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Abstract: In this paper, the problem of stability, recursive feasibility and convergence conditions of stochastic model predictive control for linear discrete-time systems affected by a large class of correlated disturbances is addressed. Based on indirect feedback of the state, we develop a stochastic model predictive control that guarantees convergence, average cost bound and chance constraint satisfaction. The results rely on the computation of probabilistic reachable and invariant sets using the notion of correlation bound. This control algorithm results from a tractable deterministic optimal control problem formulation with a value function that upper-bounds the expected quadratic cost of the predicted state trajectory and control sequence. The proposed methodology only relies on the assumption of the existence of bounds on the mean and the covariance matrices of the disturbance sequence distribution.

Keywords: Stochastic model predictive control, Probabilistic reachability, Probabilistic invariance.

1. INTRODUCTION

Model predictive control (MPC) is a well established receding horizon control technique, particularly suitable to cope with hard constraints on controls and states, see Mayne et al. (2000); Camacho and Alba (2013); Kouvaritakis and Cannon (2016); Rawlings et al. (2017) and references therein. In particular, MPC strongly relies on a model to make predictions and to ensure the stability of the closed-loop behavior, while satisfying the constraints. Unfortunately, dynamical models can never fully loyally represent a real system. The mismatches between model and reality can be a problem since they may lead to instability and/or constraints violation, which represents a threat to systems safety.

Indeed, deterministic formulations are inherently inadequate to systematically deal with uncertainties; see the surveys Mesbah (2016); Farina et al. (2016). The worst-case approach to deal with the unavoidable uncertainties have been then employed, leading to robust MPC formulations for regulation, Mayne and Langson (2001), and tracking Limon et al. (2010). Although this approach is very efficient to ensure robust stability and constraints satisfaction, it suffers from some drawbacks as the conservatism of the resulting control or the often unrealistic assumptions on boundedness of the uncertainties. This modelling framework, in fact, is not suitable to cope with stochastic descriptions of uncertainty, which often provide a better description of the probabilistic nature of real-world systems.

These drawbacks have pushed the community to enquire for another approach to account for the probabilistic occurrence of uncertainties in the control design, Mesbah (2016), to deal with the stochastic nature of the uncertainties and to reduce the conservativeness of the control. Indeed, stochastic MPC (SMPC) has recently emerged with the aim of systematically incorporating the probabilistic descriptions of uncertainties into a stochastic optimal control problem.

An enormous amount of work has been done in this area with results that are most often very conclusive. In many works concerning SMPC, however, the stochastic disturbance is modelled by an independent, identically distributed sequence of random variables with known mean and variance. This is the case, for instance, for the methods concerning: stochastic tube MPC, Cannon et al. (2010); Hewing and Zeilinger (2018); discounted probabilistic constraints, Yan et al. (2021); SMPC for controlling the average number of constraints violation, Korda et al. (2014); probabilistic MPC, Farina et al. (2013); and recursively feasible SMPC using indirect feedback, Hewing et al. (2020). We can also cite Cannon et al. (2009a,b); Bernardini and Bemporad (2011); Oldewurtel et al. (2013); Zhang et al. (2014). The assumption of independence in time, and thus uncorrelation between disturbance real-

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izations, though, is in general unrealistic. In addition, the disturbance distribution mean and covariance are in general not accessible in practice, nor necessarily constant in time.

In this work, we consider linear systems excited by disturbances which realisations are correlated in time. Only bounds on the mean and the correlation matrices are required to exist, even stationarity is not necessary. Based on recent results on the probabilistic reachable and invariance sets for correlated disturbances developed in Fiacchini and Alamo (2021), we adapt the tube based SMPC formulation in Hewing et al. (2020) and extend some results in Hewing and Zeilinger (2018); Farina et al. (2013) to the correlated disturbance case under analysis, based only on the knowledge of bounds on its first and second moments. Thus, under this "weak" assumption we propose a tube based SMPC algorithm and derive its nominal asymptotic stability, recursive feasibility, in addition to the satisfaction of the chance constraints and state convergence with asymptotic average cost bound.

Notations: The set of natural numbers is denoted $\mathbb{N}$, the set of symmetric matrices in $\mathbb{R}^{n\times n}$ is denoted $\mathbb{S}^n$, $\|\cdot\|_M$ represents the norm weighted over the $\mathbb{R}^{n\times n}$ positive definite matrix M and $\Gamma \succ 0$ ($\Gamma \succeq 0$) denotes that Γ is a symmetric definite (semi-definite) positive matrix. If $\Gamma \succeq 0$ then $\Gamma^{\frac{1}{2}}$ is the matrix satisfying $(\Gamma^{\frac{1}{2}})^2 = \Gamma$. For all $\Gamma \succeq 0$ and $r \geq 0$, the ellipsoidal set $\mathscr{E}(\Gamma, r)$ is defined by $\{x = \Gamma^{\frac{1}{2}} z \in \mathbb{R}^n : z^\top z \leq r\}$; if moreover $\Gamma \succ 0$, then $\mathscr{E}(\Gamma, r) = \{x \in \mathbb{R}^n : x^\top \Gamma^{-1} x \leq r\}$. The spectral radius of $P \in \mathbb{R}^{n\times n}$ is $\rho(P)$. Given two sets $X, Y \subseteq \mathbb{R}^n$, their Pontryagin difference is $X \ominus Y = \{x \in X | \ x + y \in X, \ \forall y \in Y\}$. The χ squared cumulative distribution function of order n is denoted $\chi^2_n(x)$. Probabilities and conditional probabilities are denoted $\Pr\{A\}$ and $\Pr\{A|B\}$; the expected value of A is denoted $\mathrm{E}\{A\}$.

2. PROBLEM FORMULATION

Consider the discrete-time LTI systems given by

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ represents an additive disturbance given by a sequence of random variables that can be correlated in time. Throughout the paper, we make the underlining assumption that perfect knowledge of the state is available and that the pair (A, B) is stabilizable.

The objective is to design a stochastic model predictive controller that stabilizes (1) and ensures the satisfaction of chance constraints of the form

$$\Pr\{x \in \mathbb{X} | x_0\} \geq 1 - p_x, \quad \Pr\{u \in \mathbb{U} | x_0\} \geq 1 - p_u, \tag{2}$$

where $\mathbb{X}$ and $\mathbb{U}$ are convex sets with the origin in their interior and p_x and p_u represent the tolerated violation probability of each constraint. As in Hewing et al. (2020), the probabilities are to be understood with respect to knowledge at the initial time step $t = 0$.

Throughout the paper, no assumption on $\{w_k\}_{k\in\mathbb{N}}$ is posed other than the existence of a bound on the mean and correlation matrices and an exponentially vanishing cross-correlation. Neither stationarity, i.i.d. assumption, nor the knowledge of the mean or the variance of the $\{w_k\}_{k\in\mathbb{N}}$ sequence are required in opposition with what done in the literature Hewing et al. (2020); Hewing and Zeilinger (2018); Farina et al. (2013); Yan et al. (2021). This is crucial in practice, as no exact knowledge of the matrices nor guarantee on the stationarity are often available.

Assumption 1. There exist $m, b, \gamma \in \mathbb{R}$, with $\gamma \in [0, 1)$, such that the sequence w_k satisfies

$$\mu_k^\top \mu_k \leq m, \quad \forall k \in \mathbb{N}, \tag{3a}$$

$$\|\mathrm{cov}(w_i, w_j)\|_2^2 \leq b\, \gamma^{j-i}, \quad i \leq j, \tag{3b}$$

with $\mu_k = \mathrm{E}\{w_k\}$ and $\mathrm{cov}(w_i, w_j) = \mathrm{E}\{(w_i - w_j)(w_i - w_j)^\top\}$, for all $k, i, j \in \mathbb{N}$.

3. PRELIMINARIES

In this section, we consider a system given by

$$e_{k+1} = A_K e_k + w_k \tag{4}$$

where $e_k \in \mathbb{R}^n$, $A_K = A + BK \in \mathbb{R}^{n\times n}$, w_k is an additive disturbance sequence similar to the one in (1) and K is assumed to render A_K Schur stable (i.e. $\rho(A_K) < 1$).

3.1 Correlation bound

We first recall the following result, provided in Fiacchini and Alamo (2021).

Proposition 1. If Assumption 1 is satisfied, then non-negative α, β, $\gamma \in \mathbb{R}$ and $\tilde{\Gamma} \in \mathbb{S}^n$ exist, with $\gamma \in [0, 1)$ and $\tilde{\Gamma} \succ 0$, such that

$$\Gamma_{k,k} \prec \tilde{\Gamma}, \quad \forall k \in \mathbb{N}, \tag{5a}$$

$$\Gamma_{i,j} \tilde{\Gamma}^{-1} \Gamma_{i,j}^\top \preceq (\alpha + \beta\gamma^{j-i})\tilde{\Gamma}, \quad \forall i \leq j, \tag{5b}$$

hold, with $\Gamma_{i,j} = \mathrm{E}\{w_i w_j^\top\}$, for all $i, j \in \mathbb{N}$.

Note that only the bounds on the mean and the covariance of $\{w_k\}_{k\in\mathbb{N}}$ are required to obtain the bounds (5).

We give here a notion that has been introduced in (Fiacchini and Alamo, 2021, Definition 1) and plays a key role in the characterization and determination of probabilistic reachable and invariant sets for the considered systems.

Definition 1. (Correlation Bound). The random sequence $\{w_k\}_{k\in\mathbb{Z}}$ in (4) is said to have a correlation bound Γ_w for matrix A_K if the recursion (4), with $e_0 = 0$, satisfies

$$\mathrm{E}\{e_{k+1} e_{k+1}^\top\} \preceq A_K \mathrm{E}\{e_k e_k^\top\} A_K^\top + \Gamma_w, \quad \forall k \geq 0. \tag{6}$$

If (5a) and (5b) are satisfied, it is possible to obtain tight correlation bounds, see Fiacchini and Alamo (2021).

3.2 Probabilistic reachable and invariant sets

We recall here the notion of probabilistic reachable set.

Definition 2. (Probabilistic reachable set). It is said that $\Omega_k \subseteq \mathbb{R}^n$ with $k \in \mathbb{N}$ is a sequence of probabilistic reachable sets for system (4), with violation level $\varepsilon \in [0, 1]$, if $e_0 \in \Omega_0$ implies $\Pr\{e_k \in \Omega_k\} \geq 1 - \varepsilon$ for all $k \geq 1$.

A condition for a sequence of ellipoids to be probabilistic reachable sets in terms of correlation bound, given in Fiacchini and Alamo (2021), is presented below.

Proposition 2. Suppose that the random sequence $\{w_k\}_{k\in\mathbb{N}}$ has a correlation bound $\Gamma_w \succ 0$ for matrix A_K with

$\rho(A_K) < 1$. Given $r > 0$, consider the system (4) with $e_0 = 0$, and the following recursion

$$\Gamma_{k+1} = A_K \Gamma_k A_K^\top + \Gamma_w \tag{7}$$

with $\Gamma_0 = 0 \in \mathbb{R}^{n\times n}$. Then the sets $\mathscr{E}(\Gamma_k, r)$ are probabilistic reachable sets with violation level n/r for every $r > 0$, for $k \in \mathbb{N}$. If, moreover, w_k is a Gaussian process with null mean, then $\mathscr{E}(\Gamma_k, r)$ are probabilistic reachable sets with violation probability $1 - \chi^2_n(r)$.

We recall now the notion of probabilistic invariant set

Definition 3. (Probabilistic invariant set). The set $\Omega \in \mathbb{R}^n$ is a probabilistic invariant set for the system (4), with violation level $\varepsilon \in [0, 1]$, if $e_0 \in \Omega$ implies $\Pr\{e_k \in \Omega\} \geq 1 - \varepsilon$ for all $k \geq 1$.

A constructive condition for an ellipsoid to be a probabilistic invariant set is given in (Fiacchini and Alamo, 2021, Proposition 5). Notice that only an upper bound on the covariance term $\tilde{\Gamma}$ that ensures the satisfaction of (5a) and (5b) is required to determine a correlation bound, and then to construct the probabilistic reachable and invariant sets.

4. MODEL PREDICTIVE CONTROL PROPERTIES

The different ingredients of SMPC are first presented.

4.1 Control policy and decoupled dynamics

As usual, the considered MPC controller relies on the following dual control policy

$$u_k = v_k + K e_k \tag{8}$$

where v is the nominal control, e the bias between the nominal state and the actual one and K a state feedback gain such that $A + BK$ is Schur. In particular, suppose that $S \succ 0$ is such that the Lyapunov condition

$$(A + BK)^\top S(A + BK) - S \preceq -Q - K^\top RK, \tag{9}$$

holds. Notice that such S exists from $\rho(A + BK) < 1$. Replacing (8) in (1) yields

$$z_{k+1} = Az_k + Bv_k, \tag{10a}$$
$$e_{k+1} = (A + BK)e_k + w_k, \tag{10b}$$
$$x_k = z_k + e_k, \tag{10c}$$

where z represents the nominal state, (10a) the nominal dynamic, (10b) the error propagation dynamic. Since (10b) is asymptotically stable, the SMPC aims at finding a nominal control v_k that steers the nominal state towards the origin minimizing a quadratic criterion, and satisfying the chance constraints and a given terminal constraint.

4.2 Cost function

The cost function used in the SMPC literature often consists of a sum over the prediction horizon of weighted quadratic terms of the state and control:

$$J_u = \mathrm{E}\Big\{\|x_N\|^2_S + \sum_{i=0}^{N-1} \|x_i\|^2_Q + \|u_i\|^2_R\Big\}, \tag{11}$$

with $Q \succ 0$, $R \succeq 0$ and $(Q^{\frac{1}{2}}, A)$ an observable pair. Matrix $S \succ 0$ should be appropriately chosen to guarantee the stability of the system controlled by the MPC, for instance satisfying (9). Such a cost function presents many advantages, since it can be reduced to a deterministic quadratic function in terms of mean and covariance of x_i and u_i, when the disturbance sequence has zero mean, its elements are i.i.d. and its second moment is known, as in Hewing and Zeilinger (2018); Hewing et al. (2020); Farina et al. (2013). Unfortunately, it is not possible to do that under the assumption done in this paper, since we lack information about mean and covariance of the disturbance distribution and we do not impose i.i.d. assumptions on the disturbances, which makes (11) hard to be dealt with. Since it is not possible to directly deal with (11), we look for a cost function that bounds it and which minimization is tractable. Although this does not ensure the decrease of (11), it provides a decreasing bound for it.

We introduce now results that are going to be later used.

Lemma 3. Given $M \succ 0$ we have $\|a + b\|^2_M \leq 2(\|a\|^2_M + \|b\|^2_M)$, $\forall a, \forall b$.

Proof. We notice that for every pair of vectors a and b

$$0 \leq \|a - b\|^2_M = \|a\|^2_M + \|b\|^2_M - 2a^\top Mb.$$

Thus, $2a^\top Mb \leq \|a\|^2_M + \|b\|^2_M$. From here we finally obtain

$$\|a + b\|^2_M = \|a\|^2_M + \|b\|^2_M + 2a^\top Mb \leq 2(\|a\|^2_M + \|b\|^2_M).$$

Lemma 4. Let $P \in \mathbb{R}^{n\times n}$ be some positive semi-definite matrix and consider symmetric matrices $\underline{M} \in \mathbb{R}^{n\times n}$ and $\overline{M} \in \mathbb{R}^{n\times n}$ such that $\underline{M} \preceq \overline{M}$. Then

$$\mathrm{tr}\{P\underline{M}\} \leq \mathrm{tr}\{P\overline{M}\}. \tag{12}$$

Proof. Recall that $\underline{M} \preceq \overline{M}$ means that

$$y^\top \underline{M} y \leq y^\top \overline{M} y \tag{13}$$

for all $y \in \mathbb{R}^n$. Since P is positive semi-definite, then there exists a matrix $N \in \mathbb{R}^{n\times n}$ such that $P = N^\top N$. Defining $y = N^\top x$ for all $x \in \mathbb{R}^n$ and considering (13) we have

$$x^\top N\underline{M}N^\top x = y^\top \underline{M} y \leq y^\top \overline{M} y = x^\top N\overline{M}N^\top x$$

for all $x \in \mathbb{R}^n$, which implies that $N\underline{M}N^\top \preceq N\overline{M}N^\top$ and then also that $\mathrm{tr}\{N\underline{M}N^\top\} \leq \mathrm{tr}\{N\overline{M}N^\top\}$. From the property $\mathrm{tr}\{AB\} = \mathrm{tr}\{BA\}$ and $P = N^\top N$ then (12) holds and the result is proved. ■

Another useful lemma follows.

Lemma 5. Given $c \in \mathbb{R}^p$, $D \in \mathbb{R}^{p\times n}$, and $F \in \mathbb{R}^{p\times p} \succ 0$, suppose that the sequence $\{w_k\}_{k\in\mathbb{N}}$ admits a correlation bound Γ_w for matrix $A_K = A + BK$. Assume also that Γ_k is given by recursion (7) and consider e_k given by (10b) with $e_0 = 0$. Then the following inequality holds

$$\mathrm{E}\{\|c + De_k\|^2_F\} \leq 2(\|c\|^2_F + \mathrm{tr}\{D^\top FD\Gamma_k\}). \tag{14}$$

Proof. First, we prove that if $e_0 = 0$, then

$$\mathrm{E}\{e_k e_k^\top\} \preceq \Gamma_k \tag{15}$$

holds, where Γ_k is given by the recursion (7). We proceed by induction, by noticing first that (15) holds trivially for $k = 0$ from $e_0 = 0$. Suppose now that (15) holds for a given $k \in \mathbb{N}$. Then, from the Definition 1 and (7) it follows that

$$\begin{aligned}\mathrm{E}\{e_{k+1}e_{k+1}^\top\} &\preceq A_K \mathrm{E}\{e_k e_k^\top\} A_K^\top + \Gamma_w \\ &\preceq A_K \Gamma_k A_K^\top + \Gamma_w = \Gamma_{k+1},\end{aligned}$$

and hence (15) is satisfied for $k+1$ and, by induction, also for all $k \in \mathbb{N}$. Denote $\psi_k = \mathrm{E}\{\|c + De_k\|^2_F\}$. With this notation, and Lemma 3, we obtain

$$\begin{aligned}\psi_k &= \mathrm{E}\{\|c + De_k\|^2_F\} \leq 2\mathrm{E}\{\|c\|^2_F + \|De_k\|^2_F\} \\ &= 2\|c\|^2_F + 2\mathrm{E}\{e_k^\top D^\top FDe_k\} \\ &= 2\|c\|^2_F + 2\mathrm{E}\{\mathrm{tr}\{e_k^\top D^\top FDe_k\}\}.\end{aligned}$$

From the well known identity $\mathrm{tr}\{AB\} = \mathrm{tr}\{BA\}$ we have

$$\begin{aligned}\psi_k &\le 2\|c\|_F^2 + 2\mathrm{E}\{\mathrm{tr}\{D^\top F D e_k e_k^\top\}\} \\ &= 2\|c\|_F^2 + 2\mathrm{tr}\{D^\top F D \mathrm{E}\{e_k e_k^\top\}\}.\end{aligned}$$

Since $\mathrm{E}\{e_k e_k^T\} \le \Gamma_k$, we finally conclude from Lemma 4:

$$\psi_k \le 2\|c\|_F^2 + 2\mathrm{tr}\{D^\top F D \Gamma_k\}.$$ ■

The following proposition presents an upper bounding function of the cost (11), that depends on the nominal state and control input z and v and on the correlation bound together with recursions (7).

Proposition 6. Consider the linear system (1), where the disturbance admits the correlation bound Γ_w for matrix $A_K = A + BK$. Consider also the control policy (8), the decoupling (10), the recursion (7), and the value function (11). If $z_0 = x_0$ (i.e. $e_0 = 0$), then (11) is bounded from above by the following cost function

$$\begin{aligned}J = 2\Big(&\|z_N\|_S^2 + \sum_{i=0}^{N-1} \|z_i\|_Q^2 + \|v_i\|_R^2 \\ &+ \mathrm{tr}\{S\Gamma_N\} + \sum_{i=0}^{N-1} \mathrm{tr}\{(Q + K^\top R K)\Gamma_i\}\Big). \qquad (16)\end{aligned}$$

Proof. By applying (14) to each term of (11) we get, for all $i = 0, \dots, N-1$, the following inequalities

$$\mathrm{E}\{\|x_i\|_Q^2\} = \mathrm{E}\{\|z_i + e_i\|_Q^2\} \le 2(\|z_i\|_Q^2 + \mathrm{tr}\{Q\Gamma_i\}) \qquad (17)$$

$$\mathrm{E}\{\|x_N\|_S^2\} = \mathrm{E}\{\|z_N + e_N\|_S^2\} \le 2(\|z_N\|_S^2 + \mathrm{tr}\{S\Gamma_N\}) \qquad (18)$$

$$\mathrm{E}\{\|u_i\|_R^2\} = \mathrm{E}\{\|v_i + K e_i\|_R^2\} \le 2(\|v_i\|_R^2 + \mathrm{tr}\{K^\top R K \Gamma_i\}). \qquad (19)$$

Summing (17) and (19) for each $i = 0, \dots, N-1$, together with (18), the claim follows and the proof is established. ■

Since (10a) and (10b) are completely decoupled, the feedback K is known and (10a) depends on the nominal control input v only. Thus, it is possible to ignore the correlation bound propagation cost terms of (16) (i.e. $\mathrm{tr}\{S\Gamma_N\} + \sum_{i=0}^{N-1} \mathrm{tr}\{(Q + K^\top R K)\Gamma_i\}$)) on the MPC optimization problem cost.

4.3 Deterministic formulation of chance constraints

The model predictive control algorithm that is to be built, has to ensure the satisfaction of the chance constraints (2). Instead of directly working on these constraints for the intractability and non-convexity of the problem they pose, we consider tightened hard constraints on the nominal state and control which satisfaction guarantees the satisfaction of (2) with at least the requested probability, as often done, see Cannon et al. (2010); Farina et al. (2013); Hewing and Zeilinger (2018); Hewing et al. (2020).

In our case, the tightening of the original constraints is done by leveraging the sequence of reachable sets given by Proposition 2, resulting in the following hard constraints on the nominal state and control input

$$z_k \in \mathbb{Z}_k = \mathbb{X} \ominus \mathscr{E}(\Gamma_k, r_x), \qquad (20)$$

$$v_k \in \mathbb{V}_k = \mathbb{U} \ominus K\mathscr{E}(\Gamma_k, r_u), \qquad (21)$$

for all $k = 0, \dots, N-1$, where r_x and r_u depend on the violation probabilities tolerances p_x and p_u. The satisfaction of (20) and (21) is enough to guarantee (2). See Proposition 2. The terminal set $\mathbb{Z}_f$ has to be characterized with some properties related to probabilistic invariance ensuring that $z_N \in \mathbb{Z}_f$ implies the satisfaction of (2) with $x = x_k$ and $u = Kx_k$ for all $k \ge N$. For this, consider first $\mathbb{X}_u := \{x : Kx \in \mathbb{U}\}$ and the sets $\mathbb{Z}_x$ and $\mathbb{Z}_u$ defined by

$$\begin{aligned}\mathbb{Z}_u &:= \mathbb{X}_u \ominus \mathscr{E}(W_{r_u}, 1), \\ \mathbb{Z}_x &:= \mathbb{X} \ominus \mathscr{E}(W_{r_x}, 1),\end{aligned}$$

where W_{r_u} and W_{r_x} are given by Proposition by (Fiacchini and Alamo, 2021, Proposition 5) with $r = r_u$ and $r = r_x$, respectively and $A_K = A + BK$. The terminal set $\mathbb{Z}_f$ is the maximal positively invariant set, contained in $\mathbb{Z}_x \cap \mathbb{Z}_u$, for system (10a) under the state feedback controller $v_k = Kz_k$.

5. SMPC SCHEME

Combining all the ingredients, the resulting tractable stochastic MPC optimization problem to be solved at any time t for the linear discrete-time stochastic system (1) is stated as follows:

$$\min_{v_0,\dots,v_{N-1}} \left\{ \|z_N\|_S^2 + \sum_{k=0}^{N-1} \|z_k\|_Q^2 + \|v_k\|_R^2 \right\} \qquad (22)$$

subject to

$$\begin{aligned}
&z_{k+1} = Az_k + Bv_k, && \forall k = 0, \dots, N-1 \quad (23)\\
&\Gamma_{k+1} = A\Gamma_k A^\top + \Gamma_w, && \forall k = 0, \dots, N-1 \quad (24)\\
&z_k \in \mathbb{Z}_k = \mathbb{X} \ominus \mathscr{E}(\Gamma_k, r_x), && \forall k = 0, \dots, N-1 \quad (25)\\
&v_k \in \mathbb{V}_k = \mathbb{U} \ominus K\mathscr{E}(\Gamma_k, r_u), && \forall k = 0, \dots, N-1 \quad (26)\\
&z_N \in \mathbb{Z}_f, && (27)\\
&(z_0, \Gamma_0) \in \{(z_1(t-1), \Gamma_1(t-1)\}, && (28)
\end{aligned}$$

where $z_1(t-1)$ and $\Gamma_1(t-1)$ represent the nominal state predicted one step ahead at $t-1$ and the correlation bound propagated one step ahead at $t-1$, if $t \ge 1$, respectively, i.e.

$$\begin{aligned}z_1(t-1) &= Az_0(t-1) + Bv_0(t-1) \\ \Gamma_1(t-1) &= (A + BK)\Gamma_0(t-1)(A + BK)^\top + \Gamma_w,\end{aligned}$$

while $(z_1(t-1), \Gamma_1(t-1)) = (x_0, 0)$ if $t = 0$. The feedback controller K and the terminal cost matrix S are determined by solving an LQR problem (9) with weight matrices Q and R.

Condition (28) has to be understood in the sense that we set the initial value of the optimisation problem (22)-(28) to the first element of the predicted trajectory sequence $z_1(t-1)$ and the propagation of the correlation bound of the error between the real and predicted state $\Gamma_1(t-1)$ at every instant $t \ge 1$. The choice of this strategy has a direct consequence on the feasibility of (22)-(28) and on the satisfaction of the chance constraints (2). These details are also discussed in Farina et al. (2013); Hewing and Zeilinger (2018); Hewing et al. (2020); Mayne (2018).

In what follows, we denote $v_k(t), z_k(t), \Gamma_k(t)$, with $k = 0, \dots, N-1$ and $z_N(t)$ the input, trajectory and covariance bounds obtained as solution of the problem (22)-(28) at time t. The explicit dependence on t, though, is avoided hereafter when clear from the context, to simplify the notation.

We make the following assumption regarding the initial feasibility of (22)-(28).

Assumption 2. We assume a perfect knowledge of the initial state (i.e. $z_0 = x_0$ or $e_0 = 0$) at $t = 0$ and that the problem (22)-(28) is initially feasible for $x_0 = z_0$ at $t = 0$.

The properties of the SMPC formulated in (22)-(28) in terms of recursive feasibility, constraints satisfaction and nominal asymptotic stability are summarized in the following proposition.

Proposition 7. If Assumption 2 is satisfied, then the problem (22)-(28) is recursively feasible, the chance constraints (2) are satisfied and the nominal system described by (10a) is asymptotically stable under the control actions that result from solving (22)-(28).

Proof. Consider first the recursive feasibility of the problem (22)-(28) under Assumption 2, that is the condition of its initial feasibility (i.e. at $t = 0$).
Assume that, at some given time instant t, a feasible solution is available with the optimal sequence $\boldsymbol{v}(t) = \{v_0(t), \ldots, v_{N-1}(t)\}$. Under this sequence, we have the satisfaction of the constraints (25), (26) for all $k = 0, \ldots, N-1$ and the terminal constraint (27) for $k \geq N$.
Given the optimal sequence $\boldsymbol{v}(t)$ at time t, and the invariance properties of the terminal set, it is always possible to build a control sequence $\overline{\boldsymbol{v}}(t)$, feasible for the problem at $t+1$, that results from shifting the optimal sequence one step back and adding the feedback term in $z_N(t)$ as the last element i.e. $\overline{\boldsymbol{v}}(t) = \{v_1(t), \ldots, v_{N-1}(t), Kz_N(t)\}$.
Indeed, being originated from the optimal sequence at t, the first $N-1$ elements of $\overline{\boldsymbol{v}}(t)$ satisfy trivially the constraints of the problem at time $t+1$. The remaining element $Kz_N(t)$ of the shifted control sequence $\overline{\boldsymbol{v}}(t)$ also satisfies the constraints by construction, because we impose $z_N(t)$ to be located in some positively invariant set for the feedback controller K, inside on which all the state and input constraints are satisfied. At last, we deduced that $\overline{\boldsymbol{v}}(t)$ is a feasible control sequence for the problem at time $t+1$, which guarantees the recursive feasibility of the proposed MPC scheme if it is initially feasible at time $t = 0$. Moreover, as proved in (Hewing et al., 2020, Lemma 1), the predicted error has the same distribution as the closed-loop error, which directly leads to chance constraints satisfaction in addition to the recursive feasibility.
Concerning the asymptotic stability result, consider the MPC optimal cost value (22), with initial state $z_0 = z_0(t)$, as a Lyapunov candidate function for the nominal system (10a) and denote it $V(z_0(t), t)$. Clearly, $V(.\,, t)$ is a positive definite function and the optimization solution, given by the control sequence $\boldsymbol{v}(t) = \{v_0(t), \ldots, v_{N-1}(t)\}$ and the predicted state trajectory $\boldsymbol{z}(t) = \{z_1(t), \ldots, z_N(t)\}$, satisfies all the constraints of the problem. Let

$$\overline{V}(z_0(t), t) = \sum_{k=0}^{N-1} \|z_k(t+1)\|_Q^2 + \|\overline{v}_k(t+1)\|_R^2 + \|z_N(t+1)\|_S^2$$

where $z_k(t+1) = z_{k+1}(t)$ for $k = 0, \ldots, N-1$ and $z_{N+1}(t) = (A+BK)z_N(t)$ is the state sequence obtained by applying the shifted control sequence $\overline{\boldsymbol{v}}(t) = \{\overline{v}_0(t+1), \ldots, \overline{v}_{N-1}(t+1)\} = \{v_1(t), \ldots, v_{N-1}(t), Kz_N(t)\}$ to the nominal system (10a) with initial state $z_0(t+1) = z_1(t)$.
Note that because of (9), we have

$$(A+BK)^\top S(A+BK) - S + Q + K^\top RK \preceq 0,. \quad (29)$$

The optimality of $V(z_0(t+1), t+1)$ and (29) yield

$$\begin{aligned} V(z_0(t+1), t+1) &\leq \overline{V}(z_0(t), t) \\ &\leq V(z_0(t), t) - \|z_0(t)\|_Q^2 - \|v_0(t)\|_R^2, \end{aligned}$$

which gives $V(z_0(t+1), t+1) - V(z_0(t), t) \leq -\|z_0(t)\|_Q^2 - \|v_0(t)\|_R^2 < 0$ for $z_0(t) \neq 0$. Now, invoking the recursive feasibility under Assumption 2 (which ensures the existence of an optimal control sequence at any time t) $V(z_0(t), t)$ is a Lyapunov function for system (10a) and the asymptotic stability of (10a) follows. ■

6. AVERAGE COST BOUND AND STATE CONVERGENCE

To evaluate the evolution of the cost function (16) along the trajectory of the system under the optimal solution of the problem (22)-(28), denote with $J^*(t)$ the value (16) for $v_k(t)$, $z_k(t)$ with $k = 0, \ldots, N-1$ and $z_N(t)$ solution of (22)-(28) at t. Recall that the explicit dependence on t is avoided in what follows, to simplify the notation, when clear from the context. Then $J^*(t)$ satisfies the following proposition.

Proposition 8. Consider system (1) under the control law (8) resulting from (22)-(28) and let $S \in \mathbb{S}^n$ satisfy (9). If $z_0(0) = x_0(0)$, then the optimal value $J^*(t)$ of (16) satisfies

$$\begin{aligned} J^*(t+1) - J^*(t) \leq &- \mathrm{E}\{\|x_0(t)\|_Q^2\} - \mathrm{E}\{\|u_0(t)\|_R^2\} \\ &+ \mathrm{tr}\{2S\Gamma_w\} \end{aligned} \quad (30)$$

for every $t \in \mathbb{N}$.

Proof. Assume the existence of an optimal sequence $\{v_0(t), \ldots, v_{N-1}(t)\}$ at time t that yields the optimal value $J^*(t)$. At time $t+1$, and considering the properties of the terminal set $\mathbb{Z}_f$, we can observe that the shifted sequence $\{v_1(t), \ldots, Kz_N(t)\}$ represents a suboptimal and feasible sequence. Denoting with $J(t+1)$ the cost induced by the suboptimal sequence at $t+1$, we have

$$\begin{aligned} J(t+1) &= J^*(t) - 2\|z_0(t)\|_Q^2 - 2\|v_0(t)\|_R^2 \\ &- 2\mathrm{tr}\{(Q + K^\top RK)\Gamma_0(t)\} + 2\|z_N(t)\|_Q^2 \\ &+ 2\|Kz_N(t)\|_R^2 - 2\|z_N(t)\|_S^2 + 2\|(A+BK)z_N(t)\|_S^2 \\ &+ 2\mathrm{tr}\{Q + K^\top RK - S \\ &+ ((A+BK)S(A+BK)^\top)\Gamma_N(t)\} + 2\mathrm{tr}\{S\Gamma_w\}. \end{aligned}$$

From Lemma 4 with $P = \Gamma_N(t)$ and $\underline{M}$ and $\overline{M}$ given by (29), it follows

$$\begin{aligned} J(t+1) \leq &J^*(t) - 2(\|z_0(t)\|_Q^2 + \|v_0(t)\|_R^2 \\ &+ \mathrm{tr}\{(Q + K^\top RK)\Gamma_0(t)\}) + 2\mathrm{tr}\{S\Gamma_w\}. \end{aligned}$$

As $J(t+1)$ is suboptimal, we have $J^*(t+1) \leq J(t+1)$, implying

$$\begin{aligned} J^*(t+1) \leq &J^*(t) - 2(\|z_0(t)\|_Q^2 + \|v_0(t)\|_R^2 \\ &+ \mathrm{tr}\{(Q + K^\top RK)\Gamma_0(t)\}) + 2\mathrm{tr}\{S\Gamma_w\} \end{aligned} \quad (31)$$

that, together with (14) in Lemma 5, yields (30) and concludes the proof. ■

Notice that the actual evolution of (16) is better represented by (31), as this inequality is sharper than (30) (consequence of Lemma 5). In addition, we remind that the result in Proposition 8 doesn't hold for (11) but for its bound given by (16) only.
The next proposition gives the average asymptotic cost

bound and is a direct extension to our case of a result from Hewing et al. (2020).

Proposition 9. Consider system (1) subject to the disturbance sequence $\{w_k\}_{k\in\mathbb{N}}$ that admits a correlation bound Γ_w for the matrix $A_K = A + BK$, under the control law (8) resulting from (22)-(28). Then we have

$$\lim_{\overline{N}\to\infty} \frac{1}{\overline{N}} \sum_{t=0}^{\overline{N}} \mathrm{E}\{\|x_0(t)\|_Q^2 + \|u_0(t)\|_R^2\} \le \mathrm{tr}\{2S\Gamma_w\}. \quad (32)$$

Proof. From Proposition 7 and (30) we have

$$J^*(\overline{N}+1) - J^*(0) \le \sum_{t=0}^{\overline{N}} \Big(- \mathrm{E}\{\|x_0(t)\|_Q^2 + \|u_0(t)\|_R^2\} + \mathrm{tr}\{2S\Gamma_w\}\Big)$$

From this, and since $J^*(t)$ is finite for every $t \in \mathbb{N}$, then

$$\begin{aligned} 0 &= \lim_{\overline{N}\to\infty} \frac{1}{\overline{N}} (J^*(\overline{N}+1) - J^*(0)) \\ &\le \lim_{\overline{N}\to\infty} \frac{1}{\overline{N}} \sum_{k=0}^{\overline{N}} (-\mathrm{E}\{\|x_k\|_Q^2 + \|u_k\|_R^2\} + \mathrm{tr}\{2S\Gamma_w\}) \\ &= \mathrm{tr}\{2S\Gamma_w\} - \lim_{\overline{N}\to\infty} \frac{1}{\overline{N}} \sum_{k=0}^{\overline{N}} \mathrm{E}\{\|x_k\|_Q^2 + \|u_k\|_R^2\} \end{aligned}$$

and the claim follows. ■

We state now a proposition about the convergence of state $x(t)$ of (1).

Proposition 10. Consider system (1) subject to the disturbance sequence $\{w_k\}_{k\in\mathbb{N}}$ that admits a correlation bound Γ_w for the matrix $A_K = A + BK$, under the control law (8) where v results from (22)-(28) and where Assumption 2 is also satisfied. Then, we have

$$\lim_{t\to\infty} \mathrm{E}\{\|x_t\|^2\} = \mathrm{E}\{\|e_\infty\|^2\} \le \mathrm{tr}\{\Gamma_\infty\} \quad (33)$$

where Γ_∞ is the matrix that satisfies the Lyapunov discrete equation $\Gamma_\infty = A_K\Gamma_\infty A_K^\top + \Gamma_w$.

Proof. From Proposition 7 we have

$$\lim_{t\to\infty} z_t = 0. \quad (34)$$

From (15), we have

$$\mathrm{E}\{\|e_t\|^2\} = \mathrm{tr}\{\mathrm{E}\{e_t e_t^\top\}\} \le \mathrm{tr}\{\Gamma_t\} \quad (35)$$

which implies, together with (34) and (10c), that

$$\begin{aligned} \lim_{t\to\infty} \mathrm{E}\{\|x_t\|^2\} &= \lim_{t\to\infty} \mathrm{E}\{\|z_t + e_t\|^2\} \\ &\le \lim_{t\to\infty} \Big\{\|z_t\|^2 + \mathrm{E}\{\|e_t\|^2\}\Big\} \le \lim_{t\to\infty} \mathrm{tr}\{\Gamma_t\} = \Gamma_\infty. \end{aligned}$$

Since, as in (Kofman et al., 2012, Subsection 2.1), the existence of Γ_∞ is ensured by $\rho(A_K) < 1$, the results is proved. ■

7. SIMULATION EXAMPLE

We test the stochastic model predictive control scheme (22)-(28) on a double integrator system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_k + w_k \quad (36)$$

with initial state $x_0 = [-3.8 \ 0]^\top$. The state and input stage costs are $Q = \mathrm{diag}([1, 10])$ and $R = 50$, respectively.

figure MPC small avec gca.png

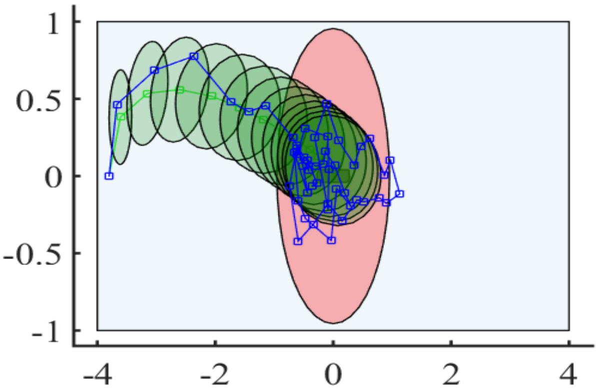

Fig. 1. Example of a stochastic tube with the nominal trajectory in green, the state trajectory in blue, the terminal set in black and the error expectation set in red, generated by the MPC scheme (22)-(28) for system (36) and for a violation probability of at most 35% for the state x of system (36) under the constraints (37).

The feedback controller K is designed as a discrete LQR controller based on the same weights Q and R.

A correlated disturbance sequence, corresponding to the output of a switched linear system excited with white noise is generated and its correlation bound,

$$\Gamma_w = \begin{bmatrix} 0.0426 & 0.0014 \\ 0.0014 & 0.1088 \end{bmatrix},$$

is determined following the procedure detailed in Fiacchini and Alamo (2021). We furthermore consider the following chance constraints

$$\Pr\{|x_1| \le 4, \ |x_2| \le 1\} \ge 1 - p_x, \quad (37a)$$
$$\Pr\{|u| \le 1.1\} \ge 1 - p_u \quad (37b)$$

with $p_x = 0.35$ and $p_u = 0.3$. The prediction horizon N is set to $N = 13$. The reachable sets for the nominal state z and the input v are determined by tightening $\mathbb{X}$ and $\mathbb{U}$, respectively, using the sequence of probabilistic reachable sets $\mathscr{E}(\Gamma_k, r)$, where Γ_k is given by the recursion (7) and r is taken equal to $r_x = \mathrm{Inv}\chi_2^2(1 - p_x)$ and $r_u = \mathrm{Inv}\chi_2^2(1 - p_u)$, respectively. As a consequence, we end up with the deterministic constraints

$$\mathbb{Z}_k = \Big\{ y \in \mathbb{R}^2 : \ |y_1| \le 4 - \sqrt{[1 \ 0]\, r_x\Gamma_k\, [1 \ 0]^\top}, \ |y_2| \le 0.75 - \sqrt{[0 \ 1]\, r_x\Gamma_k\, [0 \ 1]^\top} \Big\}$$

$$\mathbb{V}_k = \Big\{ v \in \mathbb{R} : |v| \le 1 - \sqrt{K r_u \Gamma_k K^\top} \Big\}.$$

The positively invariant set $\mathbb{Z}_f$ is obtained by applying the standard iterative procedure to compute the maximal positive invariant set for the deterministic system $z_{k+1} = (A + BK)z_k$ within the set $\mathbb{Z}_x \cap \mathbb{Z}_u$.

Fig. 1 shows the stochastic tube, the nominal trajectory, the terminal set, the state trajectory and the set where the state converges in expectation. The nominal state converges to the origin while the state realisations converge inside the set $\mathrm{E}\{\|e\|^2\} \le \mathrm{tr}\{\Gamma_\infty\} \approx 0.914$ in expectation. We can clearly see that the tube is stretching along the trajectory, as expected, to maintain the propagating error inside the reachable tube with the specified probability.

8. CONCLUSION

In this paper, we presented an indirect feedback tube based stochastic model predictive control for LTI systems affected by a large class of additive disturbances that can be correlated in time and that don't need to be stationary. This MPC algorithm only relies on the assumption of the existence of bounds on the mean and the covariance matrices of the disturbance distribution and exploits the notion of correlation bound to determine the probabilistic reachable and invariant sets that shape the stochastic tube.

Using tightening, chance constraints are reformulated into hard constraints and a suitable quadratic function on the nominal state, nominal control and the error correlation bound propagation is determined so that it leads to a tractable optimization problem. In addition, a nominal state value function that is an upper bound of the expected value of the quadratic cost on the true state and the true control input is provided. Chance constraints satisfaction, cost decrease, average cost bound and state convergence are guaranteed. A numerical simulation example to illustrate the chance constraints satisfaction and convergence results is provided.

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