitive and compliant benchmark based on the sequential uni-period Markowitz strategy is built to highlight the efficiency of our approach.

Keywords: multi-period portfolio allocation, mean-variance formulation, Wiener chaos expansion.
Classification: 62L20, 91G10, 91G60, 93E20

1 Introduction

Dynamic portfolio selection is one of the most studied topic in financial economics. The problem consists in allocating the wealth of an investor, among a basket of assets, over time. Finding the optimal portfolio is a difficult challenge since it depends on the objective of the investor. The Markowitz mean-variance formulation represents a first answer to this problem by providing fundamental basics for static portfolio allocation in a uni-period case (see [26]). Mean-variance framework offers to build a portfolio of assets such that the expected return is maximized for a given level of risk. This portfolio theory is based on the assumption that the parameters of the underlying stochastic model are known and contain no estimation error. This method is easy to apply and has the favor of asset managers. Nevertheless, when the time horizon increases, this myopic strategy which cannot see ahead of the next time period, cannot challenge the dynamic optimal portfolio obtained from the multi-period version of the problem. [27] is one of the first paper studying multi-period portfolio investment in a dynamic programming framework. In this seminal paper, the authors consider a problem with one risky asset and one risk-free asset. At each date, the investor can re-balance its wealth between the two assets, seeking to maximize an

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*Institut de Recherche en Mathématique Avancée, Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg, cedex, France. (a.cousin@unistra.fr).

†Univ. Grenoble Alpes, CNRS, Grenoble INP, LJK, 38000 Grenoble, France. (jerome.lelong@univ-grenoble-alpes.fr).

‡Univ. Grenoble Alpes, LJK, 38000 Grenoble, France and Nexialog, 75011 Paris, France. (tom.picard@grenoble-inp.org).

utility of the final time horizon wealth. They derive a simple closed-form expression for the optimal policy when there are no constraint or transaction cost. In a companion paper, [34] derives the discrete-time analog approach. The results presented in those studies and the innovative and promising aspect of the multi-period portfolio selection have stimulated the interest of the related scientific community. In the following years, the literature in multi-period portfolio selection has considerably grown, dominated by maximizing expected utility of terminal wealth of elementary forms as logarithm, exponential or CRRA functions. Dynamic programming techniques turn out to be the most suitable approaches to solve this kind of problems. Among the most noteworthy articles, [7] compares and highlights the conditions of equivalence between dynamic approaches and myopic strategies with CRRA utility functions.

However, important difficulties due the non separability of the problem in the sense of dynamic programming, have been reported in finding the optimal portfolio issued from the multi-period mean-variance approach. Nevertheless, [22] and [40] provide explicit formulation for the unconstrained multi-period mean–variance optimal portfolio both in a discrete and continuous time setting. [24] derives the optimal portfolio policy for the continuous-time mean–variance model with no-shorting constraint. [13] extends this work to provide a discrete framework. Even if these last studies have become increasingly realistic, the ignorance of transaction costs, hinders their efficient applications in real life. Transaction costs have a major impact on the optimal policy and cannot be ignored.

The integration of transaction costs has been widely studied in the uni-period mean-var case (see [5], [25], [32], [38], [39]). In a continuous time setting, the problem is not recent, especially when the time horizon is considered infinite (see [15], [16], [18], [28], [29]). In a mean-var and finite time horizon setting, [14] studies the properties of the optimal strategies and boundaries which define the buy, sell and no trade-regions. Discrete time allocation strategies submitted to transaction costs have also been widely pursued. [12] and [20] investigate optimal investment policies with proportional costs, accompanied by fixed costs for the second. They also describe them in terms of a no-trade region in which it is optimal leave the portfolio allocation unchanged. But many difficulties have been reported in the literature to design efficient and accurate methods to compute the optimal solutions. Furthermore, when solutions are proposed, they remain computationally heavy and hardly applicable. Some of them tractably solve the problem in several special cases. [6] and [31] assume that costs are convex quadratic. Other made various approximations. For instance, [17] assumes that the number of shares in risky assets are deterministic before optimizing. [10] and [36] assume an affine structure for the strategies. From another perspective, both [4] and [23] rely a rolling horizon philosophy, with the latter utilizing a Model Predictive Control (MPC) approach to identify sub-optimal strategies.

The most prevalent methods to tackle this kind of problem are based on stochastic control techniques. However, they also suffer from these criticisms. In order to limit the dimensionality, true curse in dynamic programming, [9] uses Chebyshev polynomials to interpolate the value functions on a sparse grid of the space. Many of other related studies such as [19], [35] and [37] rely on trees for modeling the rates of return. [2] goes further by assuming that every future rates of return is known by the investor. Under the same assumption, [8] derives an upper bound to measure and highlight the good performances of heuristic strategies. [6] performs an ADP (Approximate dynamic programming) by using sub-optimal solutions to approximate value functions. These sub-optimal solutions are issued from a quadratic version of the problem or from MPC method (model predictive control). [11] uses an other sub-optimal solution, called multi-stage strategy to tune the exploring phase of its backward recursion algorithm. This method provides a solution at least, as good as the sub-optimal one. Recently, [33] also adopts dynamic programming but is forced to handle different cases separately, which makes their method hardly computationally applicable.

In this paper, we attempt to fill this gap by providing a new computational scheme to solve multi-periods portfolio allocation problem submitted to transaction costs. We address this problem by proposing an innovative approach which relies on representing the set admissible portfolios by a finite dimensional Wiener chaos expansion. This numerical method estimates optimal portfo-

lios submitted to proportional transaction costs. The policies are computed thanks to a stochastic gradient descent algorithm, and require no exploration framework, which was an major step of the methods based on stochastic control algorithms (see [3]). Then a competitive benchmark, based on the sequential uni-period Markowitz strategy is built to highlight the efficiency of our approach. This benchmark relies on the independence between the sharp ratio and the risk aversion.

The main contribution of this paper is threefold. We introduce an innovative and efficient numerical method to get optimal portfolios submitted to transaction costs. Then, we study the links between risk aversion and multi-period optimal portfolios submitted to transaction costs. Finally, we provide a reliable benchmark with sequential mean-variance uni-period models in the context of transaction costs mentioned above. To the best of our knowledge, our paper is the first to provide this kind of benchmark.

The remaining of this paper is organized as follows. In Section 2, we describe our framework and notation. In Section 3, we present our methodology which aims at finding efficient portfolios for the dynamical mean-variance allocation problem, submitted to transaction costs. We also study the link between risk aversion and optimal solutions. Finally, in Section 4, we show the efficiency of our solution and investigate the impact of transaction cost by comparing performances of the presented models with benchmark models such as the sequential uni-period Markowitz approach described in Appendix C. In Appendix D, we discuss the use of a finer time grid for the Wiener chaos expansion.

2 Framework and notation

We define the filtered probability space $(\Omega, \boldsymbol{A}, \Gamma = (\mathcal{G}_t)_{t \in [0,T]}, \mathbb{Q})$, with $\Gamma = \sigma(W)$, where $W = (W^1, \ldots, W^d)$ is a Brownian motion defined on $[0,T]$ with values in $\mathbb{R}$. For $N \in \mathbb{N}$, $N > 0$, we introduce the discrete-time grid $0 = t_0 < t_1 < \cdots < t_N = T$. Let $\boldsymbol{F}$, be the discrete time filtration generated by the Brownian increments on this grid, $\mathcal{F}_n = \sigma(W_{t_k}, \ k \leq n)$ for $0 \leq n \leq N$.

Let Δ be the operator which associates its increments to a process. For any discrete time process $(\Theta_n)_n$, we write $\Theta_{k:n} = (\Theta_k, \Theta_{k+1}, \ldots, \Theta_{n-1}, \Theta_n)$ for $k, n \in \mathbb{N}$, $k < n$ and $\Delta\Theta_n = \Theta_n - \Theta_{n-1}$. We also define the normalized Brownian increments by $\Delta\widehat{W}_k = \left(\frac{W^j_{t_k} - W^j_{t_{k-1}}}{\sqrt{t_k - t_{k-1}}} \right)_{j \in \{1,\ldots,d\}}$ for $1 \leq k \leq N$.

We consider d $\boldsymbol{F}$−adapted risky assets $S = (S^1, \ldots, S^d)$ and a risk free asset S^0. We assume that the financial market they represent is complete. During the time period $[0,T]$, an investor, with an initial wealth $V_0 > 0$, can invest and re-balance its portfolio among (S, S^0), at discrete times $t_0 < t_1 <, \ldots, < t_N$. The risk free rate $(r_n)_{n \leq N}$ is assumed to be deterministic and $\forall n \in \{0, \ldots, N\}$, $\log\left(\frac{S^0_{n+1}}{S^0_n} \right) = r_n(t_{n+1} - t_n)$. We assume that the risky assets are defined by

$$\forall n \in \{1, \ldots, N\}, \ i \in \{1, \ldots, d\}, \ \log\left(\frac{S^i_{n+1}}{S^i_n} \right) = \left(\mu^i_n - \frac{(\sigma^i_n)^T \sigma^i_n}{2} \right)(t_{n+1} - t_n) + \sigma^i_n \cdot \Delta W_{n+1}, \ (1)$$

where μ and σ are $\boldsymbol{F}$-adapted processes with values in $\mathbb{R}^d$ and $\mathbb{R}^{d \times d}$ respectively. The process σ is called the volatility process and we assume that $\forall n \in \{1, \ldots, N\}$, σ_n is a.s. invertible and that $\forall p \geq 1$, $S_n \in L^p(\Omega, \mathcal{F}_n, \mathbb{P})$. This model implies that knowing $\mathcal{F}_n$, the returns $\left\{ \frac{S^i_{n+1}}{S^i_n}, \ i \in \{1, \ldots, d\} \right\}$ are independent of $\mathcal{F}_n$. Note that local volatility models fit in this framework by considering the Euler scheme of $\log(S)$. Let $\boldsymbol{F}^S = (\mathcal{F}^S_n)_{0 \leq n \leq N}$ be the natural filtration of the risky assets, $\mathcal{F}^S_n = \sigma(S_{t_k}, \ k \leq n)$. Since $(\mu^i)_{i \in \{1,\ldots,d\}}$ and $(\sigma^i)_{i \in \{1,\ldots,d\}}$ are $\boldsymbol{F}$-adapted processes and σ_n is a.s. invertible, we can easily prove by induction that $\boldsymbol{F} = \boldsymbol{F}^S$.

For $0 \leq n \leq N$ we define $\Phi_n = \sigma_n^{-1}(\mu_n - r_n)$. Let $\tau(t) = \sup\{t_n < t : n \leq N\}$. We assume that $\mathbb{E}_\mathbb{P}\left[\exp\left(\int_0^T \frac{1}{2} |\Phi_{\tau(u)}|^2 du \right) \right] < \infty$ such that the process $\left(\exp\left(-\int_0^t \Phi_{\tau(u)} \cdot dW_u - \frac{1}{2} \int_0^t |\Phi_{\tau(u)}|^2 du \right) \right)_{0 \leq t \leq T}$ is a martingale. Therefore, we know from Girsanov's theorem that there exists a probability mea-

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sure $\mathbb{Q}$ equivalent to $\mathbb{P}$, defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \Phi_{\tau(u)} \cdot dW_u - \frac{1}{2}\int_0^T |\Phi_{\tau(u)}|^2 du\right).$$

Let $(W_t^{\mathbb{Q}})_{t\in[0,T]}$ be a Γ-adapted process defined by $W_0^{\mathbb{Q}} = 0$ and

$$dW_t^{\mathbb{Q}} = dW_t + \Phi_{\tau(t)}dt,$$

then $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$. Note that

$$\forall i \in \{1,\ldots,d\}, \ \Delta W_{n+1}^{\mathbb{Q},i} \overset{\mathbb{P}}{=} \Delta W_{n+1}^i + \Phi_n^i(t_{n+1} - t_n). \tag{2}$$

Let $\boldsymbol{F}^{\mathbb{Q}} = (\mathcal{F}_n^{\mathbb{Q}})_{0\leq n\leq N}$, be the discrete time filtration generated by $W^{\mathbb{Q}}$, $\mathcal{F}_n^{\mathbb{Q}} = \sigma(W_{t_k}^{\mathbb{Q}}, \ k \leq n)$. We notice that

$$\forall n \in \{1,\ldots,N\}, \ i \in \{1,\ldots,d\}, \ \log\left(\frac{S_{n+1}^i}{S_n^i}\right) = (r_n)(t_{n+1} - t_n) + \sigma_n^i \cdot \Delta W_{n+1}^{\mathbb{Q}},$$

We can see by induction that we also have $\boldsymbol{F}^{\mathbb{Q}} = \boldsymbol{F}^S$.
We use the tilde notation to denote discounting We notice that $\tilde{S}^i = S^i/S^0$ is a $\boldsymbol{F}$-martingale under the probability $\mathbb{Q}$.

3 Mean-variance Portfolio allocation

In this section, we present an innovative approach to solve the dynamic mean-var allocation problem in presence of transaction costs.

3.1 Environment

We consider a portfolio V, of initial wealth V_0, composed of the d risky assets $S = (S^1,\ldots,S^d)$ and the risk-free asset S^0. An investor, with a risk aversion γ can re-allocate its portfolio V at discrete times $t_1 < \cdots < t_N$. We define $(\alpha_n, \alpha_n^0)_{n\in\{1,N\}}$, the quantities of risky and risk free assets hold in the portfolio, such that $\forall n \in \{0,\ldots,N\}$, $V_n = \alpha_n \cdot S_n + \alpha_n^0 S_n^0$. The processes α and α^0 are assumed to be $\boldsymbol{F}$-predictable. The agent aims to find the strategy $(\alpha_n, \alpha_n^0)_{n\in\{1,N\}}$, such that the generated portfolio V, maximizes $\mathbb{E}_{\mathbb{P}}[V_N] - \gamma\mathbb{E}_{\mathbb{P}}\left[(V_N - \mathbb{E}_{\mathbb{P}}[V_N])^2\right]$.
At time 0, we assume that the owner has all its wealth in cash, ie $\alpha_0^0 S_0^0 = V_0$ and $\forall i \in \{1,\ldots,d\}$, $\alpha_0^i = 0$. At time $n \in \{1,\ldots,N-1\}$, the investor re-balances his portfolio from (α_n, α_n^0) to $(\alpha_{n+1}, \alpha_{n+1}^0)$. He pays the transaction costs, proportional to the trade cash volume per asset and equal to $\sum_{i=1}^d \nu|\alpha_{n+1}^i - \alpha_n^i|S_n^i$. The self-financing condition is

$$\tilde{V}_n = \alpha_n \cdot \tilde{S}_n + \alpha_n^0 = \alpha_{n+1} \cdot \tilde{S}_n + \alpha_{n+1}^0 + \sum_{i=1}^d \nu\frac{|\alpha_{n+1}^i - \alpha_n^i|S_n^i}{S_n^0}.$$

So we have

$$\Delta\tilde{V}_{n+1} = \alpha_{n+1} \cdot \Delta\tilde{S}_{n+1} - \sum_{i=1}^d \nu\frac{|\alpha_{n+1}^i - \alpha_n^i|S_n^i}{S_n^0}. \tag{3}$$

Note that $\Delta\tilde{V}$ depends only α. Therefore, it is sufficient to determine α as α^0 follows the self-financing condition and V_0.

3.2 The dynamic mean-var allocation problem

We aim at finding $(\alpha_n)_{0 \leq n \leq N}$ solution to

$$\sup_{(\alpha_n)_n} \quad \mathbb{E}_{\mathbb{P}} \left[\tilde{V}_N S_N^0 - \gamma \left(\tilde{V}_N S_N^0 - \mathbb{E}_{\mathbb{P}}[\tilde{V}_N S_N^0] \right)^2 \right]$$

$$\text{subject to} \quad \tilde{V}_0 = V_0, \ (\alpha_n)_n \ \boldsymbol{F} - Pred$$

$$\tilde{V}_{n+1} = \tilde{V}_n + \alpha_{n+1} \cdot \Delta \tilde{S}_{n+1} - \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i - \alpha_n^i| S_n^i}{S_n^0}$$

Note that $\tilde{V}$ is not a $\boldsymbol{F}$-martingale but a $\boldsymbol{F}$-super-martingale under $\mathbb{Q}$. We define the cumulative cost process $\boldsymbol{C}$ by

$$\forall n \in \{1, \ldots, N\}, \ \mathcal{C}_n = \sum_{k=0}^{n-1} \sum_{i=1}^d \nu \frac{|\alpha_{k+1}^i - \alpha_k^i| S_k^i}{S_k^0}, \ \mathcal{C}_0 = 0. \tag{4}$$

Let $\tilde{X}_n = \tilde{V}_n + \mathcal{C}_n$. We clearly see that $\tilde{X}$ is a $\boldsymbol{F}$ martingale under $\mathbb{Q}$. We seek an optimal strategy α_n, solution to

$$\sup_{(\alpha_n)_n} \quad \mathbb{E}_{\mathbb{P}} \left[(\tilde{X}_N - \mathcal{C}_N) S_N^0 - \gamma \left((\tilde{X}_N - \mathcal{C}_N) S_N^0 - \mathbb{E}_{\mathbb{P}}[(\tilde{X}_N - \mathcal{C}_N) S_N^0] \right)^2 \right]$$

$$\text{subject to} \quad \tilde{X}_0 = V_0, \ (\alpha_n)_n \ \boldsymbol{F} - Pred$$

$$\tilde{X}_{n+1} = \tilde{X}_n + \alpha_{n+1} \cdot \Delta \tilde{S}_{n+1} \tag{E^γ}$$

$$\mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i - \alpha_n^i| S_n^i}{S_n^0}$$

Let $\mathcal{M}$ be the space of squared integrable $\boldsymbol{F}$-martingales under $\mathbb{Q}$ and $\mathcal{M}_S$ the sub-space of $\mathcal{M}$ defined by

$$\mathcal{M}_S = \left\{ \begin{array}{l} M \in \mathcal{M} : \ \exists (\nu_k)_{1 \leq k \leq N} \ \boldsymbol{F} - \text{predictable } s.t \ \forall n \in \{0, \ldots, N\}, \\ M_n = h + \sum_{k=0}^{n-1} \nu_{k+1} \cdot \Delta \tilde{S}_{k+1} \text{ and } \mathbb{E}[|\nu_{k+1} \sigma_k|^2] < \infty \end{array} \right\}.$$

$\mathcal{M}_S$ is the set of martingales which are a martingale transformations of $\tilde{S}$. The space $\mathcal{M}_S$ also represents the space of admissible portfolios in discrete-time. Equivalently, (E^γ) can be rewritten as

$$\sup_{\tilde{X} \in \mathcal{M}_S} \quad \mathbb{E}_{\mathbb{P}} \left[(\tilde{X}_N - \mathcal{C}_N) S_N^0 - \gamma \left((\tilde{X}_N - \mathcal{C}_N) S_N^0 - \mathbb{E}_{\mathbb{P}}[(\tilde{X}_N - \mathcal{C}_N) S_N^0] \right)^2 \right]$$

$$\text{subject to} \quad \tilde{X}_0 = V_0, \tag{$E^\gamma_{\mathcal{M}_S}$}$$

$$\mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i - \alpha_n^i| S_n^i}{S_n^0}$$

The aim of this paper is to compute an approximation of a solution to $(E^\gamma_{\mathcal{M}_S})$ The main difficulty remains to parameterize $\mathcal{M}_S$. We propose to find optimal portfolios on $\mathcal{M}_S$ by exploring $\mathcal{M}$.

3.3 Parametrisation of the problem

The method consists in finding optimal solutions on $\mathcal{M}_S$ by exploring $\mathcal{M}$. To do so, we will use the function derived in the following proposition.

Proposition 1. *Let $M \in \mathcal{M}$. The $L^2(\mathbb{Q})$ orthogonal projection of M on $\mathcal{M}_S$ is defined by*

$$Pr_{\mathbb{Q}}(M) = \left(M_0 + \sum_{k=0}^{n-1} \left(\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{k+1} \Delta \tilde{S}_{k+1}^T | \mathcal{F}_k \right]^{-1} \mathbb{E}_{\mathbb{Q}} \left[\Delta M_{k+1} \Delta \tilde{S}_{k+1} | \mathcal{F}_k \right] \right) \cdot \Delta \tilde{S}_{k+1} \right)_{0 \leq n \leq N},$$

(5)

where

$$\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta (\tilde{S}_{n+1})^T | \mathcal{F}_n \right] = \left[\tilde{S}_n^i \tilde{S}_n^j \left(e^{(\sigma_n^i)^T \sigma_n^j (t_{n+1} - t_n)} - 1 \right) \right]_{i,j \in \{1,\dots,d\}}.$$

Proof. The projection of M writes as $\left(h + \sum_{k=0}^{n-1} \alpha_{k+1} \cdot \Delta \tilde{S}_{k+1} \right)_{0 \leq n \leq N}$, with α_k^j $\boldsymbol{F}-predictable, h \in$ $\mathbb{R}$. Let $B \in \mathcal{M}_S$ such that for $n \in \{0,\dots,N\}$, $B_n = g + \sum_{k=0}^{n-1} c_{k+1} \cdot \Delta \tilde{S}_{k+1}$, $g \in \mathbb{R}$, c_{k+1}^j $\boldsymbol{F}-$ *predictable. Then we have for $n \in \{1,\dots,N\}$,*

$$\mathbb{E}_{\mathbb{Q}} \left[\left(M_n - h - \sum_{k=0}^{n-1} c_{k+1} \cdot \Delta \tilde{S}_{k+1} \right) \left(g + \sum_{k=0}^{n-1} c_{k+1} \cdot \Delta \tilde{S}_{k+1} \right) \right] = 0.$$

(6)

By taking $c_{k+1}^j = 0$, for all $j \in \{1,\dots,d\}$, we get $h = M_0$. Setting $g = 0$, we have

$$\mathbb{E}_{\mathbb{Q}} \left[\left(\sum_{l=0}^{n-1} \Delta M_{l+1} - \sum_{k=0}^{n-1} \alpha_{k+1} \cdot \Delta \tilde{S}_{k+1} \right) \left(\sum_{k=0}^{n-1} c_{k+1} \cdot \Delta \tilde{S}_{k+1} \right) \right] = 0.$$

(7)

By developing the expression and using that M and $\tilde{S}^j$ are $\boldsymbol{F}$-martingale, we obtain for $l \neq k$,

$$\mathbb{E}_{\mathbb{Q}} \left[c_{k+1}^j \Delta M_{l+1} \Delta \tilde{S}_{k+1}^j \right] = \mathbb{E}_{\mathbb{Q}} \left[c_{k+1}^j \mathbb{E}_{\mathbb{Q}} \left[\Delta M_{l+1} \Delta \tilde{S}_{k+1}^j | \mathcal{F}_{i \vee k} \right] \right] = 0.$$

(8)

Then, for $1 \leq k \leq n-1$,

$$\mathbb{E}_{\mathbb{Q}} \left[\sum_{l=0}^{n-1} \Delta M_{l+1} c_{k+1}^j \Delta \tilde{S}_{k+1}^j \right] = \mathbb{E}_{\mathbb{Q}} \left[c_{k+1}^j \Delta M_{k+1} \Delta \tilde{S}_{k+1}^j \right].$$

(9)

Finally, inserting (9) in (7) yields

$$\mathbb{E}_{\mathbb{Q}} \left[\sum_{k=0}^{n-1} c_{k+1} \cdot \Delta \tilde{S}_{k+1} \Delta M_{k+1} - \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} (\alpha_{k_1+1} \cdot \Delta \tilde{S}_{k_1+1})(c_{k_2+1} \cdot \Delta \tilde{S}_{k_2+1}) \right] = 0$$

(10)

In the same way, for $k_1 \neq k_2$,

$$\mathbb{E}_{\mathbb{Q}} \left[(\alpha_{k_1+1} \cdot \tilde{S}_{k_1+1})(c_{k_2+1} \cdot \Delta \tilde{S}_{k_2+1}) \right] = \mathbb{E}_{\mathbb{Q}} \left[\mathbb{E} \left[(\alpha_{k_1+1} \cdot \Delta \tilde{S}_{k_1+1})(c_{k_2+1} \cdot \Delta \tilde{S}_{k_2+1}) | \mathcal{F}_{k_1 \vee k_2} \right] \right] = 0.$$

(11)

Then

$$\sum_{k=0}^{n-1} \mathbb{E}_{\mathbb{Q}} \left[c_{k+1} \cdot \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{k+1} \Delta M_{k+1} | \mathcal{F}_k \right] - c_{k+1} \cdot \left(\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{k+1} \Delta \tilde{S}_{k+1}^T | \mathcal{F}_k \right] \alpha_{k+1} \right) \right] = 0.$$

(12)

So

$$\sum_{k=0}^{n-1} \mathbb{E}_{\mathbb{Q}} \left[c_{k+1} \cdot \left(\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{k+1} \Delta M_{k+1} | \mathcal{F}_k \right] - \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{k+1} \Delta \tilde{S}_{k+1}^T | \mathcal{F}_k \right] \alpha_{k+1} \right) \right] = 0.$$

(13)

As this equation holds for all $(c_k)_{1 \leq k \leq N}$ we deduce that $(\alpha_n)_n$ is defined by

$$\mathbb{E}_{\mathbb{Q}} \left[\Delta M_{n+1} \Delta \tilde{S}_{n+1} | \mathcal{F}_n \right] = \mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right] \alpha_{n+1}. \qquad \blacksquare$$

From Proposition 1, we define, for $Z \in \mathcal{M}$,

$$\alpha_{n+1}(Z) = \left(\mathbb{E}_{\mathbb{Q}} \left[\Delta \tilde{S}_{n+1} \Delta \tilde{S}_{n+1}^T | \mathcal{F}_n \right] \right)^{-1} \mathbb{E}_{\mathbb{Q}} \left[\Delta Z_{n+1} \Delta \tilde{S}_{n+1} | \mathcal{F}_n \right]. \tag{14}$$

Then, we can reformulate $(E_{\mathcal{M}_S}^\gamma)$ as

$$\sup_{Z \in \mathcal{M}} \quad \mathbb{E}_{\mathbb{P}} \left[(Pr_{\mathbb{Q}}(Z_N) - \mathcal{C}_N)S_N^0 - \gamma \left((Pr_{\mathbb{Q}}(Z_N) - \mathcal{C}_N)S_N^0 - \mathbb{E}_{\mathbb{P}}[(Pr_{\mathbb{Q}}(Z_N) - \mathcal{C}_N)S_N^0] \right)^2 \right]$$
$$\text{s.t.} \quad Z_0 = V_0, \tag{$E_{\mathcal{M}}^\gamma$}$$
$$\mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i(Z) - \alpha_n^i(Z)|S_n^i}{S_n^0}$$

3.4 Wiener chaos expansion of the solution

We will parametrize the elements of $\mathcal{M}$ through the Wiener chaos expansion of the terminal value. According to [1, Theorem 2.1], every element Y of $L^2(\Omega, \mathcal{F}_N, \mathbb{Q})$ can be represented by its Wiener chaos expansion as

$$Y = \mathbb{E}_{\mathbb{Q}}[Y] + \sum_{\lambda \in (\mathbb{N}^N)^d} \beta_\lambda H_\lambda^{\otimes}(\Delta \widehat{W}^{\mathbb{Q}}) \tag{15}$$

where

$$H_\lambda^{\otimes}(\Delta \widehat{W}^{\mathbb{Q}}) = \prod_{j=1}^d \prod_{i=1}^N H_{\lambda_i^j} \left(\frac{W_{t_i}^{\mathbb{Q},j} - W_{t_{i-1}}^{\mathbb{Q},j}}{\sqrt{t_i - t_{i-1}}} \right).$$

We define the truncated expansion of Y of order K under $\mathbb{Q}$ by $C_K^{\mathbb{Q}}(Y)$ such that

$$C_K^{\mathbb{Q}}(Y) = \mathbb{E}_{\mathbb{Q}}[Y] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \beta_\lambda H_\lambda^{\otimes}(\Delta \widehat{W}^{\mathbb{Q}}).$$

It is well-known that $\lim_{K \to +\infty} \mathbb{E}_{\mathbb{Q}} \left[|Y - C_K^{\mathbb{Q}}(Y)|^2 \right] = 0$. Let m be the number of coefficients β_λ appearing in the chaos expansion of order K, $m = \#\{\lambda \in (\mathbb{N}^N)^d | |\lambda|_1 \leq K\} = \binom{Nd+K}{Nd}$. We slightly abuse the notation $C_K^{\mathbb{Q}}$ to write

$$C_K^{\mathbb{Q}}(\beta) = \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \beta_\lambda H_\lambda^{\otimes}(\Delta \widehat{W}^{\mathbb{Q}}).$$

The basic theory of Wiener chaos expansion and the properties used here are presented in Appendix A. In particular, Proposition 20 states that for $n \leq N$, $\mathbb{E}_{\mathbb{Q}} \left[C_K^{\mathbb{Q}}(Y) | \mathcal{F}_n \right] = C_K^{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}[Y | \mathcal{F}_n])$, which can be obtained by removing the non $\mathcal{F}_n$-measurable terms from the chaos expansion of Y.
Let $(\eta_\lambda^j)_\lambda$ be the Wiener chaos expansion coefficients of $\tilde{S}_N^j$, for $j \in \{1, \ldots, d\}$

$$C_K^{\mathbb{Q}}(\tilde{S}_N^j) = S_0^j + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \eta_\lambda^j H_\lambda^{\otimes}(\Delta \widehat{W}^{\mathbb{Q}}).$$

Let $Z \in \mathcal{M}$ and $(\beta_\lambda)_\lambda$ be the Wiener chaos expansion coefficients of Z_N. Let $\mathbb{E}_{\mathbb{Q}} \left[C_K^{\mathbb{Q}}(\Delta Z_{n+1}) C_K^{\mathbb{Q}}(\Delta \tilde{S}_{n+1}^j) | \mathcal{F}_n \right]_K$ denote the truncated expansion of order K of $\mathbb{E}_{\mathbb{Q}} \left[C_K^{\mathbb{Q}}(\Delta Z_{n+1}) C_K^{\mathbb{Q}}(\Delta \tilde{S}_{n+1}^j) | \mathcal{F}_n \right]$, which is exactly

$C_K^{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left[\Delta Z_{n+1}\Delta\tilde{S}_{n+1}^j|\mathcal{F}_n\right]\right)$ according to Proposition 21 in Appendix A. Z and $\tilde{S}^j$ being $\boldsymbol{F}$-martingales, we can use Proposition 22 in Appendix A to express

$$
\begin{aligned}
C_K^{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left[\Delta Z_{n+1}\Delta\tilde{S}_{n+1}^j|\mathcal{F}_n\right]\right) &= \mathbb{E}_{\mathbb{Q}}\left[C_K^{\mathbb{Q}}(\Delta Z_{n+1})C_K^{\mathbb{Q}}(\Delta\tilde{S}_{n+1}^j)|\mathcal{F}_n\right]_K \\
&= \sum_{\substack{\lambda,\lambda'\in(\mathbb{N}^{n+1})^d \\ |\lambda|_1+|\lambda'|_1\leq K}} \beta_\lambda \eta_{\lambda'}^j H_{\lambda_{1:n}}^{\otimes}(\Delta\widehat{W}_{1:n}^{\mathbb{Q}})H_{\lambda'_{1:n}}^{\otimes}(\Delta\widehat{W}_{1:n}^{\mathbb{Q}})1_{(\lambda_{n+1}=\lambda'_{n+1}\neq 0)}\prod_{i=1}^d\lambda_{n+1}^i!.
\end{aligned}
\tag{16}
$$

Finally, we can use this expression for $E_{\mathbb{Q}}\left[\Delta Z_{n+1}\Delta\tilde{S}_{n+1}^j|\mathcal{F}_n\right]$.

Remark 2. *With the estimated chaos expansion of discounted assets, $(\eta_\lambda)_\lambda$, we can express the term $\mathbb{E}_{\mathbb{Q}}\left[\Delta Z_{n+1}\Delta\tilde{S}_{n+1}^j|\mathcal{F}_n\right]$ as a linear combination of the chaos coefficients $(\beta_\lambda)_\lambda$ of Z_N.*

We can slightly abuse the definition of the controls in (14), by identifying the martingale Z with the coefficients of the chaos expansion of its terminal value Z_N. Recalling that $(\beta_\lambda)_\lambda$ define the expansion of Z_N, we can define

$$
\alpha_{n+1}(\beta) = \mathbb{E}_{\mathbb{Q}}\left[\Delta\tilde{S}_{n+1}\Delta(\tilde{S}_{n+1})^T|\mathcal{F}_n\right]^{-1}\mathbb{E}_{\mathbb{Q}}\left[C_K^{\mathbb{Q}}(\Delta Z_{n+1})C_K^{\mathbb{Q}}(\Delta\tilde{S}_{n+1})|\mathcal{F}_n\right]_K.
\tag{17}
$$

Similarly, we also extend the cost function to

$$
\mathcal{C}_n(\beta) = \sum_{k=0}^{n-1}\sum_{i=1}^d \nu\frac{|\alpha_{k+1}^i(\beta)-\alpha_k^i(\beta)|S_k^i}{S_k^0}.
\tag{18}
$$

Let's define in this context, the portfolio value function as

$$
\forall\beta\in\mathbb{R}^m,\ \mathcal{R}(\beta) = Pr_{\mathbb{Q}}\left(C_{\mathbb{Q}}^K(\beta)\right)_N - \mathcal{C}_N(\beta).
\tag{19}
$$

We also express the objective function as

$$
\forall\beta\in\mathbb{R}^m,\ F_\gamma(\beta) = \mathcal{R}(\beta)S_N^0 - \gamma\left((\mathcal{R}(\beta)-\mathbb{E}_{\mathbb{P}}\left[\mathcal{R}(\beta)\right])S_N^0\right)^2.
\tag{20}
$$

Note that F_γ is a random function. With these new functions, we can approximation the original problem $(E_{\mathcal{M}_S}^\gamma)$ by a finite dimensional optimisation problem

$$
\begin{aligned}
\sup_{\beta\in\mathbb{R}^m}\quad &\mathbb{E}_{\mathbb{P}}\left[F_\gamma(\beta)\right] \\
\text{s.t.}\quad &\mathbb{E}_{\mathbb{Q}}\left[C_K^{\mathbb{Q}}(\beta)\right] = V_0
\end{aligned}
\tag{J^γ}
$$

The constraint $\mathbb{E}_{\mathbb{Q}}\left[C_K^{\mathbb{Q}}(\beta)\right] = V_0$ is easily satisfied by setting the first coefficient equal to V_0, $\beta_0 = V_0$.

Definition 3. *A γ-optimal strategy is a strategy $(\alpha_n(\beta^\star))_{0\leq n\leq N}$ such that $\beta^\star$ solves (J^γ). The portfolio associated to a γ-optimal strategy is called a γ-optimal portfolio.*

Proposition 4. *There exist a $\mathcal{F}_N$-measurable random variable D and $\boldsymbol{F}$-adapted processes B^i, K^i for $i\in\{1,\ldots,d\}$ and taking values in $\mathbb{R}^m$, such that $\forall i\in\{1,\ldots,d\}$, $n\in\{1,\ldots,N\}$, $\mathbb{E}_{\mathbb{P}}\left[|B_n^i|^p\right] + \mathbb{E}_{\mathbb{P}}\left[|K_n^i|^p\right] + \mathbb{E}_{\mathbb{P}}\left[|D|^p\right] < \infty$ for any $p\geq 1$ and*

$$
\alpha_n^i(\beta) = B_n^i\cdot\beta;\ \mathcal{R}(\beta) = V_0 + D\cdot\beta - \sum_{i=1}^d\sum_{n=0}^{N-1}\nu|K_n^i\cdot\beta|.
\tag{21}
$$

Proof. According to the linearity in β of the control functions, defined in (17), there exists a $\mathcal{F}_N$-measurable random variable D with values in $\mathbb{R}^m$ such that $Pr_\mathbb{Q}\left(C_\mathbb{Q}^K(\beta)\right)_N = V_0 + D \cdot \beta$ and $\exists \{B^i, i \in \{1, \ldots, d\}\}$, $\boldsymbol{F}$-adapted processes with values in $\mathbb{R}^m$ such that $\forall i \in \{1, \ldots, d\}$, $n \in \{0, \ldots, N-1\}$, $\alpha_n^i(\beta) = B_n^i \cdot \beta$. By calling $K_n^i = B_{n+1}^i - B_n^i$, we obtain $\mathcal{C}_N(\beta^\star) = \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |K_n^i \cdot \beta|$.

$\forall i \in \{1, \ldots, d\}$, $n \in \{0, \ldots, N-1\}$, $j \in \{1, \ldots, m\}$, $(B_n^i)_j \in \text{span}\left\{H_\lambda^{\bigotimes}(\Delta\widehat{W}^\mathbb{Q}), \lambda \in \mathbb{R}^m\right\}$, $(K_n^i)_j \in \text{span}\left\{H_\lambda^{\bigotimes}(\Delta\widehat{W}^\mathbb{Q}), \lambda \in \mathbb{R}^m\right\}$ so $\forall p \geq 1$, $\mathbb{E}_\mathbb{P}\left[|B_n^i|^p\right]$ and $\mathbb{E}_\mathbb{P}\left[|K_n^i|^p\right] < \infty$. And we have $\forall j \in \{1, \ldots, m\}$, $D_j = \sum_{n=0}^{N-1} \sum_{i=1}^d (B_{n+1}^i)_j \Delta\tilde{S}_{n+1}^i$. $\tilde{S}_n \in L^p(\Omega, \mathcal{F}_n, \mathbb{P})$, then $\mathbb{E}_\mathbb{P}[|D_j|^p] < \infty$. ∎

Proposition 5. *The problem* (J^γ) *admits a solution.*

Proof. Let $\beta \in \mathbb{R}^m \neq 0_{\mathbb{R}^m}$. By using the decomposition of Proposition 4, we can deduce the following property on the function $\mathcal{R}$,

$$\forall \beta \in \mathbb{R}^m, \ \beta \neq 0_{\mathbb{R}^m}, \ \mathcal{R}(\beta) - V_0 = |\beta|\left(\mathcal{R}\left(\frac{\beta}{|\beta|}\right) - V_0\right). \tag{22}$$

Then $\forall \beta \in \mathbb{R}^m \neq 0_{\mathbb{R}^m}$, we have

$$\mathbb{E}_\mathbb{P}\left[F_\gamma(\beta) - V_0\right] = |\beta|\mathbb{E}_\mathbb{P}\left[\mathcal{R}\left(\frac{\beta}{|\beta|}\right) - V_0\right] - \gamma|\beta|^2 \text{Var}\left[\mathcal{R}\left(\frac{\beta}{|\beta|}\right) - V_0\right]$$

$$= |\beta|\mathbb{E}_\mathbb{P}\left[F\left(\frac{\beta}{|\beta|}\right) - V_0\right] + \gamma\left(|\beta| - |\beta|^2\right)\text{Var}\left[\mathcal{R}\left(\frac{\beta}{|\beta|}\right)\right].$$

By considering the optimums of continuous functions on compact sets, we define

$$v = \inf_{|\beta|=1} \ \text{Var}\left[\mathcal{R}(\beta)\right] \quad, \ u = \sup_{|\beta|=1} \ \mathbb{E}_\mathbb{P}\left[F\left(\frac{\beta}{|\beta|}\right) - V_0\right].$$

Since the market is assumed to be complete, it is not possible to build a risk free portfolio with risky assets, $v > 0$. Finally, $\forall \beta \in \mathbb{R}^m$, $|\beta| > 1$,

$$\mathbb{E}_\mathbb{P}\left[F_\gamma(\beta) - V_0\right] \leq |\beta| u + \gamma\left(|\beta| - |\beta|^2\right)v.$$

With this inequality, we deduce that $\lim_{|\beta| \to +\infty} \mathbb{E}_\mathbb{P}\left[F_\gamma(\beta)\right] = -\infty$. The objective function $\beta \to \mathbb{E}_\mathbb{P}\left[F_\gamma(\beta)\right]$ is continuous and $\mathbb{E}_\mathbb{P}\left[F_\gamma(0)\right] = V_0 > 0$, then it attains its maximum. We can conclude that (J^γ) admits a solution. ∎

3.5 Resolution

Applying a stochastic descent gradient algorithm directly to solve (J^γ) is quite challenging. The objective function $\beta \to F_\gamma(\beta)$ depends on the law of the portfolio and makes the stochastic gradient descent not suitable. In order to solve (E^γ) and (J^γ) We would like to embed them into tractable equivalent ones.

Proposition 6. *The problem* (E^γ) *is equivalent to*

$$\sup_{(\alpha_n)_n, \theta \in \mathbb{R}} \quad \mathbb{E}_\mathbb{P}\left[\tilde{V}_N S_N^0 - \gamma\left((\tilde{V}_N - \theta)S_N^0\right)^2\right]$$

$$s.t. \quad \tilde{V}_0 = V_0, \ (\alpha_n)_n \ \boldsymbol{F} - Pred \tag{$\mathcal{E}^\gamma$}$$

$$\tilde{V}_{n+1} = \tilde{V}_n + \alpha_{n+1} \cdot \Delta\tilde{S}_{n+1} - \sum_{i=1}^d \nu\frac{|\alpha_{n+1}^i - \alpha_n^i|S_n^i}{S_n^0}$$

Proof. Let $(T_n)_n$ be a $\boldsymbol{F}$-adapted process. We define the function

$$\mathcal{U} : \mathbb{R} \to \mathbb{R}$$

$$\theta \longmapsto \mathbb{E}_{\mathbb{P}}[T_N] - \gamma \mathbb{E}_{\mathbb{P}}\left[\left((T_N - \theta)S_N^0\right)^2\right]$$

The function $\mathcal{U}$ is a second order polynomial such that $\lim_{|\theta| \to \infty} \mathcal{U}(\theta) = -\infty$. Then, $\mathcal{U}$ attains its maximum at a unique point $\theta^\star$ defined by $\nabla \mathcal{U}(\theta^\star) = 0$, $\theta^\star = \mathbb{E}_{\mathbb{P}}[T_N]$. ∎

Let G_γ be the random function defined by $G_\gamma : (\beta, \theta) \in \mathbb{R}^m \times \mathbb{R} \longmapsto \mathcal{R}(\beta)S_N^0 - \gamma\left((\mathcal{R}(\beta) - \theta)S_N^0\right)^2$. By using the Wiener chaos expansion in the equivalent problem of Proposition 6, we obtain the following proposition.

Proposition 7. *The problem (J^γ) is equivalent to*

$$\sup_{\beta \in \mathbb{R}^m, \theta \in \mathbb{R}} \quad \mathbb{E}_{\mathbb{P}}\left[G_\gamma(\beta, \theta)\right]$$

$$s.t. \quad \mathbb{E}_{\mathbb{Q}}\left[C_K^{\mathbb{Q}}(\beta)\right] = V_0 \tag{$\mathcal{J}^\gamma$}$$

Let's introduce $\forall n \ \{0, \ldots, N - 1\}$,

$$\mathcal{Z}_{n+1} = \left\{\lambda \in (\mathbb{N}^{n+1})^d, \ |\lambda| \le K, \ \lambda_{n+1} \ne 0\right\} \text{ and } \mathcal{Z} = \left\{(\beta \in \mathbb{R}^m, \ (\beta_\lambda)_{\lambda \in \mathcal{Z}_n} \ne 0_{\mathbb{R}^{\sharp(\mathcal{Z}_n)}}, \ \forall n \in \{1, \ldots, N - 1\}\right\}.$$

Note that $(\beta_\lambda)_{\lambda \in \mathcal{Z}_n}$ are the coefficients which intervene in the computation in (16) and consequently in the computation of α_{n+1} according to (17). According to (17) too, we also notice that for $\beta \in \mathcal{Z}$ then $\forall n \in \{1, \ldots, N - 1\}$, $\alpha_n(\beta) \ne 0$ a.s. Conversely, for $\beta \notin \mathcal{Z}$ then $\exists \ n \in \{1, \ldots, N - 1\}$, $\alpha_n(\beta) = 0$ a.s. This strategy involves no investment during a time period between t_1 and t_{N-1}. It does not represent a coherent strategy therefore it is no matter of interest here. We will now assume that $\beta \in \mathcal{Z}$.

Assuming that $\mathcal{R}$ is almost surely differentiable on $\mathcal{Z}$, it can be proved that $(\beta, \theta) \longmapsto \mathbb{E}_{\mathbb{P}}[G_\gamma(\beta, \theta)] = \mathbb{E}_{\mathbb{P}}\left[\mathcal{R}(\beta)S_N^0 - \gamma\left((\mathcal{R}(\beta) - \theta)S_N^0\right)^2\right]$ is differentiable on $\mathcal{Z} \times \mathbb{R}$. We refer to Appendix B for a detailed study on differentiability.

Let us define $\Psi_\gamma : (\beta, \theta) \longmapsto \nabla_{\beta,\theta}\left(G_\gamma(\beta, \theta)\right).$
$(\beta, \theta) \longmapsto \mathbb{E}_{\mathbb{P}}[G_\gamma(\beta, \theta)]$ is differentiable on $\mathcal{Z} \times \mathbb{R}$. $\mathcal{Z} \times \mathbb{R}$ is an open set of $\mathbb{R}^m \times \mathbb{R}$. Elements of $\mathbb{R}^m \setminus \mathcal{Z}$ are not coherent strategies and are not optimal strategies. Therefore if $(\beta^\star, \theta^\star)$ is solution to $(\mathcal{J}^\gamma)$ then $\mathbb{E}_{\mathbb{P}}[\Psi_\gamma(\beta^\star, \theta^\star)] = 0$. We have seen that

$$\mathbb{E}_{\mathbb{P}}\left[\nabla_\theta\left(\mathcal{R}(\beta)S_N^0 - \gamma\left((\mathcal{R}(\beta) - \theta^\star)S_N^0\right)^2\right)\right] = 0 \iff \theta^\star = \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta)].$$

We can then deduce that if $\beta^\star$ is the chaos expansion decomposition of an optimal portfolio then $\beta^\star$ is solution to

$$\mathbb{E}_{\mathbb{P}}\left[\Psi_\gamma\left(\beta^\star, \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^\star)]\right)\right] = 0. \tag{T^γ}$$

Proposition 8. *If $\mathcal{R}$ is almost surely differentiable on $\mathcal{Z}$ then the solutions to (T^γ) are locally optimal.*

Proof. $\mathcal{R}$ is concave w.r.t. β. $G_Y : (Y, \theta) \in \mathbb{R}^2 \to \mathbb{E}_{\mathbb{P}}\left[YS_N^0 - \gamma\left((Y - \theta)S_N^0\right)^2\right]$ is concave. If $\mathbb{E}_{\mathbb{P}}[G_Y]$ is monotone increasing in Y then $\mathbb{E}_{\mathbb{P}}[G_\gamma(\beta, \theta)]$ is concave in β. $\mathbb{E}_{\mathbb{P}}[G_Y]$ is increasing in Y if and only if $\mathbb{E}_{\mathbb{P}}[Y] \le \frac{1}{2\gamma} + \theta$.
Let $\beta^\star$ be a solution to (T^γ). $\mathcal{R}$ is continuous then we can find $\varepsilon > 0$ such that $\forall \ \beta \in \mathbb{R}^m$ s.t. $|\beta - \beta^\star| \le \varepsilon$, we have $|\mathcal{R}(\beta) - \mathcal{R}(\beta^\star)| \le \frac{1}{4\gamma}$. Then $\forall(\beta, \theta)$ such that $|\beta - \beta^\star| \le \varepsilon$, $|\theta - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^\star)]| \le \frac{1}{4\gamma}$, we have

$$|\theta - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta)]| \le |\theta - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^\star)]| + \mathbb{E}_{\mathbb{P}}[|\mathcal{R}(\beta^\star) - \mathcal{R}(\beta)|] \le \frac{1}{2\gamma}.$$

Then $\mathbb{E}_{\mathbb{P}}[G_\gamma(\beta, \theta)]$ is concave on $\{(\beta, \theta) : |\beta - \beta^\star| \le \varepsilon, \ |\theta - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^\star)]| \le \frac{1}{4\gamma}\}$. ∎

Using the previous results and the assumption (2), we can apply a gradient descent algorithm to find a local optimum of $(\mathcal{J}^\gamma)$.

3.6 Influence of risk aversion

In this section, we discuss the influence of the risk aversion on optimal multi-period portfolios submitted to transaction costs. One main result claims that the sharp ratio of a zeros gradient portfolio submitted to transaction costs is independent from its risk aversion. A second main result affirms that all optimal portfolios have the same sharp ratio.

Definition 9. *A γ-zero gradient strategy or γ-locally optimal strategy is a strategy $(\alpha_n(\beta^\star))_{0 \le n \le N}$ such that $\beta^\star$ solves (T^γ). The portfolio associated to a γ-zero gradient strategy is called a γ-zero gradient portfolio.*

Proposition 10. *Assume $\mathcal{R}$ is a.s. differentiable. Then, the risk aversion, the sharp ratio and the volatility of a γ-zero gradient portfolio $V^\star$ submitted to transaction costs are related by*

$$\gamma = \frac{\text{Sharp}(V_N^\star)}{2\mathbb{V}ar[V_N^\star]^{\frac{1}{2}}}. \tag{23}$$

Proof. We have seen in (T^γ) that if $V^\star$ is a γ-zero gradient portfolio, then $\exists\, \beta^\star \in \mathbb{R}^m$ such that $V_N^\star = \mathcal{R}(\beta^\star)S_N^0$ and $\mathbb{E}_\mathbb{P}[\Psi(\beta^\star, \mathbb{E}_\mathbb{P}[\mathcal{R}(\beta^\star)])] = 0$. This equality leads to

$$\mathbb{E}_\mathbb{P}\left[\nabla \mathcal{R}(\beta^\star) S_N^0 \left(1 - 2\gamma S_N^0 (\mathcal{R}(\beta^\star) - \mathbb{E}_\mathbb{P}[\mathcal{R}(\beta^\star)])\right)\right] = 0.$$

We recall that $\mathcal{R}(\beta^\star) = Pr_\mathbb{Q}\left(C_\mathbb{Q}^K(\beta^\star)\right) - \mathcal{C}_N(\beta^\star)$. As $\mathcal{R}$ is a.s differentiable and using the decomposition of Proposition 4,

$$\forall\, \beta \in \mathbb{R}^m,\ \nabla\mathcal{R}(\beta) = D - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu \times \text{sign}(K_n^i \cdot \beta) K_n^i. \tag{24}$$

We deduce that

$$\forall\, \beta \in \mathbb{R}^m,\ \beta \cdot \nabla\mathcal{R}(\beta) = \mathcal{R}(\beta) - V_0.$$

Then with

$$\beta^\star \cdot \mathbb{E}_\mathbb{P}\left[\nabla\mathcal{R}(\beta^\star) S_N^0 \left[1 - 2\gamma S_N^0 (\mathcal{R}(\beta^\star) - \mathbb{E}_\mathbb{P}[\mathcal{R}(\beta^\star)])\right]\right] = 0,$$

we obtain

$$\mathbb{E}_\mathbb{P}\left[\mathcal{R}(\beta^\star) S_N^0 \left[1 - 2\gamma S_N^0 (\mathcal{R}(\beta^\star) - \mathbb{E}_\mathbb{P}[\mathcal{R}(\beta^\star)])\right]\right] = V_0 S_N^0.$$

We deduce that

$$\mathbb{E}_\mathbb{P}\left[\mathcal{R}(\beta^\star) S_N^0\right] - 2\gamma\mathbb{E}_\mathbb{P}\left[\left(\mathcal{R}(\beta^\star) S_N^0\right)^2\right] + 2\gamma\mathbb{E}_\mathbb{P}\left[\mathcal{R}(\beta^\star) S_N^0\right]^2 = V_0 S_N^0.$$

Finally,

$$\gamma = \frac{\text{Sharp}(V_N^\star)}{2\mathbb{V}\text{ar}[V_N^\star]^{\frac{1}{2}}}. \tag{25}$$

$\blacksquare$

Corollary 11. *All γ-optimal portfolios have the same sharp ratio.*

Proof. Let $V_N^\star$ be such a γ-optimal portfolio. As an optimal portfolio, $V_N^\star$ is also a γ-zero gradient portfolio. Then according to Proposition 10,

$$\mathbb{E}_\mathbb{P}[V_N^\star] - V_0 S_N^0 = 2\gamma\mathbb{V}\text{ar}[V_N^\star]$$

Inserting this term in the mean-variance objective function leads to

$$\mathbb{E}_{\mathbb{P}}\left[V_N^\star - \gamma\left(V_N^\star - \mathbb{E}_{\mathbb{P}}[V_N^\star]\right)^2\right] = \frac{\mathbb{E}_{\mathbb{P}}[V_N^\star]}{2} - \frac{V_0 S_N^0}{2}. \tag{26}$$

We deduce that if V_N' is another γ-optimal portfolio then (26) gives $\mathbb{E}_{\mathbb{P}}[V_N^\star] = \mathbb{E}_{\mathbb{P}}[V_N']$. But the two portfolios also attain the same mean-variance value then

$$\mathbb{E}_{\mathbb{P}}\left[V_N^\star\right] - \gamma\mathbb{V}\mathrm{ar}\left[V_N^\star\right] = \mathbb{E}_{\mathbb{P}}\left[V_N'\right] - \gamma\mathbb{V}\mathrm{ar}\left[V_N'\right].$$

With this last equality, we can conclude they have the same variance and as a result, the same sharp ratio. ∎

Remark 12. *Maximizing the mean-variance objective function of zeros gradient portfolios, is equivalent to maximize their sharp ratio.*

Now, let's prove this important proposition on the impact of the risk aversion.

Proposition 13. *Let $(\gamma, \gamma') \in \mathbb{R}^*$, then $(\alpha_n^\star)_n$ is a γ-zero gradient strategy if and only if $(\frac{\gamma}{\gamma'}\alpha_n^\star)_n$ is a γ'-zero gradient strategy.*

Proof. Let $(\alpha_n^\star)_n$ be a γ-zero gradient strategy with the associated chaos coefficients $\beta^\star$. Then $\beta^\star$ is solution to (T^γ). Then with the previous notations

$$\mathbb{E}_{\mathbb{P}}\left[\nabla\mathcal{R}(\beta^\star)S_N^0\left[1 - 2\gamma S_N^0\left(\mathcal{R}(\beta^\star) - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^\star)]\right)\right]\right] = 0. \tag{27}$$

By using the decomposition of Proposition 4, $\mathcal{R}$ is a.s differentiable on $\mathbb{R}^m$ and

$$\nabla\mathcal{R}(\beta) = D - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu \times \mathrm{sign}(K_n^i \cdot \beta)K_n^i. \tag{28}$$

Let's now consider the portfolio

$$\mathcal{R}(\frac{\gamma}{\gamma'}\beta^\star) = V_0 + \frac{\gamma}{\gamma'}D \cdot \beta^\star - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu|K_n^i \cdot \frac{\gamma}{\gamma'}\beta^\star|.$$

We obtain the following equality

$$\mathcal{R}(\frac{\gamma}{\gamma'}\beta^\star) - \frac{\gamma}{\gamma'}\mathcal{R}(\beta^\star) = V_0(1 - \frac{\gamma}{\gamma'}).$$

We deduce that

$$\frac{\gamma}{\gamma'}\left(\mathcal{R}(\beta^\star) - \mathbb{E}_{\mathbb{P}}[\mathcal{R}(\beta^\star)]\right) = \mathcal{R}(\frac{\gamma}{\gamma'}\beta^\star) - \mathbb{E}_{\mathbb{P}}\left[\mathcal{R}(\frac{\gamma}{\gamma'}\beta^\star)\right]. \tag{29}$$

By using the expression (28), we also notice that

$$\nabla\mathcal{R}(\frac{\gamma}{\gamma'}\beta^\star) = D - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu \times \mathrm{sign}\left(K_n^i \cdot \frac{\gamma}{\gamma'}\beta^\star\right)K_n^i = \nabla\mathcal{R}(\beta^\star). \tag{30}$$

Finally, by inserting (29) and (30) in equality (27), we obtain

$$\mathbb{E}_{\mathbb{P}}\left[\nabla\mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^\star\right)S_N^0\left[1 - 2\gamma' S_N^0\left(\mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^\star\right) - \mathbb{E}_{\mathbb{P}}\left[\mathcal{R}\left(\frac{\gamma}{\gamma'}\beta^\star\right)\right]\right)\right]\right] = 0, \tag{31}$$

which finishes the proof. ∎

Corollary 14. *Let $(\gamma, \gamma') \in \mathbb{R}^*$, then $(\alpha_n^\star)_n$ is a γ-optimal strategy if and only if $(\frac{\gamma}{\gamma'}\alpha_n^\star)_n$ is a γ'-optimal strategy.*

Proof. Let $(\alpha_n^\star)_n$ be a γ-optimal strategy with the associated chaos coefficients $\beta^\star$. Then $\beta^\star$ is a solution to (J^γ). Firstly, according to Proposition 13, $\frac{\gamma}{\gamma'}\beta^\star$ is a solution to $(T^{\gamma'})$ with

$$\mathbb{E}_\mathbb{P}[F_{\gamma'}(\frac{\gamma}{\gamma'}\beta^\star)] = \mathbb{E}_\mathbb{P}\left[\mathcal{R}(\frac{\gamma}{\gamma'}\beta^\star)S_N^0 - \gamma'\left(\left(\mathcal{R}(\frac{\gamma}{\gamma'}\beta^\star) - \mathbb{E}_\mathbb{P}\left[\mathcal{R}(\frac{\gamma}{\gamma'}\beta^\star)\right]\right)S_N^0\right)^2\right].$$

But as $\beta \to \mathcal{R}(\beta) - V_0$ is a positive homogeneous function, ie $\forall\, u > 0,\ \mathcal{R}(u\beta) - V_0 = u\left(\mathcal{R}(\beta) - V_0\right)$, then

$$\mathbb{E}_\mathbb{P}[F_{\gamma'}(\frac{\gamma}{\gamma'}\beta^\star)] - V_0 S_N^0 = \mathbb{E}_\mathbb{P}\left[\frac{\gamma}{\gamma'}\left(\mathcal{R}(\beta^\star) - V_0\right)S_N^0 - \gamma'\left(\frac{\gamma}{\gamma'}\left(\mathcal{R}(\beta^\star) - \mathbb{E}_\mathbb{P}\left[\mathcal{R}(\beta^\star)\right]\right)S_N^0\right)^2\right]$$

$$= \frac{\gamma}{\gamma'}\left(\mathbb{E}_\mathbb{P}[F_\gamma(\beta^\star)] - V_0 S_N^0\right).$$

If $\exists\, \beta^1 \in \mathbb{R}^m$ such that $\mathbb{E}_\mathbb{P}[F_{\gamma'}(\frac{\gamma}{\gamma'}\beta^\star)] < \mathbb{E}_\mathbb{P}[F_{\gamma'}(\beta^1)]$ then using the previous equality $\mathbb{E}_\mathbb{P}[F_\gamma(\beta^\star)] < \mathbb{E}_\mathbb{P}[F_{\gamma'}(\frac{\gamma'}{\gamma}\beta^1)]$, which is impossible with the optimally of $\beta^\star$. We deduce that $\frac{\gamma}{\gamma'}\beta^\star$ is a solution to $(J^{\gamma'})$. The converse statement is obvious.

We have seen in (17), that the controls $(\alpha_n^\star)_n$ can be expressed as linear functions of $\beta^\star$. We can conclude that $(\alpha_n^\star)_n$ is an optimal solution for (E^γ) if and only if $(\frac{\gamma}{\gamma'}\alpha_n^\star)_n$ is an optimal solution for $(E^{\gamma'})$. ∎

Proposition 15. *Let $(\alpha_n^\star)_n$ be a γ-zero gradient strategy and $(V_n^\star)_n$ the associated γ-zero gradient portfolio. Then for all $u > 0$, $\mathrm{Sharp}(V_N^\star) = \mathrm{Sharp}(V_N^u)$, where $(V_n^u)_n$ is the associated portfolio of the strategy $(u\alpha_n^\star)_n$.*

Proof. By using that $\beta \to \mathcal{R}(\beta) - V_0$ is a positive homogeneous function, we have

$$\mathrm{Sharp}(V_N^\star) = \frac{\mathbb{E}_\mathbb{P}\left[\tilde{V}_N^\star S_N^0\right] - V_0 S_N^0}{\mathbb{V}\mathrm{ar}\left[\tilde{V}_N^\star S_N^0\right]^{\frac{1}{2}}} = \frac{u\mathbb{E}_\mathbb{P}\left[\mathcal{R}(\beta^\star)S_N^0\right]}{u\mathbb{V}\mathrm{ar}\left[\mathcal{R}(\beta^\star)S_N^0\right]^{\frac{1}{2}}} = \mathrm{Sharp}(V_N^u),$$

that finishes the proof. ∎

Proposition 16. *The sharp ratio of a zeros gradient portfolio is independent of its risk aversion.*

Proof. Let $(\gamma, \gamma') \in \mathbb{R}^\star$. According to Proposition 13, $(\alpha_n^\star)_n$ is a γ-zero gradient strategy if and only if $(\frac{\gamma}{\gamma'}\alpha_n^\star)_n$ is a γ'-zero gradient strategy. We respectively call $(V_n^\star)_n$ and $(V_n')_n$, the portfolios built with the strategy $(\alpha_n^\star)_n$ and $\frac{\gamma}{\gamma'}(\alpha_n^\star)_n$. Proposition 15 claims that $\mathrm{Sharp}(V_N^\star) = \mathrm{Sharp}(V_N')$. Then, the sharp ratio is not a function of γ nor γ'. ∎

Proposition 17. *Let $\mathcal{V} > 0$ and $(V_n^\star)_n$ be a γ-zero gradient portfolio. Then there exists γ' and a γ'-zeros gradient portfolio V' with $\mathrm{Sharp}(V_N') = \mathrm{Sharp}(V_N^\star)$ and with volatility $\mathcal{V}$.*

Proof. According to Proposition 15 and 13, for all $u > 0$, $(u\alpha_n^\star)_n$ is a $\frac{\gamma}{u}$-zeros gradient strategy and the associated $\frac{\gamma}{u}$-zeros gradient portfolio V_N^u has the same sharp ratio as $V_N^\star$. Consequently, according to Proposition 10,

$$\frac{\gamma}{u} = \frac{\mathrm{Sharp}(V_N^\star)}{2\mathbb{V}\mathrm{ar}(V_N^u)^{\frac{1}{2}}}.$$

Then by choosing $u = \frac{2\gamma\mathcal{V}}{\mathrm{Sharp}(V_N^\star)}$, the portfolio $(u\alpha_n^\star)_n$ verifies the conditions. ∎

Proposition 18. *Let $\gamma,\ \gamma' > 0$ s.t. $\gamma \neq \gamma'$. Let $V^\star$ (resp. V') be a γ-optimal portfolio (resp. γ'-optimal portfolio). Then, $\mathrm{Sharp}(V_N^\star) = \mathrm{Sharp}(V_N')$.*

Proof. Let $(\alpha_n')_n$ be the γ'-optimal strategy associated to V'. Let $V^\#$ the portfolio associated to the strategy $(\frac{\gamma'}{\gamma}\alpha_n')_n$. According to Proposition 14, $V^\#$ is a γ-optimal portfolio. Furthermore, using Proposition 15, we have $\mathrm{Sharp}(V_N') = \mathrm{Sharp}(V_N^\#)$. According to Proposition 11, γ-optimal portfolios have the same sharp ratio. So $\mathrm{Sharp}(V_N^\star) = \mathrm{Sharp}(V_N^\#)$, and finally $\mathrm{Sharp}(V_N^\star) = \mathrm{Sharp}(V_N')$. ∎

4 Numerical illustration

In this section, we implement the approach presented above and the sequential uni-period Markowitz model which is used as benchmark, see C.2. We consider the case where costs are ignored and considered. In order to maximise the rate of return of a portfolio while controlling its volatility at time T, an agent consecutively maximizes at time $0 = t_0 < \cdots < t_N = T$, its uni-period mean-var objective function. We assume that the agent has a uni-period risk aversion parameter γ_u. The agent has to consecutively solve for $n \in \{0, \ldots, N-1\}$,

$$\sup_{\alpha_{n+1}, \alpha_{n+1}^0 \in \mathbb{R}^d \times \mathbb{R}} \mathbb{E}_{\mathbb{P}}\left[V_{n+1} | \mathcal{F}_n\right] - \gamma_u \mathbb{V}\mathrm{ar}_{\mathbb{P}}\left[V_{n+1} | \mathcal{F}_n\right]$$

$$\text{subject to} \quad \Delta V_{n+1} = \alpha_{n+1} \cdot \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_{n+1}^0 - \sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| S_n^i \qquad \left(D_n^{\gamma_u}\right)$$

Further details on the implemented benchmark model and its link with risk aversion are presented in Appendix C.2. Propositions 26 and 27 state that the optimal portfolios derived from the sequential uni-period Markowitz model are not affected by risk aversion. Then, we evaluate the performance of both models by comparing their Sharpe ratios without considering risk aversion. We assume that the risky assets follow the dynamics (1) with constant parameters. Formally, the drift terms, the volatility matrices and the risk free rate are chosen deterministic and constant, ie. $\forall\, n \in \{0, \ldots, N\}\ \mu_n = \mu,\ \sigma_n = \sigma,\ r_n = r$. Along a first part, we compare the performances by choosing the same risk aversion parameter γ. As the sharp ratios of the estimated portfolios is independent of the aversion parameter (see Propositions 16, 18 , 26, 27), it can be used as an indicator of performance. Then, along a second part, we apply a framework for matching the risk aversion between uni-period and multi-period models. We are able to illustrate and compare the behaviour of our solutions on two realisations.

4.1 Model Parameters

We consider $d = 3$ assets, evolving during $N = 368$ days. Transactions are only available every 92 days. The model is described in Section 2. Instead of specifying a volatility matrix, we fix the marginal volatilities $(\hat{\sigma}^i)_{i \in \{1,2,3\}}$ and a correlation matrix ρ as in Tables 1 and 2.

Table 1: Model parameters

	S_1	S_2	S_3
μ	0.06	0.02	0.14
$\hat{\sigma}$	0.1	0.06	0.2

Table 2: Correlation matrix ρ of risky assets

	S_1	S_2	S_3
S_1	1	-0.2	0.3
S_2	-0.2	1	-0.2
S_3	0.3	-0.2	1

We fix a constant risk free rate $r = 0.001$. The initial portfolio wealth is $V_0 = 100$. The implementation parameters are summarized in the following table

Table 3: Implementation parameters

nb.traj MonteCarlo	nb.traj. calibration	nb.traj. test	N	p	Chaos degree
10^7	10^5	10^5	368	92	2

The first step uses a high number of trajectories to estimate the Wiener chaos expansion of the risky assets. Then, we split a new sample into two parts. The first part is used to run the descent gradient algorithm to calibrate and find an optimal portfolio, while the second part is used to compute performance indicators for comparing the models. In presence of costs, we assume that the first position is free of charge.

4.2   Same risk aversion parameter

In this first experiment, we choose the same risk aversion parameter $\gamma = \gamma_u$ for the different approaches. Since the objective functions are not the same between multi-period and uni-period models, the agents have not the same risk aversion. Therefore volatility and rates of return are not comparable. Nevertheless the sharp ratio is a relevant indicator for comparing performances of optimal portfolios based on different risk aversions. Indeed, the sharp ratios of estimated portfolios are independent from the risk aversion in every models according to Propositions 16, 18, 26 and 27.
The implementation parameters are summarized in the following table.

Table 4: Implementation parameters for Multiperiod model

risk aversion	batch size	iteration	learning rate	cost(%)
0.05	100	1000	8.5	1

We compare five models; two sequential uni-period versions and two multi-period versions, where cost are on one hand ignored and on the other, considered. Moreover, we add in the benchmark the famous equal weight portfolio to measure the performance of the other approaches. We refer to our approach as *Multi-period with costs*. We refer to Appendix C for further details on the four other benchmark models. Formally, our aim is to evaluate the performance of our method compared to the more basic approaches commonly used. The main results can be found in Table 5.

Table 5: Estimated metrics for evaluating models performances

	rate of return(%)	vol(%)	Min-Var	sharp ratio
Multi-period ignoring cost	13.24	12.17	105.83087	1.07939
Multi-period with costs	12.31	11.00	106.26365	1.11033
Sequential uni-period ignoring cost	10.44	10.00	105.43191	1.03316
Sequential uni-period with costs	11.00	10.72	105.24599	1.01606
equal weight	5.69	5.70	104.06893	0.98125

We analyse here, the difference in sharp ratios. By focusing on uni-period models, it is interesting to notice that ignoring costs seems to be better than considering them. The myopic effect, joined with the consideration of costs, may make the optimal strategy rigid and inflexible. The agent's myopic behavior explains why they do not see the benefit of paying costs for short-term positions. Therefore considering cost almost freezes the strategy and can explain that the performance is not as good as if we have ignored them. Obviously these remarks, are dependent of the chosen parameter ν, which determines the weight of costs in the transaction.
In contrast, considering costs in the multi-period version is a significant improvement. According to the choices of parameters, a difference of 0.03 in sharp ratio is not negligible. A multi-period model targets a final value and must adapt its positions according to the variations of the environment. These changes in positions are typically more significant than in myopic strategies, and thus, the impact of costs becomes more critical. Ignoring costs can have a considerable impact on strategy, leading to performance deterioration.
Undoubtedly, the myopic effect has a negative impact on performances. The difference in sharp

ratios between uni-period and multi-period models is substantial. Therefore, we advise to use multi-period models, despite their greater complexity. In that case, costs must not be ignored as highlighted by the variability of the strategies.

4.3   Same risk aversion level

In a second experiment, we want to compare the optimal portfolios submitted to transaction costs in uni-period and multi-period settings with the same level of risk. According to Proposition 17, we can link the risk aversions γ and γ_u in both models to ensure the same level of risk. This experiment aims to illustrate the comparison in terms of rates of returns. We can also directly compare portfolio trajectories.

4.3.1   Performances

Our objective is to obtain a locally optimal multi-period portfolio that carries the same level of risk as the optimal sequential uni-period portfolio. We adopt the approach described in Proposition 17. We use the $\gamma = 0.05$ locally optimal multi-period portfolio estimated in the previous section to build another locally optimal multi-period portfolio with the same volatility as the optimal uni-period portfolio. With a uni-period risk aversion $\gamma_u = 0.05$, we have obtained a volatility of 10.72% for the optimal uni-period portfolio. The γ_u locally optimal multi-period portfolio estimated in the previous experiment has a harp ratio equal to 1.11033. According to Proposition 17, we can build another locally optimal multi-period portfolio with the same sharp ratio and a volatility of 10.72%. To reach this volatility, a multi-period risk aversion parameter $\gamma = 0.0518$ is estimated by using (23). The main results can be found in Table 6

Table 6: Results

	risk aversion	rate of return(%)	Min-Var	sharp ratio
Multi-period with costs	0.0518	11.89	105.9372	1.11033
Sequential uni-period with costs	0.0500	11.00	105.24599	1.01606

The multi-period model is obviously the most performing. The difference between the rates of return is almost 1%. This difference is important according to a level of risk of 12.72%.
After comparing the performances of the optimal portfolios with the same risk aversion. We aim to illustrate their behaviour by showing trajectories from two different situations, A and B. To ensure consistency, we use the same framework as before to match the risk aversions.

4.3.2   Behaviour on realisation A

To illustrate the comparison we present in Figures 1, one particular sample path of assets.
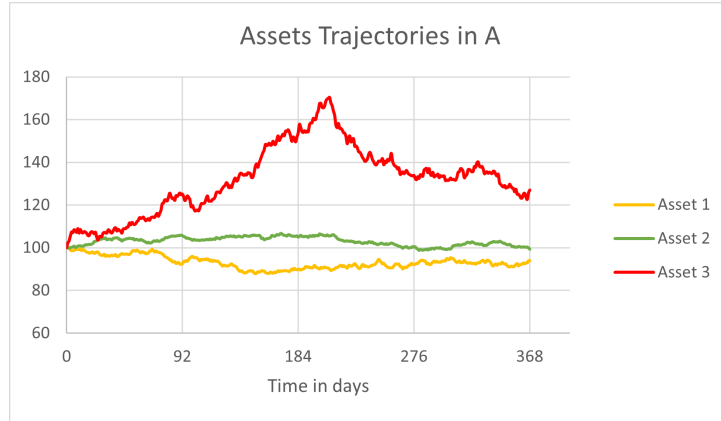
Figure 1: Asset trajectories in A

It can be observed that the curves for assets 1 and 2 are relatively flat and symmetric. Asset 2 remains above its initial value. On the other hand, asset 1 remains below its initial value. Asset 3 experiences significant growth, reaching a peak of 170 before collapsing towards the end of the period to finish just above 120.

We respectively present in Figures 2, 3, 4, 5, the portfolios values, the cumulative costs and the controls of the uni-period model considering cost against the multi-period model considering cost, whose performances have been presented in Table 6.
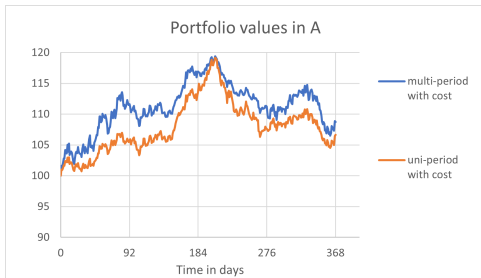
Figure 2: Portfolio values in A
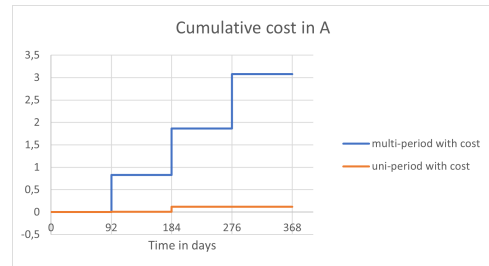
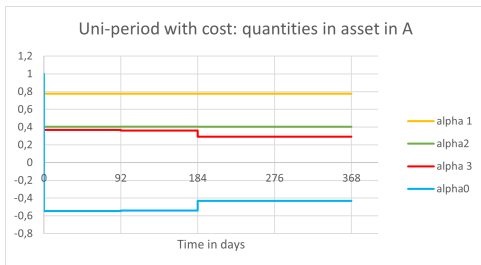Figure 3: Cumulative cost in A

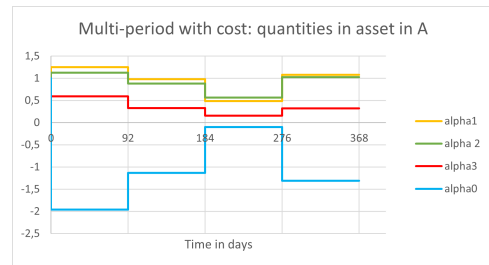Figure 4: Controls of uni-period model with costs in A

Figure 5: Controls of multi-period model with costs in A

By analysing Figure 2, both portfolio trajectories follow the trend of asset 3. This is not surprising given the flat evolution of assets 1 and 2. The multi-period portfolio outperforms at every moment, for this particular realisation.

Figures 3 highlights that the uni-period strategy pays few cost compared to the multi-period one. This can be attributed to the myopic vision of the uni-period strategy, which results in a very flat strategy that is not very sensitive to asset and wealth variations. It can not embrace the benefit of sacrificing money in paying costs to anticipate future evolution. The high costs required to make a reversal of strategy further reinforce this inflexibility, which partly explains the low sharp ratio estimated in Section 4.2. The long position of this portfolio explains the high dependence with asset 3 and the decline of its value after the middle of the period.

The multi-period portfolio follows a completely different policy. While initially adopting a more aggressive long strategy than the uni-period portfolio, the level of risk taken is significantly higher. The risk taken is much more important. The portfolio performs well during the growth of asset 3 until day 276, at which point the strategy begins to reverse as assets are gradually sold. Subsequently, in response to the decline of asset 3, the strategy undergoes another shift, with new asset quantities being purchased.

4.3.3 Behaviour on realisation B

To illustrate the comparison, we present in Figures 6, a new realisation of the assets. This scenario looks like the previous one, as Asset 1 and 2 exhibit minimal changes while Asset 3 experiences a large increase without any significant decrease.
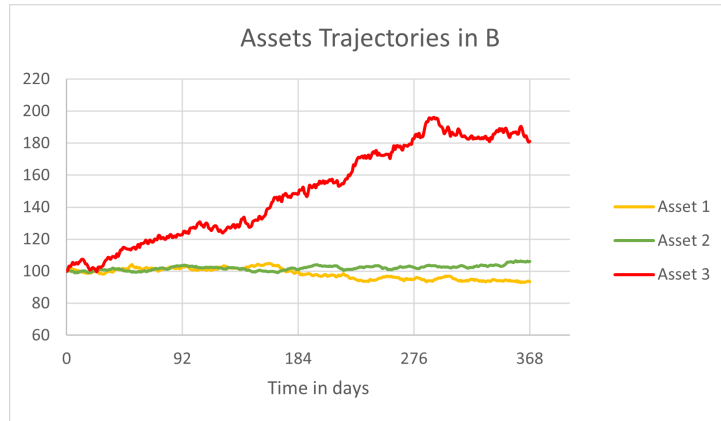

Figure 6: Asset trajectories in B

We present respectively in Figures 7, 8, 9, 10, the portfolios values, the cumulative costs and the controls of the uni-period model against the multi-period model with costs, whose performance results have been presented in Table 6.
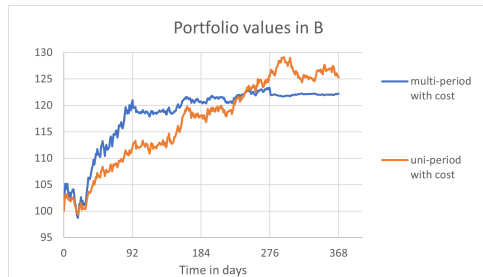
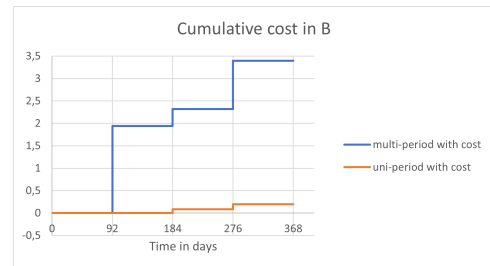

Figure 7: Portfolio values in B
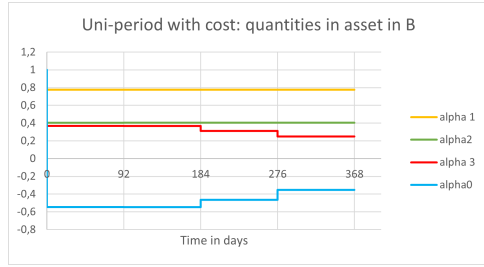

Figure 8: Cumulative cost in B

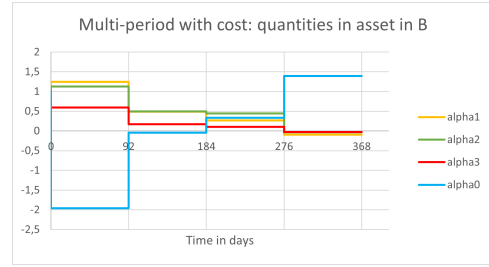Figure 9: Controls of uni-period model with costs in B

Figure 10: Controls of multi-period model with costs in B

Although both portfolios experience growth over the period, the uni-period portfolio with costs outperforms the multi-period portfolio at the end. It is important to note that this outcome is specific to this particular trajectory but cannot be generalised. The uni-period portfolio appears to closely track the trend of asset 3, which continues to perform toward the end of the period. As for the realisation A, the positions of the uni-period strategy maintains predominantly long and flat positions. The multi-period strategy is considerably different and more versatile. Following 276 days of long positions, all assets are sold. This reverse of positions explains the plateau reached. After performing , the portfolio eliminates all risk to insure a positive performance. Even if Asset 3 were to continue its upward trend, this approach remains prudent. The portfolio achieves a higher value than anticipated, making it logical to seek to protect these gains.

5   Conclusion

In this paper, we have presented an efficient numerical method to solve multi-stage portfolio allocation that involve multiple assets and transaction costs. We applied a stochastic descent gradient algorithm to find the Wiener chaos expansion of an optimal portfolio. The method can be extended to handle realistic constraints as no-shorting and is computationally tractable. explored theve also studied the link between risk aversion and optimal portfolios subject to transaction costs, with two findings standing out. Firstly, we have proved that the Sharpe ratio of a locally optimal portfolio with transaction costs does not depend on its risk aversion. Secondly, we have established that all optimal portfolios have the same Sharpe ratio. We have used this result to compare our approach to a competitive benchmark, based on the sequential uni-period mean-variance strategy. We have highlighted the efficiency of our approach and we have showcased our benchmark by analyzing the performance of our models on two selected trajectories. Costs must not be ignored in multi-period setting since reverses of strategy are frequent. On the other hand, considering costs for uni-period models is still a topic of debate. The myopic effect with the consideration of costs may freeze the strategy and negatively impact performance. As expected, the multi-period model is more intricate, but outperforms the uni-period models.

Acknowledgments

A   Wiener chaos expansion properties

In this Appendix, we present several fundamental properties on Wiener chaos expansion and Hermite polynomials, use-full in our study. We refer to [1], [21] [30] for theoretical details.

Let H_i be the i-th Hermite polynomial defined by

$$H_0(x) = 1 : \ H_i(x) = (-1)^i e^{\frac{x^2}{2}} \frac{d^i}{dx^i}(e^{\frac{-x^2}{2}}), \text{ for } i \geq 1.$$

For $\lambda \in \mathbb{N}^n$, $x \in \mathbb{R}^n$, we define

$$H_\lambda^\otimes(x) = \prod_{i=1}^n H_{\lambda_i}(x_i).$$

We have the following properties

- 1. For $i \geq 1$, $H_i' = iH_{i-1}$ with $H_{-1} = 0$.

- 2. For $x, y \in \mathbb{R}$,
$$H_i(x+y) = \sum_{r=0}^i \binom{i}{r} x^r H_{i-r}(y).$$

- 3. Let X, Y be two random variables with joint Gaussian distribution such that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2) = 1$. Then $\forall \ i, j \geq 0$, we have
$$\mathbb{E}\left[H_i(X)H_j(Y)\right] = 1_{(i=j)} i! (\mathbb{E}\left[XY\right])^i.$$

- 4. Let $\lambda, \lambda' \in \mathbb{R}^n$ and X, Y, be two independent n-multivariate i.i.d standard normal vectors, then
$$\mathbb{E}\left[H_\lambda^\otimes(X) H_{\lambda'}^\otimes(Y)\right] = 0.$$

- 5. Let $\lambda, \lambda' \in \mathbb{R}^n$ and X a n-multivariate i.i.d standard normal vector, then
$$\mathbb{E}\left[H_\lambda^\otimes(X) H_{\lambda'}^\otimes(X)\right] = 1_{(\lambda=\lambda')} \prod_{i=1}^n \lambda_i! \ .$$

- Let $\lambda \in \mathbb{R}^n \neq 0_{\mathbb{R}^n}$ and X be a n-multivariate i.i.d standard normal vector. Then we have
$$\mathbb{E}\left[H_\lambda^\otimes(X)\right] = 0.$$

The following theorem represents the basics of our approach. It provides a discrete-time general representation with a convergent series expansion. This expansion is introduced and proved in [1, Theorem 2.1].

Theorem 19. *Let $Y \in L^2(\Omega, \mathcal{F}_N, \mathbb{Q})$, then the following Wiener chaos expansion holds*

$$Y = \mathbb{E}_\mathbb{Q}[Y] + \sum_{\lambda \in (\mathbb{N}^N)^d} \beta_\lambda \prod_{j=1}^d \prod_{i=1}^N H_{\lambda_i^j}\left(\frac{W_{t_i}^{\mathbb{Q}j} - W_{t_{i-1}}^{\mathbb{Q}j}}{\sqrt{t_i - t_{i-1}}}\right)$$
$$= \mathbb{E}_\mathbb{Q}[Y] + \sum_{\lambda \in (\mathbb{N}^N)^d} \beta_\lambda H_\lambda^\otimes(\Delta \widehat{W}^\mathbb{Q}).$$

We call $C_K^\mathbb{Q}(Y)$ the truncated expansion of order K

$$C_K^\mathbb{Q}(Y) = \mathbb{E}_\mathbb{Q}[Y] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \beta_\lambda H_\lambda^\otimes(\Delta \widehat{W}^\mathbb{Q}).$$

We have $\lim_{K \to +\infty} \mathbb{E}_{\mathbb{Q}} \left[|Y - C_K^{\mathbb{Q}}(Y)|^2 \right] = 0$. Let us define $m \in \mathbb{N}$ be the number of coefficients λ appearing in the chaos expansion. We have $m = \#\{\lambda \in (\mathbb{N}^N)^d : |\lambda|_1 \leq K\} = \binom{Nd+K}{Nd}$. We slightly abuse the above definition and also write

$$C_K^{\mathbb{Q}}(\beta) = \mathbb{E}_{\mathbb{Q}}[Y] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \beta_\lambda H_\lambda^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}).$$

Proposition 20. *Let* $Y \in L^2(\Omega, \mathcal{F}_N, \mathbb{Q})$, *then*

$$\mathbb{E}_{\mathbb{Q}} \left[C_K^{\mathbb{Q}}(Y) | \mathcal{F}_n \right] = C_K^{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}} \left[Y | \mathcal{F}_n \right]) = C_{K,n}^{\mathbb{Q}}(Y), \tag{32}$$

where

$$C_{K,n}^{\mathbb{Q}}(Y) = \mathbb{E}_{\mathbb{Q}}[Y] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K, \\ \forall j \leq d, \forall l \geq n+1 : \lambda_l^j = 0}} \beta_\lambda H_\lambda^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}) = \mathbb{E}_{\mathbb{Q}}[Y] + \sum_{\substack{\lambda \in (\mathbb{N}^n)^d \\ |\lambda|_1 \leq K}} \beta_\lambda H_\lambda^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}).$$

The Wiener chaos expansion of the conditional expectation of the variable Y is obtained by truncating the terms which are not $\mathcal{F}_n$-measurable.

Proposition 21. *Let A, B two random variables of $L^4(\Omega, \mathcal{F}_N, \mathbb{Q})$, with respectively Wiener chaos expansions such that*

$$A = \sum_{\lambda_1 \in (\mathbb{N}^N)^d} \eta_{\lambda_1} H_{\lambda_1}^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}), \; B = \sum_{\lambda_2 \in (\mathbb{N}^N)^d} \beta_{\lambda_2} H_{\lambda_2}^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}).$$

Let $(C_K^{\mathbb{Q}}(A)C_K^{\mathbb{Q}}(B))_K$ stand for the truncated expansion of order K of $C_K^{\mathbb{Q}}(A)C_K^{\mathbb{Q}}(B)$. Then, $C_K^{\mathbb{Q}}(AB)$ is exactly $(C_K^{\mathbb{Q}}(A)C_K^{\mathbb{Q}}(B))_K$.

Proof. Let us write the chaos expansion of $AB = \sum_{\lambda \in (\mathbb{N}^N)^d} \mu_\lambda H_\lambda^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}})$. AB also writes $\sum_{\lambda_1, \lambda_2 \in (\mathbb{N}^N)^d} \eta_{\lambda_1} \beta_{\lambda_2} H_{\lambda_1}^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}) H_{\lambda_2}^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}})$. $\left\{ H_\lambda^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}), \; \lambda \in (\mathbb{N}^N)^d \right\}$ is an orthonormal basis so the decomposition is unique. We can then identify

$$\forall \; \lambda \in (\mathbb{N}^N)^d, \; \mu_\lambda = \sum_{\substack{\lambda_1, \lambda_2 \in (\mathbb{N}^N)^d \\ |\lambda_1|_1 + |\lambda_2|_1 = |\lambda|_1}} \eta_{\lambda_1} \beta_{\lambda_2}.$$

Therefore, we deduce that

$$C_K^{\mathbb{Q}}(AB) = \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \mu_\lambda H_\lambda^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}) = \sum_{\substack{\lambda_1, \lambda_2 \in (\mathbb{N}^N)^d \\ |\lambda_1|_1 + |\lambda_2|_1 \leq K}} \eta_{\lambda_1} \beta_{\lambda_2} H_{\lambda_1}^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}) H_{\lambda_2}^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}) = (C_K^{\mathbb{Q}}(A)C_K^{\mathbb{Q}}(B))_K.$$

Proposition 22. *Let A and B, two $\boldsymbol{F}$-martingales. Let us consider the following Wiener chaos expansions of their final value.*

$$C_K^{\mathbb{Q}}(A_N) = \mathbb{E}_{\mathbb{Q}}[A] + \sum_{\substack{\lambda \in (\mathbb{N}^N)^d \\ |\lambda|_1 \leq K}} \eta_\lambda H_\lambda^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}),$$

$$C_K^{\mathbb{Q}}(B_N) = \mathbb{E}_{\mathbb{Q}}[B] + \sum_{\substack{\lambda' \in (\mathbb{N}^N)^d \\ |\lambda'|_1 \leq K}} \beta_{\lambda'} H_{\lambda'}^{\bigotimes}(\Delta \widehat{W}^{\mathbb{Q}}).$$

Then, we have the following expansion of order K,

$$C_K^{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}\left[\Delta A_{n+1}\Delta B_{n+1}|\mathcal{F}_n\right]) = \sum_{\substack{\lambda,\lambda' \in (\mathbb{N}^{n+1})^d \\ |\lambda|_1+|\lambda'|_1 \leq K}} \eta_\lambda \beta_{\lambda'} H_{\lambda_{1:n}}^{\bigotimes}(\Delta\widehat{W}_{1:n}^{\mathbb{Q}}) H_{\lambda'_{1:n}}^{\bigotimes}(\Delta\widehat{W}_{1:n}^{\mathbb{Q}}) 1_{(\lambda_{n+1}=\lambda'_{n+1}\neq 0)} \prod_{j=1}^{d} \lambda_{n+1j}! \; .$$

Proof. As A and B are $\boldsymbol{F}$-martingales and using Proposition 20, we have

$$C_K^{\mathbb{Q}}(\Delta A_{n+1}) = \mathbb{E}_{\mathbb{Q}}\left[C_K^{\mathbb{Q}}(A_N)|\mathcal{F}_{n+1}\right] - \mathbb{E}_{\mathbb{Q}}\left[C_K^{\mathbb{Q}}(A_N)|\mathcal{F}_n\right] = C_{K,n+1}^{\mathbb{Q}}(A_N) - C_{K,n}^{\mathbb{Q}}(A_N).$$

This equality also holds for ΔB_n. We deduce the following expansion

$$C_K^{\mathbb{Q}}(\Delta A_{n+1}) = \sum_{\substack{\lambda \in (\mathbb{N}^{n+1})^d \\ |\lambda|_1 \leq K, \\ \lambda_{n+1}\neq 0}} \eta_\lambda H_\lambda^{\bigotimes}(\Delta\widehat{W}^{\mathbb{Q}}), \text{ and } C_K^{\mathbb{Q}}(\Delta B_{n+1}) = \sum_{\substack{\lambda \in (\mathbb{N}^{n+1})^d \\ |\lambda|_1 \leq K, \\ \lambda_{n+1}\neq 0}} \beta_\lambda H_\lambda^{\bigotimes}(\Delta\widehat{W}^{\mathbb{Q}}).$$

The announced formula yields from Proposition 21 and the conditional expectation property of Wiener chaos expansions. $\blacksquare$

B   Differentiability

In this appendix, we study the differentiability of $\mathcal{R}$, $(\beta,\theta) \to \mathbb{E}_{\mathbb{P}}[G_\gamma(\beta,\theta)] = \mathbb{E}_{\mathbb{P}}\left[\mathcal{R}(\beta)S_N^0 - \gamma\left((\mathcal{R}(\beta)-\theta)S_N^0\right)^2\right]$. Let us introduce the following three spaces for $n \leq 1$,

$$\mathcal{P}_n = \left\{\sum_{k=1}^{K} b_k H_k(\Delta W_n), \; (b_k)_k \; \mathcal{F}_{k-1} - \text{measurable}, \; K \in \mathbb{N}^*, \; \exists k \in \{1,..K\}, \; \mathbb{Q}(b_k = 0) = 0\right\},$$

$$\mathcal{E}_n = \left\{\sum_{l=0}^{L} a_l e^{d_l \Delta W_n}, \; (a_l)_l, \; (c_l)_l \; \mathcal{F}_{k-1} - \text{measurable}, \; L \in \mathbb{N}^*, \; \exists l \in \{1,..L\}, \; \mathbb{Q}(a_l = 0) = \mathbb{Q}(c_l = 0) = 0\right\}.$$

$$\mathcal{P}_n \otimes \mathcal{E}_n = \left\{\sum_{l=1}^{L} p_l \epsilon_l, \; L \in \mathbb{N}^*, \; p_l \in \mathcal{P}_n, \; \epsilon_l \in \mathcal{E}_n\right\}.$$

Note that if $X_1 \in \mathcal{P}_n \otimes \mathcal{E}_n$, $X_2 \in \mathcal{P}_n \otimes \mathcal{E}_n$, then $X_1 X_2 \in \mathcal{P}_n \otimes \mathcal{E}_n$. We also define

$$Y_n = \mathbb{E}_{\mathbb{Q}}\left[\left(\frac{\tilde{S}_{n+1}}{\tilde{S}_n} - 1\right)\left(\frac{\tilde{S}_{n+1}}{\tilde{S}_n} - 1\right)^T |\mathcal{F}_n\right] = \left[e^{\sigma_n^k \cdot \sigma_n^j (t_{n+1}-t_n)} - 1\right]_{k,j},$$

$$\mathcal{B}_n = \mathbb{E}_{\mathbb{Q}}\left[C_K^{\mathbb{Q}}(\Delta\tilde{X}_{n+1})C_K^{\mathbb{Q}}(\Delta\tilde{S}_{n+1})|\mathcal{F}_n\right]_K.$$

Note that Y_n is a.s. invertible.

Proposition 23. *If $\forall i \in \{1,\ldots,d\}$, $\forall X_1 \in (\mathcal{P}_n \otimes \mathcal{E}_n)^d$, $\forall X_2 \neq 0$ a.s, $\mathcal{F}_{n-1} - $ measurable, $\mathbb{Q}\left((Y_n^{-1})_i \cdot X_1 = X_2|\mathcal{F}_{n-1}\right) = 0$, then $\mathcal{R}$ is almost surely differentiable on $\mathcal{Z}$.*

Proof. Let us prove that the $\mathbb{Q}(\alpha_{n+1}^i = \alpha_n^i) = 0$, $\forall 1 \leq i \leq d$, $0 \leq n \leq N-1$. We have seen that for $\beta \in \mathcal{Z}$, $\mathbb{Q}(\alpha_n^i = 0) = 0$. Then we assume now that $\alpha_n^i \neq 0$ a.s. Let Diag be the operator which associates to a vector $(d^1,\ldots,d^k)$ the matrix $\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_k \end{bmatrix}$.

We have $\{\alpha_{n+1}^i = \alpha_n^i\} \iff \left[\mathbb{E}_{\mathbb{Q}}\left[\Delta\tilde{S}_{n+1}\Delta\tilde{S}_{n+1}^T|\mathcal{F}_n\right]^{-1}\mathbb{E}_{\mathbb{Q}}\left[C_K^{\mathbb{Q}}(\Delta\tilde{X}_{n+1})C_K^{\mathbb{Q}}(\Delta\tilde{S}_{n+1})|\mathcal{F}_n\right]_K\right]_i = \alpha_n^i$

$$\iff \left[\text{Diag}(\tilde{S}_n)^{-1}Y_n^{-1}\text{Diag}(\tilde{S}_n)\mathcal{B}_n\right]_i = \alpha_n^i.$$

For $(\beta_\lambda)_\lambda \in \mathcal{Z}$, $\forall j \in \{1, \ldots, d\}$ and knowing $\mathcal{F}_{n-1}$, $\mathcal{B}_n^j \in \mathcal{P}_n$ a.s. and $\tilde{S}_n^j \in \mathcal{E}_n$. Then we have

$$\left[\mathrm{Diag}(\tilde{S}_n)^{-1} Y_n^{-1} \mathrm{Diag}(\tilde{S}_n) \mathcal{B}_n\right]_i = \sum_{j=1}^d (Y_n^{-1})_{ij} (\tilde{S}_n^i)^{-1} X_j,$$

with $X_j \in \mathcal{P}_n \otimes \mathcal{E}_n$. We deduce that $(\tilde{S}_n^i)^{-1} X_j \in \mathcal{P}_n \otimes \mathcal{E}_n$. With the assumption made on the form of the volatility and Y_{n-1}, we deduce that $\mathbb{Q}(\alpha_{n+1}^i = \alpha_n^i | \mathcal{F}_{n-1}) = 0$. Therefore $\mathbb{Q}(\alpha_{n+1}^i = \alpha_n^i) = \mathbb{E}_\mathbb{Q}\left[\mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{[\alpha_{n+1}^i = \alpha_n^i]} | \mathcal{F}_{n-1}\right]\right] = 0$. Finally, $\mathcal{R}$ is as. differentiable on $\mathcal{Z}$. ∎

Proposition 24. *Under the assumptions of Proposition 23, the function*

$$(\beta, \theta) \longmapsto \mathbb{E}_\mathbb{P}[G_\gamma(\beta, \theta)] = \mathbb{E}_\mathbb{P}\left[\mathcal{R}(\beta) S_N^0 - \gamma \left((\mathcal{R}(\beta) - \theta) S_N^0\right)^2\right]$$

is differentiable on $\mathcal{Z} \times \mathbb{R}$.

Proof. Let $r > 0$, $r_\theta > 0$ and $\beta \in \mathbb{R}^m$, $\theta \in \mathbb{R}$ such that $|\beta| \le r$, $|\theta| \le r_\theta$. According to Proposition 4, we have

$$|\mathcal{R}(\beta)| \le V_0 + |D||\beta| + \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |K_n^i| |\beta|.$$

Since we have $\forall i \in \{1, \ldots, d\}$, $n \in \{0, \ldots, N-1\}$, $\forall p \ge 1$, $\mathbb{E}_\mathbb{P}\left[|K_n^i|^p\right] + \mathbb{E}_\mathbb{P}\left[|D|^p\right] < \infty$, we have $\mathbb{E}_\mathbb{P}\left[\sup_{|\beta| \le r} |\mathcal{R}(\beta)|\right] < \infty$ and $\mathbb{E}_\mathbb{P}\left[\sup_{|\beta| \le r} |\mathcal{R}(\beta)|^2\right] < \infty$. We bound

$$|G_\gamma(\beta, \theta)| \le |\mathcal{R}(\beta)| + 2\gamma |\mathcal{R}(\beta)|^2 + 2\gamma \theta^2$$
$$\le \sup_{|\beta| \le r} |\mathcal{R}(\beta)| + 2\gamma \sup_{|\beta| \le r} |\mathcal{R}(\beta)|^2 + 2\gamma r_\theta^2.$$

So $\mathbb{E}_\mathbb{P}[|G_\gamma(\beta, \theta)|] < \infty$. According to Proposition 23, $\mathcal{R}$ is a.s. differentiable on $\mathcal{Z}$, then G is also a.s. differentiable on $\mathcal{Z} \times \mathbb{R}$. We also have

$$|\nabla \mathcal{R}(\beta)| = |D - \sum_{i=1}^d \sum_{n=0}^{N-1} \nu \times \mathrm{sign}(K_n^i \cdot \beta) K_n^i| \le |D| + \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |K_n^i|.$$

Following the same arguments, we prove that $\forall j \in \{1, \ldots, m\}$

$$|\nabla_{\beta_j} G_\gamma(\beta, \theta)| \le |\nabla \mathcal{R}(\beta)|_j + 2\gamma |\nabla \mathcal{R}(\beta)|_j |\mathcal{R}(\beta) - \theta|$$
$$\le |D_j| + \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |(K_n^i)_j| + 2\gamma \left(|D_j| + \sum_{i=1}^d \sum_{n=0}^{N-1} \nu |(K_n^i)_j|\right) \left(\sup_{|\beta| \le r} |\mathcal{R}(\beta)| + r_\theta\right).$$

We conclude using Cauchy Schwartz' inequality that $\mathbb{E}_\mathbb{P}[\sup_{|\beta| \le r, |\theta| \le r_\theta} |\nabla_{\beta_j} G_\gamma(\beta, \theta)|] < \infty$. We also dominate

$$|\nabla_\theta G_\gamma(\beta, \theta)| \le 2\gamma |\mathcal{R}(\beta) - \theta| \le 2\gamma \sup_{|\beta| \le r} |\mathcal{R}(\beta)| + r_\theta.$$

We apply Lebesgue's theorem to conclude that $(\beta, \theta) \to \mathbb{E}_\mathbb{P}[G_\gamma(\beta, \theta)]$ is differentiable on $\mathcal{Z} \times \mathbb{R}$ and $\nabla (\mathbb{E}_\mathbb{P}[G(\beta, \theta)]) = \mathbb{E}_\mathbb{P}[\nabla G_\gamma(\beta, \theta)]$. ∎

C   Benchmark models

Here, we present the different models which are used to benchmark and compare the multi-period proposed approach. The first model described in the multi-period approach which ignores transaction costs. A second comparison is done with the sequential uni-period mean–variance Markowitz framework. We present two version of this approach, when cost are ignored and considered. We also study the link of risk aversion on optimal solutions.

C.1 Multi-period allocation Ignoring transaction costs

An easy approach consists in applying the framework described in Section 3, while ignoring costs. The costs are removed from the portfolio value but do not impact the strategy. Theoretically, we look for an optimal strategy whose costs are refunded. The control $(\alpha_n^\star)_n$, solution to (E^γ), are obtained by temporally setting $\nu = 0$ during the resolution. Formally, we directly solve

$$
\begin{aligned}
&\sup_{(\alpha_n)_{n \in \{1,N\}}} && \mathbb{E}_{\mathbb{P}}\left[U(\tilde{X}_N S_N^0)\right] \\
&\text{subject to} && \tilde{X}_0 = V_0, \ \forall n, (\alpha_n^0, \alpha_n)_n \ \boldsymbol{F} - Pred \\
& && \tilde{X}_{n+1} = \tilde{X}_n + \alpha_{n+1} \cdot \Delta \tilde{S}_{n+1}
\end{aligned} \qquad (E_0^\gamma)
$$

Then, the real quantity in risk free asset at stake is deduced with the auto-financing relation and by removing costs with the real value of ν. Formally, after maximizing (E_0^γ), the generated cost $\mathcal{C}_N$ is computed and removed to obtain the real portfolio value $\tilde{V}_N = \tilde{X}_N - \mathcal{C}_N$. This strategy is implemented in order to observe the effect of costs on optimal strategies.

C.2 Benchmark: a sequential uni-period Markowitz portfolio allocation

In this section, we present an alternative approach, intended to serve as a benchmark. We would like to compare the performance of our method to a method traditionally applied by asset managers. We propose to implement the sequential uni-period mean–variance Markowitz framework. This approach also corresponds to the one-time-step Model Predictive Control (MPC) described in [23]. In order to maximise the rate of return of a portfolio while controlling its volatility at time T, an agent consecutively maximizes at time $0 = t_0 < \cdots < t_N = T$, its uni-period mean-var objective function. We assume that the agent has an uni-period risk aversion parameter γ_u. The agent has to consecutively solve for $n \in \{0, \ldots, N-1\}$,

$$
\begin{aligned}
&\sup_{\alpha_{n+1} \in \mathbb{R}^d, \alpha_{n+1}^0 \in \mathbb{R}} && \mathbb{E}_{\mathbb{P}}\left[V_{n+1} | \mathcal{F}_n\right] - \gamma_u \mathbb{V}\mathrm{ar}_{\mathbb{P}}\left[V_{n+1} | \mathcal{F}_n\right] \\
&\text{subject to} && \Delta V_{n+1} = \alpha_{n+1} \cdot \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_{n+1}^0 - \sum_{i=1}^d \nu |\alpha_{n+1}^i - \alpha_n^i| S_n^i
\end{aligned}
$$

Equivalently, the problem can be written

$$
\begin{aligned}
&\sup_{\alpha_{n+1} \in \mathbb{R}^d, \alpha_{n+1}^0 \in \mathbb{R}} && \mathbb{E}_{\mathbb{P}}\left[X_{n+1} - \mathcal{C}_{n+1} S_{n+1}^0 | \mathcal{F}_n\right] - \gamma_u \mathbb{V}\mathrm{ar}_{\mathbb{P}}\left[X_{n+1} - \mathcal{C}_{n+1} S_{n+1}^0 | \mathcal{F}_n\right] \\
&\text{subject to} && \Delta X_{n+1} = \alpha_{n+1} \cdot \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_n^0 \\
& && \mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^i - \alpha_n^i| S_n^i}{S_n^0}
\end{aligned} \qquad (D_n^{\gamma_u})
$$

We present two versions of this model. A naive version, where the asset manager pays the cost but does not take the amount into account in its strategy (by ignoring them), is presented in Section C.2.1. In a second version, described in Section C.2.2, the investor consecutively solves $(D_n^{\gamma_u})$ and bases his strategy according to the cost he will have to pay.

C.2.1 Ignoring costs

We aim to propose here, a simplified solution of the initial sequential uni-period problem $(D_n^{\gamma_u})$. The underlying idea is the same as the multi-period version described in Section C.1. At each time step, the agent maximizes its mean-var utility function as though costs are refunded. Costs

do not impact the strategy and are just removed from the portfolio value at the end. The agent has to consecutively solve for $n \in \{0, \dots, N-1\}$,

$$\sup_{\alpha_{n+1} \in \mathbb{R}^d, \alpha_{n+1}^0 \in \mathbb{R}} \quad \mathbb{E}_\mathbb{P}[X_{n+1}|\mathcal{F}_n] - \gamma_u \mathbb{V}\mathrm{ar}_\mathbb{P}[X_{n+1}|\mathcal{F}_n] \qquad (\mathcal{Y}_n^{\gamma_u})$$
$$\text{subject to} \qquad \Delta X_{n+1} = \alpha_{n+1} \cdot \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_n^0$$

Let $(\alpha_{n+1}^\star, \alpha_{n+1}^{0\,\star})$ be a solution and $(X^\star)$, the associated portfolio. The real portfolio is computed after removing the generated costs such that $V_{n+1}^\star = X_{n+1}^\star - \mathcal{C}_{n+1} S_{n+1}^0$. According to the dynamics of the assets, we have for $n \in \{0, \dots, N-1\}$,

$$\begin{cases} \mathbb{E}_\mathbb{P}\left[S_{n+1}^i|\mathcal{F}_n\right] = S_n^i e^{\mu_n^i (t_{n+1}-t_n)}, \ \forall \ 1 \le i \le d , \\ \mathbb{C}\mathrm{ov}\left[S_{n+1}^i, S_{n+1}^j|\mathcal{F}_n\right] = S_n^i S_n^j e^{(\mu_n^i + \mu_n^j)(t_{n+1}-t_n)}(e^{(\sigma_n^i)^T \sigma_n^j (t_{n+1}-t_n)} - 1), \ \forall \ 1 \le i, j \le d . \end{cases}$$

By calling $A_n = \mathrm{Diag}((e^{\mu_n^i (t_{n+1}-t_n)})_{1 \le i \le d})$ and $B_n = [e^{(\mu_n^i + \mu_n^j)(t_{n+1}-t_n)}(e^{(\sigma_n^i)^T \sigma_n^j (t_{n+1}-t_n)} - 1)]_{ij}$, We have

$$\alpha_{n+1}^\star = \arg \sup_{\alpha_{n+1} \in \mathbb{R}^d} \quad \alpha_{n+1}^T A_n S_n + (X_n - \alpha_{n+1}^T S_n)\frac{S_{n+1}^0}{S_n^0} - \gamma_u \alpha_{n+1}^T S_n^T B_n S_n \alpha_{n+1} \qquad (33)$$

$$\Leftrightarrow A_n S_n - S_n \frac{S_{n+1}^0}{S_n^0} - 2\gamma_u (S_n^T B_n S_n)\alpha_{n+1}^\star = 0 \quad \Leftrightarrow \alpha_{n+1}^\star = \frac{A_n S_n - S_n e^{r_n(t_{n+1}-t_n)}}{2\gamma_u S_n^T B_n S_n}. \qquad (34)$$

Remark 25. *It is equivalent to remove costs at each time step or to remove them at the end because the quantity in the risk free asset does not impact the strategy.*

Comparing the multi-period strategy with this framework is a difficult task. The risk aversion parameters γ and γ_u, in the two models do not refer to the same risk aversion for the agent. We need to find a correct matching, or a measure of performance, independent from risk aversion. Obviously, this measure is the sharp ratio.

Proposition 26. *The sharp ratio of the sequential uni-period Markowitz strategy which ignores costs, does not depend on the risk aversion parameter γ_u.*

Proof. We prove by recurrence $\forall \ 0 \le n \le N$, $\mathcal{H}_n : \exists$ a $\mathcal{F}_n-$measurable random variable $\mathcal{X}_n$ not function of γ_u such that $V_n^\star = V_0 S_n^0 + \frac{\mathcal{X}_n}{2\gamma_u}$.
$\mathcal{H}_0$ is true with $\mathcal{X}_0 = 0$. We assume $\mathcal{H}_n$ true for $n > 0$. We have

$$\tilde{V}_{n+1}^\star = \tilde{V}_n^\star + \alpha_{n+1}^\star \cdot \Delta \tilde{S}_{n+1} - \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^{i\,\star} - \alpha_n^{i\,\star}|S_n^i}{S_n^0}$$

$$= \frac{1}{S_n^0}\left(V_0 S_n^0 + \frac{\mathcal{X}_n}{2\gamma_u}\right) + (\alpha_{n+1}^\star)^T \Delta \tilde{S}_{n+1} - \sum_{i=1}^d \nu \frac{|\alpha_{n+1}^{i\,\star} - \alpha_n^{i\,\star}|S_n^i}{S_n^0}.$$

By using that $\tilde{R}d_n = \mathrm{Diag}(\frac{\tilde{S}_n^i}{\tilde{S}_{n-1}^i})$, and the form of the solution in (34), we rewrite

$$(\alpha_{n+1}^\star)^T \Delta \tilde{S}_{n+1} = (\alpha_{n+1}^\star)^T(\tilde{R}d_{n+1} - Id)\tilde{S}_n = \tilde{S}_n^T(\tilde{R}d_{n+1} - Id)\alpha_{n+1}^\star$$

$$= \frac{1}{S_n^0}\frac{\tilde{S}_n^T(\tilde{R}d_{n+1} - Id)(A_n S_n - S_n e^{r_n(t_{n+1}-t_n)})}{2\gamma_u S_n^T B_n S_n}.$$

and $\displaystyle\sum_{i=1}^d \nu \left(\frac{|\alpha_{n+1}^{i\,\star} - \alpha_n^{i\,\star}|S_n^i}{S_n^0}\right) = \sum_{i=1}^d \nu \left|\frac{A_n^i S_n^i - S_n^i e^{r_n(t_{n+1}-t_n)}}{2\gamma_u S_n^T B_n S_n} - \frac{A_{n-1}^i S_{n-1}^i - S_{n-1}^i e^{r_{n-1}(t_n - t_{n-1})}}{2\gamma_u S_{n-1}^T B_{n-1} S_{n-1}}\right| \frac{S_n^i}{S_n^0}.$

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We prove $\mathcal{H}_{n+1}$ by denoting

$$\mathcal{X}_{n+1} = \frac{S_{n+1}^0}{S_n^0}\left(\mathcal{X}_n + \frac{S_n^T(Rd_{n+1} - Id)(A_n S_n - S_n e^{r_n(t_{n+1}-t_n)})}{S_n^T B_n S_n}\right)$$

$$- \frac{S_{n+1}^0}{S_n^0}\sum_{i=1}^d \nu \left|\frac{A_n^i S_n^i - S_n^i e^{r_n(t_{n+1}-t_n)}}{S_n^T B_n S_n} - \frac{A_{n-1}^i S_{n-1}^i - S_{n-1}^i e^{r_{n-1}(t_n - t_{n-1})}}{S_{n-1}^T B_{n-1} S_{n-1}}\right| S_n^i.$$

Using this form we deduce that $\mathbb{E}_{\mathbb{P}}[V_N^\star] - V_0 S_N^0 = \frac{\mathbb{E}_{\mathbb{P}}[\mathcal{X}_N]}{2\gamma_u}$, and $\mathbb{V}\mathrm{ar}[V_N^\star] = \frac{\mathbb{V}\mathrm{ar}[\mathcal{X}_N]}{4\gamma_u^2}$. Finally

$$\mathrm{Sharp}\,[V_N^\star] = \frac{\frac{\mathbb{E}_{\mathbb{P}}[V_N^\star] - V_0}{V_0} - \frac{V_0 S_N^0 - V_0}{V_0}}{\mathbb{V}\mathrm{ar}[\frac{V_N^\star - V_0}{V_0}]^{\frac{1}{2}}} = \frac{\mathbb{E}_{\mathbb{P}}[\mathcal{X}_N]}{\mathbb{V}\mathrm{ar}[\mathcal{X}_N]^{\frac{1}{2}}},$$

which does not depend on γ_u. ∎

C.2.2 Taking costs into account

This model is more sophisticated than the previous one. Costs are taken into account in the objective function but the strategy remains myopic. We recall that the agent has to consecutively solve $(D_n^{\gamma_u})$ for $n \in \{0, \ldots, N-1\}$,

$$\sup_{\alpha_{n+1}, \alpha_{n+1}^0 \in \mathbb{R}^d \times \mathbb{R}} \quad \mathbb{E}_{\mathbb{P}}\left[V_{n+1}|\mathcal{F}_n\right] - \gamma_u \mathbb{V}\mathrm{ar}_{\mathbb{P}}\left[V_{n+1}|\mathcal{F}_n\right]$$

$$\text{subject to} \qquad \Delta V_{n+1} = \alpha_{n+1}^T \Delta S_{n+1} + \alpha_{n+1}^0 \Delta S_{n+1}^0 - \sum_{i=1}^d \nu|\alpha_{n+1}^i - \alpha_n^i|S_n^i$$

In the presence of costs, we do not provide an explicit solution of $(D_n^{\gamma_u})$.

In order to use this model as a benchmark, let us prove an analog result to Proposition 26.

Proposition 27. *The sharp ratio of the sequential uni-period Markowitz strategy considering costs, does not depend on the risk aversion parameter γ_u.*

Proof. With the notation of the previous section, $(D_n^{\gamma_u})$ can be rewritten as

$$\sup_{\alpha_{n+1} \in \mathbb{R}^d} \quad \alpha_{n+1}^T A_n S_n + \left(V_n - \alpha_{n+1}^T S_n - \sum_{i=1}^d -\nu(|\alpha_{n+1}^i - \alpha_n^i|S_n^i)\right)\frac{S_{n+1}^0}{S_n^0} - \gamma_u \alpha_{n+1}^T S_n^T B_n S_n \alpha_{n+1}$$

$$\text{subject to} \quad V_{n+1} = \alpha_{n+1} S_{n+1} + \alpha_{n+1}^0 S_{n+1}^0$$

Let us consider the objective functions

$$\forall\, n \leq N,\ f_n(\alpha) = \alpha_{n+1}^T A_n S_n + \left(V_n - \alpha_{n+1}^T S_n - \sum_{i=1}^d -\nu(|\alpha_{n+1}^i - \alpha_n^i|S_n^i)\right)\frac{S_{n+1}^0}{S_n^0} - \gamma_u \alpha_{n+1}^T S_n^T B_n S_n \alpha_{n+1}.$$

f_n is strictly concave and $\lim_{||\alpha||_2 \to +\infty} f_n(\alpha) = -\infty$, then $(D_n^{\gamma_u})$ has an unique solution. We call $\alpha_{n+1}^\star$ the solution of $(D_n^{\gamma_u})$. f_n is differentiable on $\mathbb{R}^d \setminus \mathcal{O}_n$, where $\mathcal{O}_n = \{\alpha \in \mathbb{R}^d, \exists i \in \{1, \ldots, d\}$ such that $\alpha^i = \alpha_n^i\}$. f_n admits a sub-differential ∂f_n, at any point $\alpha \in \mathbb{R}^d$

$$\partial f_n(\alpha) = \left\{\begin{array}{l} A_n S_n - 2\gamma_u(S_n^T B_n S_n)\alpha_{n+1} - \mathrm{Diag}(1 + \nu\epsilon)S_n\frac{S_{n+1}^0}{S_n^0},\ \text{with} \\ \epsilon^i = \mathrm{sign}(\alpha^i - \alpha_n^i)\ \text{if}\ \alpha^i \neq \alpha_n^i,\ \epsilon^i \in [-1, 1]\ \text{otherwise} \end{array}\right\}.$$

We have $0 \in \partial f_n(\alpha_{n+1}^\star)$. Let us show by recurrence $\forall\, 0 \leq n \leq N$, $\mathcal{H}_n : \exists$ a $\mathcal{F}_n$−measurable random variable $\mathcal{X}_n$ not function of γ_u such that $\alpha_n^\star = \frac{\mathcal{X}_n}{2\gamma_u}$.

$\mathcal{H}_0$ is true with $\mathcal{X}_0 = 0$. We assume $\mathcal{H}_n$ true for $n > 1$. If $\exists\ i \in \{1, \ldots, d\}$ such that $\alpha_{n+1}^{i\star} = \alpha_n^{i\star}$ then it is sufficient to choose $\mathcal{X}_{n+1}^i = \mathcal{X}_n^i$. If $\alpha_{n+1}^{i\star} \neq \alpha_n^{i\star}$, then

$$\alpha_{n+1}^{i\star} = \frac{A_n^{ii}S_n^i - (1 - \nu\,\text{sign}(\alpha_{n+1}^{i\star} - \alpha_n^{i\star}))S_n^i e^r}{2\gamma_u S_n^T B_n S_n}.$$

Let us define $\mathcal{Z}$ a $\mathcal{F}_n$-measurable random variable such that $\alpha_{n+1}^i = \frac{\mathcal{Z}}{2\gamma_u}$. Then with the form of $\alpha_n^{i\star}$, we have

$$\mathcal{Z} = \frac{A_n^{ii}S_n^i - (1 - \nu\,\text{sign}(\mathcal{Z} - \mathcal{X}_n^i))S_n^i e^r}{S_n^T B_n S_n}.$$

We can deduce that $\mathcal{Z}$ is not a function of γ_u. Therefore $\mathcal{Z}$ is the chosen candidate to be $\mathcal{X}_{n+1}$. $\mathcal{H}_{n+1}$ is then true, and $\mathcal{H}_n$ true for all $n \geq 0$. Then is is easy to check that

$$\text{Sharp}(V_N^\star) = \frac{\mathbb{E}_{\mathbb{P}}\left[S_N^0\left(V_0 + \sum_{n=0}^N (\alpha_{n+1}^\star)^T \tilde{S}_n - \sum_{i=1}^d \nu|\alpha_{n+1}^{i\star} - \alpha_n^{i\star}|\tilde{S}_n^i\right)\right] - V_0 S_N^0}{\mathbb{V}\text{ar}\left[S_N^0\left(V_0 + \sum_{n=0}^N (\alpha_{n+1}^\star)^T \tilde{S}_n - \sum_{i=1}^d \nu|\alpha_{n+1}^{i\star} - \alpha_n^{i\star}|\tilde{S}_n^i\right)\right]^{\frac{1}{2}}}$$

$$= \frac{\mathbb{E}_{\mathbb{P}}\left[\left(\sum_{n=0}^N (\mathcal{X}_{n+1})^T \tilde{S}_n - \sum_{i=1}^d \nu|\mathcal{X}_{n+1}^i - \mathcal{X}_n^i|\tilde{S}_n^i\right)\right]}{\mathbb{V}\text{ar}\left[\left(\sum_{n=0}^N \mathcal{X}_{n+1}^T \tilde{S}_n - \sum_{i=1}^d \nu|\mathcal{X}_{n+1}^i - \mathcal{X}_n^i|\tilde{S}_n^i\right)\right]^{\frac{1}{2}}},$$

which is not a function of γ_u. ∎

D   Using of a finer grid to compute the chaos expansion

In this section we assume that N is a multiple of $p \in \mathbb{N}$. Let us consider two time grids, $\mathcal{T} = \{t_0, \ldots, t_N\}$ and $\tilde{\mathcal{T}} = \{t_0, t_p, \ldots, t_{kp}, \ldots, t_N\}$. Transactions are only possible at time in $\tilde{\mathcal{T}}$, every p date time in $\mathcal{T}$ but processes are observable in between. We define two associated filtrations

$$G_n = \sigma\{W_k - W_{k-1}, 1 \leq k \leq n\}, \quad \mathcal{F}_n = \sigma\{W_{kp} - W_{(k-1)p}, 1 \leq k \leq \lfloor \tfrac{n}{p} \rfloor\}.$$

We rewrite the problem (E^γ) according to the finest grid.

$$\sup_{\alpha_1, \ldots \alpha_{kp}, \ldots, \alpha_N} \quad \mathbb{E}_{\mathbb{P}}\left[(\tilde{X}_N - \mathcal{C}_N)S_N^0 - \gamma\left(\left(\tilde{X}_N - \mathcal{C}_N - \mathbb{E}_{\mathbb{P}}[\tilde{X}_N - \mathcal{C}_N]\right)S_N^0\right)^2\right]$$

$$\text{subject to} \quad \tilde{X}_0 = V_0, \ (\alpha_n)_n\ \boldsymbol{G} - Pred$$

$$\tilde{X}_{n+1} = \tilde{X}_n + \alpha_{\lfloor \frac{n+1}{p} \rfloor p} \cdot \Delta \tilde{S}_{n+1} \qquad\qquad (A^\gamma)$$

$$\mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu \frac{|\alpha_{\lfloor \frac{n+1}{p} \rfloor p}^i - \alpha_{\lfloor \frac{n}{p} \rfloor p}^i|S_n^i}{S_n^0}$$

We call $(\alpha_1^\star, \ldots, \alpha_{kp}^\star, \ldots, \alpha_N^\star)$ a solution of (A^γ). Since the transactions are only available at time in $\tilde{\mathcal{T}}$, we want to know whether representing processes with chaos on grid $\tilde{\mathcal{T}}$ is a loss of information or not.

We assume here that the assets follows the dynamics presented in Section 2 with constant drift parameters and volatility matrix, ie. $\forall\ n \in \{0, \ldots, N\}$, $\sigma_n = \sigma$, $r_n = r$, $\mu_n = \mu$. Under this assumption and according to the following proposition, the response is no.

Proposition 28. $\alpha_n^\star$ is $\mathcal{F}_{n-1}$ measurable.

Proof. Let us define for all $p \leq n \leq N$, the diagonal matrices $\tilde{R}d_n = \text{Diag}\left(\frac{\tilde{S}_n^i}{\tilde{S}_{n-p}^i}\right)_{1 \leq i \leq d}$. First we can prove by recurrence that for all $k \leq \lfloor \frac{N}{p} \rfloor$, $\tilde{S}_{kp}$ is $\mathcal{F}_{kp}$ measurable.

H_k: $\tilde{S}_{kp}$ is $\mathcal{F}_{kp}$ measurable.

H_0 is true since $\tilde{S}_0$ is deterministic and therefore $G_0 = \mathcal{F}_0$-measurable.

Let us assume $H_1, \ldots, H_k$ true for $k > 1$. Then $\forall\, 1 \le i \le d$, $\tilde{S}^i_{(k+1)p} = \tilde{S}^i_{kp} e^{(\sigma^i)^T \Delta W^{\mathbb{Q}}_{(k+1)p}}$. $\tilde{S}^i_{kp}$ is $\mathcal{F}_{kp}$ measurable, so $\tilde{S}^i_{(k+1)p}$ is $\mathcal{F}_{(k+1)p}$ measurable and H_{k+1} is true.

Now, Let us prove by recurrence the proposition $\forall n \ge 1$, $\mathcal{H}_n$: $\alpha^\star_n$ is $\mathcal{F}_{n-1}$ measurable.

$\mathcal{H}_1$ is true because α_1 is deterministic and therefore $G_0 = \mathcal{F}_0$ measurable. Lets now assume $\mathcal{H}_2, \ldots, \mathcal{H}_n$ true for $n > 1$.

If $n \not\equiv 0$ $[p]$ then $\alpha^\star_n = \alpha^\star_{n+1}$. We deduce that $\alpha^\star_{n+1}$ is $\mathcal{F}_{n-1}$ measurable but $\mathcal{F}_{n-1} = \mathcal{F}_n$ so $\alpha^\star_{n+1}$ is $\mathcal{F}_n$ measurable.

If $n \equiv 0$ $[p]$, $\exists k \le \lfloor \frac{N}{p} \rfloor - 1$ such that $n = kp$. Its has been proven in Proposition 6 that the problem (A^γ) is equivalent to the following

$$\sup_{\alpha_1, \ldots \alpha_{kp}, \ldots, \alpha_N, \theta \in \mathbb{R}} \quad \mathbb{E}_{\mathbb{P}}\left[(\tilde{X}_N - \mathcal{C}_N) S^0_N - \gamma \left((\tilde{X}_N - \mathcal{C}_N - \theta) S^0_N \right)^2 \right]$$

$$\text{subject to} \quad \tilde{X}_0 = V_0, \ (\alpha_n)_n \ \boldsymbol{G} - Pred$$

$$\tilde{X}_{n+1} = \tilde{X}_n + \alpha_{\lfloor \frac{n+1}{p} \rfloor p} \cdot \Delta \tilde{S}_{n+1} \qquad (\mathcal{A}^\gamma)$$

$$\mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu \frac{|\alpha^i_{\lfloor \frac{n+1}{p} \rfloor p} - \alpha^i_{\lfloor \frac{n}{p} \rfloor p}| S^i_n}{S^0_n}$$

For $\theta \in \mathbb{R}$, we consider the analog problem

$$\sup_{\alpha_1, \ldots, \alpha_{kp}, \ldots, \alpha_N} \quad \mathbb{E}_{\mathbb{P}}\left[(\tilde{X}_N - \mathcal{C}_N) S^0_N - \gamma \left((\tilde{X}_N - \mathcal{C}_N - \theta) S^0_N \right)^2 \right]$$

$$\text{subject to} \quad \tilde{X}_0 = V_0, \ (\alpha_n)_n \ \boldsymbol{G} - Pred$$

$$\tilde{X}_{n+1} = \tilde{X}_n + \alpha_{\lfloor \frac{n+1}{p} \rfloor p} \cdot \Delta \tilde{S}_{n+1} \qquad (A^\gamma_\theta)$$

$$\mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{i=1}^d \nu \frac{|\alpha^i_{\lfloor \frac{n+1}{p} \rfloor p} - \alpha^i_{\lfloor \frac{n}{p} \rfloor p}| S^i_n}{S^0_n}$$

This stochastic control problem can be rewritten with value functions b, such that $\alpha^\star_{(k+1)p}$ is solution of

$$b_{kp}(\tilde{X}_{kp}, \tilde{S}_{kp}, \mathcal{C}_{kp}, \alpha_{kp}, \theta) = \sup \alpha_{(k+1)p} \in \mathbb{R} \quad q_{kp}(\tilde{X}_{kp}, \tilde{S}_{kp}, \mathcal{C}_{kp}, \alpha_{(k+1)p}, \alpha_{kp}, \theta),$$

$$\text{with} \quad \begin{cases} q_N(x, s, c, \alpha, \alpha' \theta) = (x - c) S^0_N - \gamma \left((x - c - \theta) S^0_N \right)^2, \\[2mm] q_{N-p}(x, s, c, \alpha, \alpha', \theta) = \mathbb{E}_{\mathbb{P}}\left[b_N(x + \alpha^T(\tilde{R}d_N - Id)s, \tilde{R}d_N s, c + \nu \frac{|\alpha - \alpha'|^T s}{S^0_{N-p}}, \alpha, \theta) \right], \\[2mm] q_{lp}(x, s, c, \alpha, \alpha', \theta) = \mathbb{E}_{\mathbb{P}}\left[b_{(l+1)p}(x + \alpha^T(\tilde{R}d_{(l+1)p} - Id)s, \tilde{R}d_{(l+1)p} s, c + \nu \frac{|\alpha - \alpha'|^T s}{S^0_{lp}}, \alpha, \theta) \right], \text{ For } l < \lfloor \frac{N}{p} \rfloor. \end{cases}$$

With these three possible forms, and using that $\tilde{R}d_{(k+1)p}$ is independent from G_{kp}, we can say that $\alpha^\star_{(k+1)p}(\theta)$ is a deterministic function of $(\tilde{S}_{kp}, \tilde{X}_{kp}, \alpha_{(k-1)p}, \theta)$. We also notice that $\tilde{X}_{kp} = V_0 + \sum_{i=0}^{k-1} \alpha_{(i+1)p}(\tilde{S}_{(i+1)p} - \tilde{S}_{ip})$. Then we can deduce that $\alpha^\star_{(k+1)p}(\theta)$ is a deterministic function of $(\tilde{S}_{kp}, \tilde{S}_{(k-1)p}, \ldots, S_0, \alpha_{kp}, \alpha_{(k-1)p}, \ldots, \alpha_0)$. By applying Proposition 6, $\exists \theta^\star \in \mathbb{R}$ such that $(\alpha^\star_1(\theta^\star), \ldots, \alpha^\star_N(\theta^\star))$ is solution of the main problem (A^γ). Equivalently, one can say that $\exists\, \theta^\star$ such that $(\alpha^\star_1(\theta^\star), \ldots, \alpha^\star_{kp}(\theta^\star), \ldots, \alpha_N(\theta^\star))$, a solution of $(A^\gamma_{\theta^\star})$, is equal to $(\alpha^\star_1, \ldots, \alpha^\star_{kp}, \ldots, \alpha^\star_N)$, solution of (A^γ). $\theta^\star$ is deterministic then $\alpha^\star_{(k+1)p}(\theta^\star)$ is a deterministic function of

$(\tilde{S}_{kp}, \tilde{S}_{(k-1)p}, \ldots, S_0, \alpha_{(k-1)p}(\theta^\star), \alpha_{(k-2)p}(\theta^\star), \ldots, \alpha_0)$. By applying the assumption of recurrence, we can conclude that $\alpha^\star_{(k+1)p}(\theta^\star) = \alpha^\star_{(k+1)p} = \alpha^\star_{kp+1}$ is $\mathcal{F}_{kp}$ measurable. But $\alpha^\star_{kp+1} = \alpha^\star_{n+1}$ and $\mathcal{F}_{kp} = \mathcal{F}_n$ so H_{n+1} is true. The proposition is then true. $\blacksquare$

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