Monographs, a Category of Graph Structures

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Abstract. Does a graph necessarily have nodes? May an edge be adjacent to itself and be a self-loop? These questions arise in the study of *graph structures*, i.e., monadic many-sorted signatures and the corresponding algebras. A simple notion of *monograph* is proposed that generalizes the standard notion of directed graph and can be drawn consistently with them. It is shown that monadic many-sorted signatures can be represented by monographs, and that the corresponding algebras are isomorphic to the monographs typed by the corresponding signature monograph. Monographs therefore provide a simple unifying framework for working with monadic algebras. Their simplicity is illustrated by deducing some of their categorial properties from those of sets.

Keywords: Universal algebra · Monadic signatures · Category theory

1 Introduction

Many different notions of graphs are used in mathematics and computer science: simple graphs, directed graphs, multigraphs, hypergraphs, etc. One favourite notion in the context of logic and rewriting is that also known as *quivers*, i.e., structures of the form (N, E, s, t) where N, E are sets and s, t are functions from E (edges) to N (nodes), identifying the source and target tips of every edge (or arrow). One reason for this is that the category of quivers is isomorphic to the category of Σ_g-algebras, where Σ_g is the signature with two sorts `nodes` and `edges` and two operator names `src` and `tgt` of type `edges` → `nodes`. In conformity with this tradition, by *graph* we mean quiver throughout this paper.

In order to conveniently represent elaborate data structures it is often necessary to enrich the structure of graphs with other objects: nodes or edges may be labelled with elements from a fixed set, or with the elements of some algebra, or graphs may be typed by another graph (i.e., a graph comes with a morphism from itself to this other graph, considered as its type). An interesting example can be found in [4] with the notion of E-graphs, since some of these new objects are also considered as edges or nodes. More precisely, an E-graph is an algebra whose signature Σ_e can be represented by the following graph:
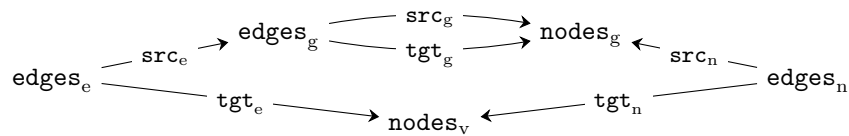
The names given to the sorts and operators help to understand the structure of the Σ_e-algebras: the edges_g relate the nodes_g among themselves, the edges_n relate the nodes_g to the nodes_v, and the edges_e relate the edges_g to the nodes_v. These extra edges allow values (elements of nodes_v) to be attached to edges and nodes of the inner graph. But then we see that in E-graphs some edges can be adjacent to other edges. This is non standard, but we may still accept such structures as some form of graph, if only because we understand how they can be drawn.

Hence the way of generalizing the notion of graphs seems to involve a generalization of the signature of graphs considered as algebras. This path has been followed by Michael Löwe in [7], where a *graph structure* is defined as a monadic many-sorted signature. Indeed in the examples above, and in many examples provided in [7], all operators have arity 1 and can therefore be considered as edges from their domain to their range sort. Is this the reason why they are called graph structures? But the example above shows that, if Σ_e-algebras are interpreted as graphs of some form, these are very different from the graph Σ_e. Besides, it is not convenient that our understanding of such structures should be based on syntax, i.e., on the particular names given to objects in the signature.

Furthermore, it is difficult to see how the algebras of some very simple monadic signatures can be interpreted as graphs of any form. Take for instance Σ_g and reverse the target function to $\text{tgt} : \text{nodes} \rightarrow \text{edges}$. Then there is a symmetry between the sorts nodes and edges, which means that in an algebra of this signature nodes and edges would be objects of the same nature. Is this still a graph? Can we draw it? Worse still, if the two sorts are collapsed into one, does it mean that a node/edge can be adjacent to itself?

We may address these problems by restricting graph structures to some class of monadic signatures whose algebras are guaranteed to behave in an orthodox way, say by exhibiting clearly separated edges and nodes. But this could be prone to arbitrariness, and it would still present another drawback: that the notion of graph structure does not easily give rise to a category. Indeed, it is difficult to define morphisms between algebras of different signatures, if only because they can have any number of carrier sets.

The approach adopted here is rather to reject any *structural* distinction between nodes and edges, to gather them all in a *single* carrier set and to rely on a *unique* function to distinguish them. For this reason, the resulting structures are called *monographs*. The definitions of monographs and their morphisms, given in Section 3, are thus quite simple although for reasons that will only be made clear in Section 5 we use sequences of ordinal length, defined in Section 2.

Prior to examining their mathematical properties, we show in Section 4 that these structures deserve to be considered as "graphs" since they can be represented by drawings, provided of course that they are finite (in a strong sense). In particular, such drawings correspond to the standard way of drawing a graph, for those monographs that can be identified with standard graphs.

The relationship between monographs and graph structures (monadic signatures) is explored in Section 5, using relevant notions from category theory

defined in Section 2. This gives rise to an isomorphism-dense embedding of monographs into many-sorted monadic signatures, that will expose a fundamental difference between them. This result is used in Section 6 to exhibit isomorphisms between the categories of (partitioned) algebras of all graph structures and all slice categories of monographs, i.e., the categories of typed monographs.

As a result of their simplicity the category of monographs and some of its subcategories can easily be shown to share a number of properties with **Graphs**, as illustrated in Section 7. Concluding remarks are given in Section 8. The missing proofs can be found in [3].

2 Notations and Definitions

For any sets A, B, any relation $r \subseteq A \times B$ and any subset $X \subseteq A$, we write $r[X]$ for the set $\{y \in B \mid x \in X \wedge (x,y) \in r\}$. For any $x \in A$, by abuse of notation we write $r[x]$ for $r[\{x\}]$. We write r^{-1} for the relation $\{(y,x) \mid (x,y) \in r\} \subseteq B \times A$. If r is functional we write $r(x)$ for the unique element of $r[x]$, and if $q \subseteq B' \times C$ is also functional for sets B' and C such that $r[A] \subseteq B'$, we write $q \circ r$ for the functional relation $\{(x, q(r(x))) \in A \times C \mid x \in A\}$.

A *function* $f : A \to B$ is a triple (A, r, B) where A and B are sets, respectively called the *domain* and *codomain* of f, and $r \subseteq A \times B$ is a functional relation. Hence A must be the set $\{x \mid (x,y) \in r\}$ and B may be any superset of the image $r[A]$, also denoted $f[A]$ (and generally $f[X]$ and $f(x)$ stand for $r[X]$ and $r(x)$ respectively). If $A = \varnothing$ and the codomain B can be determined from the context then we write ε for $f = (\varnothing, \varnothing, B)$. If $g = (B, q, C)$ is a function, then $g \circ f \stackrel{\text{def}}{=} (A, q \circ r, C)$. We may also compose a functional relation with a function, that then denotes its underlying functional relation, so that $q \circ f = g \circ r = q \circ r$.

More generally, any object and its obvious underlying object will be written similarly, i.e., the forgetful functor will be omitted, whenever the ambiguity can easily be lifted from the context. Category theoretic concepts and notations will be consistent with [1], unless stated otherwise. In particular, id_A denotes the identity morphism of the object A in any category, except in **Sets** where it is denoted Id_A (the identity function of A) as a way of reminding the reader that A is a set. In **Sets** the standard product $\times$, projections π_1 and π_2 and coproduct $+$ are used. For functional relations f, g with the same domain A, let $\langle f, g \rangle(x) \stackrel{\text{def}}{=} (f(x), g(x))$ for all $x \in A$; if $f : A \to B$ and $g : A \to C$ are functions then $\langle f, g \rangle : A \to B \times C$ is the unique function such that $\pi_1 \circ \langle f, g \rangle = f$ and $\pi_2 \circ \langle f, g \rangle = g$.

Isomorphism between objects in a category, or between categories, is denoted by the symbol $\simeq$. For any two categories $\boldsymbol{A}$ and $\boldsymbol{B}$, a functor $F : \boldsymbol{A} \to \boldsymbol{B}$ is *faithful* (resp. *full*) if F is injective (resp. surjective) from the set of $\boldsymbol{A}$-morphisms from X to Y to the set of $\boldsymbol{B}$-morphisms from FX to FY, for all $\boldsymbol{A}$-objects X and Y. If F is faithful and injective on objects, then it is an *embedding*. F is *isomorphism-dense* if for every $\boldsymbol{B}$-object Y there exists an $\boldsymbol{A}$-object X such that $FX \simeq Y$. Categories $\boldsymbol{A}$ and $\boldsymbol{B}$ are *equivalent*, written $\boldsymbol{A} \cong \boldsymbol{B}$, if there is a full, faithful and isomorphism-dense functor from one to the other.

For any object T of $\mathbf{A}$, the *slice category*[1] $\mathbf{A}/T$ has as objects the morphisms of codomain T of $\mathbf{A}$, as morphisms from object $f : A \to T$ to object $g : B \to T$ the morphisms $k : A \to B$ of $\mathbf{A}$ such that $g \circ k = f$, and the composition of morphisms in $\mathbf{A}/T$ is defined as the composition of the underlying morphisms in $\mathbf{A}$. T is a *terminal object* of $\mathbf{A}$ if for every object A of $\mathbf{A}$ there is a unique $f : A \to T$. It is easy to see that id_T is a terminal object of $\mathbf{A}/T$, and that if T is a terminal object of $\mathbf{A}$ then $\mathbf{A} \simeq \mathbf{A}/T$.

Since edges will have arbitrary lengths, including infinite ones, we use the notion of ordinal and refer to [11] for their properties. An *ordinal* is a set α such that every element of α is a subset of α, and such that the restriction of the membership relation $\in$ to α is a strict well-ordering of α (every non empty subset of α has a minimal element). Every member of an ordinal is an ordinal, and we write $\lambda < \alpha$ for $\lambda \in \alpha$. For any two ordinals α, β we have either $\alpha < \beta$, $\alpha = \beta$ or $\alpha > \beta$. Every ordinal α has a successor $\alpha \cup \{\alpha\}$, denoted $\alpha + 1$. Natural numbers n are identified with finite ordinals, so that $n = \{0, 1, \ldots, n-1\}$ and $\omega = \{0, 1, \ldots\}$ is the smallest infinite ordinal.

Definition 2.1 (E**-sequences** s **of length** λ**, elements** s_ι**,** $x \mid s$**).** *For any set E and ordinal λ, an E-sequence s of length λ is an element of E^λ, i.e., a function $s : \lambda \to E$. For any $s \in E^\lambda$ and $\iota < \lambda$, the image $s(\iota)$ is written s_ι. If λ is finite and non zero then s can be described as $s = s_0 \cdots s_{\lambda-1}$. For any $x \in E$ we write $x \mid s$ and say that x occurs in s if there exists $\iota < \lambda$ such that $s_\iota = x$.*

Take for instance $E = \{x, y\}$, then $s = xyx$ describes the E-sequence of length $3 = \{0, 1, 2\}$ such that $s_0 = s_2 = x$ and $s_1 = y$. We have $x \mid xyx$ and $y \mid xyx$.

Note that there is no set of *all* E-sequences, hence the following notions.

Definition 2.2 (**sets** $E^{<\alpha}$**, functions** $f^{<\alpha}$**).** *For any set E and ordinal α, let $E^{<\alpha} \overset{\mathrm{def}}{=} \bigcup_{\lambda < \alpha} E^\lambda$. For any set F and function $f : E \to F$, let $f^{<\alpha} : E^{<\alpha} \to F^{<\alpha}$ be the function defined by $f^{<\alpha}(s) \overset{\mathrm{def}}{=} f \circ s$ for all $s \in E^{<\alpha}$.*

Thus $s = xyx$ is an element of $E^{<4}$, $E^{<5} \ldots$ but not of $E^{<3} = E^0 \cup E^1 \cup E^2$. If $f : E \to E$ is the function that swaps x and y ($f = (x\ y)$ in cycle notation), then $f^{<4}(s)_0 = (f \circ s)_0 = f(s_0) = y$, etc., hence $f^{<4}(s) = yxy$.

We have $E^{<0} = \varnothing$ and $E^{<1} = E^0 = \{\varepsilon\}$, i.e., the function ε (with codomain E) is the only E-sequence of length 0. It is obvious that for any $s \in E^{<\alpha}$ and any ordinal $\beta \geqslant \alpha$, we have $s \in E^{<\beta}$ and $f^{<\beta}(s) = f^{<\alpha}(s)$. If $f : E \to F$ and $g : F \to G$ then $(g \circ f)^{<\alpha} = g^{<\alpha} \circ f^{<\alpha}$.

If s and s' are respectively E- and F-sequences of length λ, then they are both functions with domain λ hence there is a function $\langle s, s' \rangle$ of domain λ. Thus $\langle s, s' \rangle$ is an $(E \times F)$-sequence of length λ, and then $\pi_1^{<\alpha}(\langle s, s' \rangle) = \pi_1 \circ \langle s, s' \rangle = s$ and similarly $\pi_2^{<\alpha}(\langle s, s' \rangle) = s'$ for all $\alpha > \lambda$.

If $f : E \to F$ and $g : E \to G$ then $\langle f, g \rangle : E \to F \times G$, hence for all $s \in E^{<\alpha}$ of length $\lambda < \alpha$ we have $\langle f, g \rangle^{<\alpha}(s) = \langle f, g \rangle \circ s = \langle f \circ s, g \circ s \rangle = \langle f^{<\alpha}(s), g^{<\alpha}(s) \rangle$ is an $(F \times G)$-sequence of length λ.

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[1] This is called the *comma category of $\mathbf{A}$ over T* in [5, Definition 4.19], and the *category of objects over T* in [1, Exercise 3K]. We adopt the terminology and notation of [4].

3 Categories of Monographs

We may now define the structure of monographs and some related notions. These definitions will be illustrated by a running example and will be followed by a number of basic facts on monographs and a comparison with graphs.

In graphs, edges are arbitrary objects to which are associated two arbitrary objects called nodes. We may thus associate to each edge a sequence of nodes of length 2. In monographs, the corresponding sequences may contain arbitrary edges and have arbitrary lengths.

Definition 3.1 (monographs, grade, edges E_a, length $\ell_a(x)$, trace $\mathrm{tr}(A)$).
For any ordinal α, an α-monograph A is a pair (E, a) where E is a set whose elements are called edges *of A, and $a \subseteq E \times E^{<\alpha}$ is a functional relation, called the* map *of A. A pair $A = (E, a)$ is a* monograph *if it is an α-monograph for some ordinal α; we then say that α is an ordinal* for *A. The* grade *of A is the smallest ordinal for A. Monographs will usually be denoted by upper-case letters (A, B, ...), their map by the corresponding lower-case letter (a, b, ...) and their set of edges E_a, E_b, ...*

The length *$\ell_a(x)$ of an edge $x \in E_a$ is the length of $a(x)$, i.e., the unique ordinal λ such that $a(x) \in E_a^\lambda$. The* trace *of A is the set $\mathrm{tr}(A) \stackrel{\mathrm{def}}{=} \ell_a[E_a]$. For any set O of ordinals, an O-monograph[2] A is a monograph such that $\mathrm{tr}(A) \subseteq O$.*

Take for instance the monograph $A = (\{x, y\}, a)$ where $a = \{(x, xyx), (y, yxy)\}$ (i.e., $a(x) = xyx$ and $a(y) = yxy$), then A is a 4-monograph and more precisely a $\{3\}$-monograph since $\mathrm{tr}(A) = \{3\}$, but A is not a 3-monograph, hence the grade of A is 4.

A monograph A is essentially defined by its map a, since $E_a = \{x \mid (x, y) \in a\}$. But a is only a functional relation and not a function; there is no codomain to artificially separate monographs that have the same map. This means in particular that any α-monograph is a β-monograph for all $\beta \geqslant \alpha$. It is easy to see that there always exists an ordinal *for* any two monographs, and indeed for any set of monographs (e.g. the sum of their grades).

Definition 3.2 (adjacency, self-loops, nodes N_a, standard monographs).
For any monograph A and edges $x, y \in E_a$, x is adjacent *to y if $y \mid a(x)$. A* self-loop *is an edge x that is adjacent only to x, i.e., such that $a(x)$ is an $\{x\}$-sequence. A* node *is an edge of length 0, and the set of nodes of A is written N_a. A is* standard *if $a(x)$ is a N_a-sequence for all $x \in E_a$.*

We see on the running example that $a(x) = xyx$, hence x is adjacent to y and to itself, but is not a self-loop. Similarly, $a(y) = yxy$ yields that y is adjacent to x and to itself, but is not a self-loop. A has two edges but no nodes, hence A is not standard.

A monograph A is not normally defined by its adjacency relation $y \mid a(x)$ on edges, since the sequences $a(x)$ may not be uniquely determined by this relation.

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[2] Note that any ordinal α is a set of ordinals, and that a monograph A is an α-monograph iff $\mathrm{tr}(A) \subseteq \alpha$.

The adjacency relation may not be symmetric: a node is never adjacent to any edge, while edges may be adjacent to nodes.

Definition 3.3 (morphisms of monographs). *A morphism f from monograph $A = (E_a, a)$ to monograph $B = (E_b, b)$, denoted $f : A \to B$, is a function $f : E_a \to E_b$ such that $f^{<\alpha} \circ a = b \circ f$, where α is any[3] ordinal for A.*

Building on the running example, we consider the function $f = (x\ y)$ from E_a to E_a, we see that $f^{<4} \circ a(x) = f^{<4}(xyx) = yxy = a(y) = a \circ f(x)$ and similarly that $f^{<4} \circ a(y) = f^{<4}(yxy) = xyx = a(x) = a \circ f(y)$, hence $f^{<4} \circ a = a \circ f$ and f is therefore a morphism from A to A. Since $f \circ f = \mathrm{Id}_{E_a}$ is obviously the identity morphism of A then f is an isomorphism.

The length of edges are preserved by morphisms: if f is a morphism from A to B then for all $x \in E_a$, $\ell_b(f(x))$ is the length of the E_b-sequence $b \circ f(x) = f^{<\alpha} \circ a(x)$, whose length is the same as the E_a-sequence $a(x)$, i.e., $\ell_a(x) = \ell_b(f(x))$. Hence $\mathrm{tr}(A) \subseteq \mathrm{tr}(B)$, and the equality holds if f is surjective. This means that if B is an O-monograph then so is A. This also means that the grade of B is at least that of A, hence that every ordinal for B is an ordinal for A.

Given morphisms f from A to B and g from B to C, we see that $g \circ f$ is a morphism from A to C by letting α be an ordinal for B, so that

$$(g \circ f)^{<\alpha} \circ a = g^{<\alpha} \circ f^{<\alpha} \circ a = g^{<\alpha} \circ b \circ f = c \circ g \circ f.$$

Definition 3.4 (categories of monographs). *Let* **MonoGr** *be the category of monographs and their morphisms. Let* **SMonoGr** *be its full subcategory of standard monographs. For any set O of ordinals, let O-**MonoGr** (resp. O-**SMonoGr**) be the full subcategory of O-monographs (resp. standard O-monographs).*

A monograph A is finite *if E_a is finite. Let* **FMonoGr** *be the full subcategory of finite ω-monographs.*

It is customary in Algebraic Graph Transformation to call *typed graphs* the objects of **Graphs**/G, where G is a graph called *type graph*, see e.g. [4]. A type graph is therefore seen as the specification of a category of typed graphs. In particular, the terminal graph G_t specifies the whole category of graphs since **Graphs** $\simeq$ **Graphs**/G_t. We will extend this terminology to monographs and refer to the objects of **MonoGr**/T as the *monographs typed by T*. The monograph T is then considered as a *type monograph* and hence as a specification for a category of typed monographs.

It is obvious from the above that for any set O of ordinals, if T is an O-monograph then **MonoGr**/$T = O$-**MonoGr**/T. Similarly, if $f : A \to B$ then

$$f^{-1}[N_b] = \{x \in E_a \mid \ell_b(f(x)) = 0\} = N_a,$$

hence if $b \circ f(x)$ is a N_b-sequence for some $x \in E_a$ then for all $\iota < \ell_a(x)$,

$$f(a(x)_\iota) = (f \circ a(x))_\iota = (f^{<\alpha}(a(x)))_\iota = (f^{<\alpha} \circ a(x))_\iota = (b \circ f(x))_\iota \in N_b,$$

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[3] Imposing the grade of A for α here would be a useless constraint. Note that the equation $f^{<\alpha} \circ a = b \circ f$ holds for all ordinals α for A iff it holds for one.

hence $a(x)_\iota \in \mathrm{N}_a$ and $a(x)$ is therefore a N_a-sequence. Thus A is standard whenever B is standard. This proves that for any standard monograph (resp. standard O-monograph) T we have $\mathbf{MonoGr}/T = \mathbf{SMonoGr}/T$ (resp. $\mathbf{MonoGr}/T = O\text{-}\mathbf{SMonoGr}/T$).

The introduction of the present section suggests a similitude between graphs and standard $\{0, 2\}$-monographs. It is actually easy to define a functor $\mathsf{M} :$ $\mathbf{Graphs} \to \{0, 2\}\text{-}\mathbf{SMonoGr}$ by mapping any graph $G = (N, E, s, t)$ to $\mathsf{M}G = (N{+}E, g)$ where $g(x) = \varepsilon$ for all $x \in N$ and $g(e) = s(e)t(e)$ for all $e \in E$ (and similarly graph morphisms are transformed into morphisms of monographs through a coproduct of functions). It is easy to see that M is an equivalence of categories[4]. This means that for any graph G we have $\mathbf{Graphs}/G \cong \mathbf{MonoGr}/\mathsf{M}G$, hence that typed graphs can be represented as typed monographs.

4 Drawing Monographs

Obviously we may endeavour to draw a monograph A only if E_a is finite and if its edges have finite lengths, i.e., if A is a finite ω-monograph. If we require that any monograph $\mathsf{M}G$ should be drawn as the graph G, then a node should be represented by a bullet $\bullet$ and an edge of length 2 by an arrow $\frown\!\!\!\rightarrow$ joining its two adjacent nodes. But generally the adjacent edges may not be nodes and there might be more than 2 of them, hence we adopt the following convention: an edge e of length at least 2 is represented as a sequence of connected arrows with an increasing number of tips
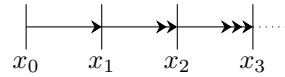
$$x_0 \qquad x_1 \qquad x_2 \qquad x_3$$

(where $a(e) = x_0 x_1 x_2 x_3 \cdots$) and such that any arrow should enter x_i at the same angle as the next arrow leaves x_i. This is important when x_i is a node since several adjacent edges may traverse the corresponding bullet, and they should not be confused. For the sake of clarity we will also represent symmetric adjacencies by a pair of crossings rather than a single one, e.g., if $a(e) = xe'y$ and $a(e') = xey$, where x and y are nodes, the drawing may be

 but not
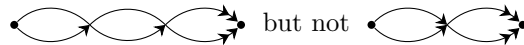
It is sometimes necessary to name the edges in a drawing. We may then adopt the convention used for drawing diagrams in a category: the bullets are replaced by the names of the corresponding nodes, and arrows are interrupted to write their name at a place free from intersection, as in
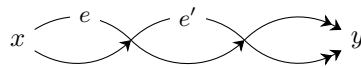
Note that no confusion is possible between the names of nodes and those of other edges, e.g., in

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[4] This can be seen as a consequence of Corollary 6.8 below and of the isomorphism between $\mathbf{Graphs}$ and the category of Σ_g-algebras.
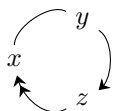
it is clear that x and z are nodes since arrow tips point to them, and that y is the name of an edge of length 3.

As is the case of graphs, monographs may not be planar and drawing them may require crossing edges that are not adjacent; in this case no arrow tip is present at the intersection and no confusion is possible with the adjacency crossings. However, it may seem preferable in such cases to erase one arrow in the proximity of the other, as in $\times$.

There remains to represent the edges of length 1. Since $a(e) = x$ is standardly written $a : e \mapsto x$, the edge e will be drawn as

$$x$$

In order to avoid confusion there should be only one arrow out of the thick dash, e.g., if $a(e) = e'$ and $a(e') = ex$ where x is a node, the drawing may be

 but not

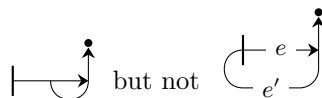since this last drawing may be interpreted as the monograph $a(e') = x$ and $a(e) = e'e'$, that is not isomorphic to the intended monograph.

With these conventions it is only possible to read a drawing of any finite ω-monograph A as the monograph A itself if all edges are named in the drawing, or as some monograph isomorphic to A otherwise. Note that this would not be true if the map a was a function rather than a functional relation, because its codomain $\mathrm{E}_a^{<\alpha}$ is not pictured. It would of course be possible to add the ordinal α to the drawing, but then would it still qualify as a drawing?

Note that the drawing of a graph or of a standard $\{0, 2\}$-monograph can be read either as a graph G or as a monograph A, and then $\mathsf{M}G \simeq A$.

Possible drawings for the self-loops of length 1 to 4 are given below.

We may also draw typed monographs, i.e., monographs A equipped with a morphism f from A to a monograph T, considered as a type. Then every edge $e \in \mathrm{E}_a$ has a type $f(e)$ that can be written at the proximity of e. For instance, let T be the monograph
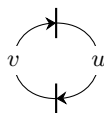
then a monograph typed by T is drawn with labels u and v as in
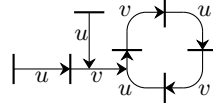

Of course, knowing that f is a morphism sometimes allows to deduce the type of an edge, possibly from the types of adjacent edges. In the present case, indicating a single type would have been enough to deduce all the others.

5   Monadic Signatures as Monographs

As mentioned in Section 1, graph structures, i.e., monadic many-sorted signatures, can be represented as graphs. More precisely, there is an obvious isomorphism between the category **Graphs** and the category of monadic signatures defined below[5].

Definition 5.1 (monadic signatures). *A* (monadic) signature *is a function* $\Sigma : \Omega \to S \times S$; *the elements of its domain* Ω, *that may be written* Σ_{op}, *are called* operator names *and the elements of S, that may be written Σ_{srt}, are called* sorts. Σ *is* finite *if both Ω and S are finite. Let $\Sigma_{\mathrm{ds}} \overset{\mathrm{def}}{=} \pi_1 \circ \Sigma$ and $\Sigma_{\mathrm{rs}} \overset{\mathrm{def}}{=} \pi_2 \circ \Sigma$, then $\Sigma_{\mathrm{ds}}(o)$ and $\Sigma_{\mathrm{rs}}(o)$ are respectively the* domain *and* range sorts *of $o \in \Omega$.*

A morphism m from signature Σ to signature Σ' is a pair $m = (m_{\mathrm{op}}, m_{\mathrm{srt}})$ of functions, where $m_{\mathrm{op}} : \Sigma_{\mathrm{op}} \to \Sigma'_{\mathrm{op}}$ and $m_{\mathrm{srt}} : \Sigma_{\mathrm{srt}} \to \Sigma'_{\mathrm{srt}}$, such that

$$\Sigma' \circ m_{\mathrm{op}} = (m_{\mathrm{srt}} \times m_{\mathrm{srt}}) \circ \Sigma.$$

Let $\mathrm{id}_\Sigma \overset{\mathrm{def}}{=} (\mathrm{Id}_{\Sigma_{\mathrm{op}}}, \mathrm{Id}_{\Sigma_{\mathrm{srt}}})$ and given two morphisms $m : \Sigma \to \Sigma'$ and $n : \Sigma' \to \Sigma''$, let $n \circ m \overset{\mathrm{def}}{=} (n_{\mathrm{op}} \circ m_{\mathrm{op}}, n_{\mathrm{srt}} \circ m_{\mathrm{srt}})$; then $\mathrm{id}_\Sigma : \Sigma \to \Sigma$ and $n \circ m : \Sigma \to \Sigma''$ are morphisms. Let $\mathbf{MonSig}$ be the category of monadic signatures and their morphisms.

The obvious isomorphism from **MonSig** to **Graphs** maps every monadic signature $\Sigma : \Omega \to S \times S$ to the graph $(S, \Omega, \Sigma_{\mathrm{ds}}, \Sigma_{\mathrm{rs}})$. But we have seen in Section 1 on E-graphs that this representation of the monadic signature Σ_{e} bears no relation with the expected graphical representations of E-graphs. It would be more natural to represent Σ_{e} as an E-graph, or more precisely as a monograph whose drawing corresponds to that of an E-graph.

Since the image $\Sigma(\Omega)$ is a subset of $S \times S$, it can be viewed as a binary relation on S, hence there exists a monograph with S as set of edges whose adjacency relation is exactly $\Sigma(\Omega)$. However, this monograph may not be unique since, as mentioned in Section 3, a monograph is not generally determined by its adjacency relation. Similarly, the direction of edges in E-graphs is not determined by the signature Σ_{e}, it is only a convention given by the names of its operators.

---

[5] For the sake of simplicity, we do not allow the overloading of operator names as in [10], which would be irrelevant anyway since we wish to abstract the syntax away, hence to consider signatures only up to isomorphisms.

For this reason it is more convenient to define a function *from* monographs *to* monadic signatures. Indeed, any monograph determines a unique adjacency relation that can then be interpreted as a signature. The sorts of this signature are exactly the edges of the monograph, and for every edge
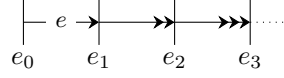
we create a first operator name from sort e to sort e_0, a second operator name from e to e_1, and so on for all elements of the sequence.

Definition 5.2 (functor $\mathsf{S} : \mathbf{MonoGr} \to \mathbf{MonSig}$). *To every monograph $T = (\mathrm{E}_t, t)$ we associate the set*

$$\Omega_t \overset{\text{def}}{=} \{(e, \iota) \mid e \in \mathrm{E}_t \wedge \iota < \ell_t(e)\}$$

of operator names, and the signature $\mathsf{S}T : \Omega_t \to \mathrm{E}_t \times \mathrm{E}_t$ defined by

$$\mathsf{S}T(e, \iota) \overset{\text{def}}{=} (e, t(e)_\iota) \ \text{for all } (e, \iota) \in \Omega_t.$$

To every morphism of monographs $f : T \to U$ we associate the morphism $\mathsf{S}f : \mathsf{S}T \to \mathsf{S}U$ defined by

- $(\mathsf{S}f)_{\mathrm{op}}(e, \iota) \overset{\text{def}}{=} (f(e), \iota) \in \Omega_u$ *for all $(e, \iota) \in \Omega_t$, and*
- $(\mathsf{S}f)_{\mathrm{srt}} \overset{\text{def}}{=} f$ *(as a function from E_t to E_u).*

Note that the signature $\mathsf{S}T$ is finite iff T is a finite ω-monograph.

Lemma 5.3. S *is an embedding from* $\mathbf{MonoGr}$ *to* $\mathbf{MonSig}$.

The next lemma uses the Axiom of Choice through its equivalent formulation known as the Numeration Theorem [11].

Lemma 5.4. S *is isomorphism-dense: for every monadic signature Σ there exists a monograph T such that $\mathsf{S}T \simeq \Sigma$.*

Proof (sketch). For any object $\Sigma : \Omega \to S \times S$ of **MonSig**, and for every sort $s \in S$ let O_s be the set of operator names in Ω whose domain sort is s. By the Numeration Theorem there exists an ordinal λ_s equipollent to O_s, i.e., such that there exists a bijection $\sigma_s : \lambda_s \to O_s$. Let $t(s)$ be the S-sequence of length λ_s such that, for all $\iota < \lambda_s$, $t(s)_\iota$ is the range sort of the operator name $\sigma_s(\iota)$, and let T be the monograph (S, t). It is then easy to see that the signature $\mathsf{S}T$ is isomorphic to Σ by mapping every operator name $(s, \iota) \in \Omega_t$ to the original operator name $\sigma_s(\iota) \in \Omega$. $\qquad\square$

The reason why monographs require edges of ordinal length now becomes apparent: the length of an edge s is the cardinality of O_s, i.e., the number of operators whose domain sort is s, and no restriction on this cardinality is ascribed to signatures. The bijections σ_s provide linear orderings of the sets O_s.

We now show on an example that the functor S is not full, hence that S is not an equivalence between the categories **MonoGr** and **MonSig**.

Example 5.5. The monadic signature Σ_g has two operators `src`, `tgt`, two sorts in $S_g = \{\texttt{nodes}, \texttt{edges}\}$ and is defined by $\Sigma_g : \texttt{src}, \texttt{tgt} \mapsto (\texttt{edges}, \texttt{nodes})$. Then $O_{\texttt{nodes}} = \varnothing$ and $O_{\texttt{edges}} = \{\texttt{src}, \texttt{tgt}\}$ has 2 elements. Let $\sigma : 2 \to O_{\texttt{edges}}$ be the bijection defined by $\sigma : 0 \mapsto \texttt{src}, 1 \mapsto \texttt{tgt}$ and t be the map defined by

$$t(\texttt{nodes}) = \varepsilon, \ t(\texttt{edges}) = \texttt{nodes}\,\texttt{nodes}$$

then $T_g = (S_g, t)$ is a monograph. The signature ST_g has the same sorts as Σ_g, two operators $(\texttt{edges}, 0)$, $(\texttt{edges}, 1)$ and is defined by

$$ST_g : (\texttt{edges}, 0), (\texttt{edges}, 1) \mapsto (\texttt{edges}, \texttt{nodes}).$$

Hence ST_g is indeed isomorphic to Σ_g. However, the only automorphism of T_g is id_{T_g}, while Σ_g has a non trivial automorphism $m = ((\texttt{src}\,\texttt{tgt}), \mathrm{Id}_{S_g})$ (in cycle notation), hence S is not surjective on morphisms.

This automorphism reflects the fact that Σ_g does not define an order between its operators `src` and `tgt`. Directing edges as arrows from `src` to `tgt` is only a matter of convention that is reflected in the choice of σ above. This contrasts with monographs, where the edges are inherently directed by the ordinals in their length. In the translation from **MonoGr** to **MonSig**, the direction of edges are necessarily lost. Note however that in this example, since `src` and `tgt` have the same range sort, the other obvious choice for σ yields the same monograph T_g.

We therefore see that in most cases there are many distinct, non isomorphic monographs that faithfully represent a single signature, depending on the chosen direction of their edges. Monographs carry more information than signatures, but the additional information is precisely the kind of information that has to be provided by means of syntax when a monadic signature is intended as a graph structure. By observing the examples given in [7, Section 3.1], we see that this syntactic information mostly consists of an order on operators, given either by indices or by calling them "source" and "target".

More precisely, Examples 3.1 to 3.4 of [7] are monadic signatures defining graphs, edge-labelled and labelled graphs (with a possibly infinite set of labels) and hypergraphs (see Example 6.10 below). Examples 3.5 and 3.6 are special representations of signatures and terms. We observe in these examples a strict partition of sorts into domain and range sorts. It is easy to see that a monograph T is standard iff the signature ST is *separated*, i.e., no sort occurs both as a domain and a range sort. Thus the range sorts are the nodes of T and the domain sorts are edges of diverse lengths that relate nodes.

We now consider in detail Example 3.7 of [7], the only example of a monadic signature that is not separated.

Example 5.6. Let Σ_a be the monadic signature defined by the set of sorts

$$S_a = \{\texttt{V}, \texttt{E}, \texttt{V-Ass}, \texttt{E-Ass}, \texttt{Graph}, \texttt{Morphism}\}$$

and the following operators:

$$
\Sigma_a : \quad
\begin{aligned}
\texttt{s}, \texttt{t} \quad &\mapsto (\texttt{E}, \texttt{V}) \\
\texttt{s}_\texttt{V}, \texttt{t}_\texttt{V} \quad &\mapsto (\texttt{V-Ass}, \texttt{V}) \\
\texttt{s}_\texttt{E}, \texttt{t}_\texttt{E} \quad &\mapsto (\texttt{E-Ass}, \texttt{E}) \\
\texttt{s}_\texttt{G}, \texttt{t}_\texttt{G} \quad &\mapsto (\texttt{Morphism}, \texttt{Graph}) \\
\texttt{abstract}_\texttt{V} \quad &\mapsto (\texttt{V}, \texttt{Graph}) \\
\texttt{abstract}_\texttt{E} \quad &\mapsto (\texttt{E}, \texttt{Graph}) \\
\texttt{abstract}_{\texttt{V-Ass}} \quad &\mapsto (\texttt{V-Ass}, \texttt{Morphism}) \\
\texttt{abstract}_{\texttt{E-Ass}} \quad &\mapsto (\texttt{E-Ass}, \texttt{Morphism})
\end{aligned}
$$

An ALR-graph is a Σ_a-algebra. It is not very clear how such structures can be considered as graphs, especially because there is no conventional way of ordering the operator name `abstract` w.r.t. `s` and `t` (we only know that sources come before targets). Textual explanations are provided in [7] to help the reader's understanding of ALR-graphs:

> ALR-graphs not only allow to represent arbitrary labeled graphs but also morphisms between graphs. Since morphisms map vertices to vertices and edges to edges, they are represented by pairs of vertex assignments and edge assignments. In order to keep track of which assignment belongs to which morphism, an abstraction operator is introduced in ALR-graphs which allows to group vertices and edges into graphs and vertex and edge assignments into morphisms. Thus, ALR-graphs as algebras w.r.t. the graph structure below are able to represent the diagram level (graphs and morphisms) and the object level (vertices, edges, and assignments) in a single structure.

The explanations given below on the corresponding monograph are much simpler. A choice of ordering `abstract` between `s` and `t` is made, that helps visualize the structure. The set of edges is of course S_a, and the map t_a is defined by:

$t_a(\texttt{Graph}) = \varepsilon$	graphs are represented by nodes
$t_a(\texttt{V}) = \texttt{Graph}$	to every vertex is associated a graph
$t_a(\texttt{E}) = \texttt{V}\,\texttt{Graph}\,\texttt{V}$	an edge joins two vertices through a graph
$t_a(\texttt{Morphism}) = \texttt{Graph}\,\texttt{Graph}$	a morphism joins two graphs
$t_a(\texttt{V-Ass}) = \texttt{V}\,\texttt{Morphism}\,\texttt{V}$	a vertex association joins two vertices through a morphism
$t_a(\texttt{E-Ass}) = \texttt{E}\,\texttt{Morphism}\,\texttt{E}$	an edge association joins two edges through a morphism.

We thus see that specifying a monadic signature by a monograph may yield a better understanding of the structure of the corresponding algebras, at least if these are meant as graph structures. The next section shows that this is always possible.

6 Monadic Algebras as Typed Monographs

Now that monographs have been embedded in graph structures, the relation that the corresponding algebras bear with these monographs may be investigated. We first need a definition of Σ-algebras and Σ-homomorphisms that, for the sake of simplicity, are restricted to monadic signatures.

Definition 6.1 (Σ-algebras). *For any monadic signature $\Sigma : \Omega \to S \times S$, a Σ-algebra $\mathcal{A}$ is a pair $((\mathcal{A}_s)_{s\in S}, (o^{\mathcal{A}})_{o\in\Omega})$ where $(\mathcal{A}_s)_{s\in S}$ is an S-indexed family of sets and $o^{\mathcal{A}} : \mathcal{A}_{\Sigma_{\mathrm{ds}}(o)} \to \mathcal{A}_{\Sigma_{\mathrm{rs}}(o)}$ is a function for all $o \in \Omega$. $\mathcal{A}$ is partitioned if $s \neq s'$ entails $\mathcal{A}_s \cap \mathcal{A}_{s'} = \varnothing$ for all $s, s' \in S$.*

A Σ-homomorphism $h : \mathcal{A} \to \mathcal{B}$ from a Σ-algebra $\mathcal{A}$ to a Σ-algebra $\mathcal{B}$ is an S-indexed family of functions $(h_s)_{s\in S}$ where $h_s : \mathcal{A}_s \to \mathcal{B}_s$ for all $s \in S$, such that

$$o^{\mathcal{B}} \circ h_{\Sigma_{\mathrm{ds}}(o)} = h_{\Sigma_{\mathrm{rs}}(o)} \circ o^{\mathcal{A}}$$

for all $o \in \Omega$. Let $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ be the Σ-homomorphism $(\mathrm{Id}_{\mathcal{A}_s})_{s\in S}$, and for any Σ-homomorphism $h : \mathcal{A} \to \mathcal{B}$ and $k : \mathcal{B} \to \mathcal{C}$, let $k \circ h : \mathcal{A} \to \mathcal{C}$ be the Σ-homomorphism $(k_s \circ h_s)_{s\in S}$. Let Σ-$\mathbf{Alg}$ be the category of Σ-algebras with Σ-homomorphisms as their morphisms, and Σ-$\mathbf{PAlg}$ be its full subcategory of partitioned algebras.

Following Example 5.5, we notice that $\mathrm{T_g} = \,$ ⟳ $\simeq \mathsf{MG_t}$ (where $\mathrm{G_t}$ is the terminal graph, see Section 3), hence we have

$$\mathbf{Graphs} \simeq \mathbf{Graphs}/\mathrm{G_t} \cong \mathbf{MonoGr}/\mathsf{MG_t} \simeq \mathbf{MonoGr}/\mathrm{T_g}.$$

But we know that $\mathbf{Graphs} \simeq \Sigma_\mathrm{g}$-$\mathbf{Alg}$ and that $\Sigma_\mathrm{g} \simeq \mathsf{ST_g}$. The following result allows us to replace Σ_g by $\mathsf{ST_g}$.

Lemma 6.2. *If $\Sigma \simeq \Sigma'$ then Σ-$\mathbf{Alg} \simeq \Sigma'$-$\mathbf{Alg}$ and Σ-$\mathbf{PAlg} \simeq \Sigma'$-$\mathbf{PAlg}$.*

Note that Σ-$\mathbf{Alg}$ is not isomorphic to Σ-$\mathbf{PAlg}$ since many distinct algebras may be Σ-isomorphic to the same partitioned algebra. There is however a trivial equivalence between these categories.

Lemma 6.3. *For every signature Σ, Σ-$\mathbf{PAlg} \cong \Sigma$-$\mathbf{Alg}$*

By Lemma 6.2 we thus obtain $\mathbf{MonoGr}/\mathrm{T_g} \cong \mathsf{ST_g}$-$\mathbf{Alg}$, the monographs typed by $\mathrm{T_g}$ are essentially the $\mathsf{ST_g}$-algebras. We are now going to generalize this property to all monographs T with the following functor from the category of monographs typed by T to the category of partitioned $\mathsf{S}T$-algebras.

Definition 6.4 (functor $\mathsf{A}_T : \mathbf{MonoGr}/T \to \mathsf{S}T$-$\mathbf{PAlg}$). *Given a monograph T, we define the function A_T that maps every object $f : A \to T$ of $\mathbf{MonoGr}/T$ to the partitioned $\mathsf{S}T$-algebra $\mathsf{A}_T f$ defined by*

- *$(\mathsf{A}_T f)_e \overset{\mathrm{def}}{=} f^{-1}[e]$ for all $e \in \mathrm{E}_t$, and*
- *$(e, \iota)^{\mathsf{A}_T f}(x) \overset{\mathrm{def}}{=} a(x)_\iota$ for all $x \in f^{-1}[e]$ and $(e, \iota) \in \Omega_t$.*
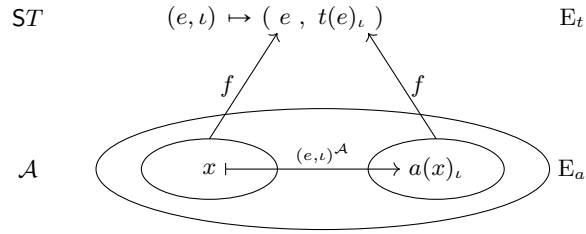
Fig. 1. The $\mathsf{S}T$-algebra $\mathcal{A} = \mathsf{A}_T f$ where $f : A \to T$

Besides, A_T also maps every morphism $k : f \to g$ of $\mathbf{MonoGr}/T$, where $f : A \to T$ and $g : B \to T$, to the $\mathsf{S}T$-homomorphism $\mathsf{A}_T k$ from $\mathsf{A}_T f$ to $\mathsf{A}_T g$ defined by

$$(\mathsf{A}_T k)_e \stackrel{\text{def}}{=} k|_{f^{-1}[e]} \text{ for all } e \in \mathrm{E}_t.$$

The $\mathsf{S}T$-algebra $\mathcal{A} = \mathsf{A}_T f$ can be pictured as in Figure 1. In $\mathsf{S}T$, every operator name (e, ι) has e and $t(e)_\iota$ as domain and range sort respectively, and these two sorts are also edges in T. Hence these edges have inverse images by f in E_a (they form a partion of the set E_a). The inverse image of an edge e of T is considered as the carrier set $\mathcal{A}_e$, since e is a sort of $\mathsf{S}T$. The operator name (e, ι) is interpreted as the function $(e, \iota)^{\mathcal{A}}$ that maps every $x \in f^{-1}[e]$ to the edge $a(x)_\iota$.

Example 6.5. Consider the monograph $\mathrm{T_g}$, the corresponding signature $\mathsf{ST_g}$ from Example 5.5, and the monograph $A = (\{x, y, z\}, a)$ where $a(x) = y\,z$ and $a(y) = a(z) = \varepsilon$, i.e., x is an edge from node y to node z. There is a unique morphism $f : A \to \mathrm{T_g}$, that is $f(x) = \mathtt{edges}$ and $f(y) = f(z) = \mathtt{nodes}$.

Let $\mathcal{A}$ be the $\mathsf{ST_g}$-algebra $\mathsf{A}_{\mathrm{T_g}} f$. By Definition 6.4 we have

$$\mathcal{A}_{\mathtt{edges}} = f^{-1}[\mathtt{edges}] = \{x\} \text{ and } \mathcal{A}_{\mathtt{nodes}} = f^{-1}[\mathtt{nodes}] = \{y, z\}.$$

$\mathsf{ST_g}$ has two operator names $(\mathtt{edges}, 0)$ and $(\mathtt{edges}, 1)$, both of domain sort $\mathtt{edges}$ and range sort $\mathtt{nodes}$, and their interpretation in $\mathcal{A}$ is given by

$$(\mathtt{edges}, 0)^{\mathcal{A}}(x) = a(x)_0 = y \text{ and } (\mathtt{edges}, 1)^{\mathcal{A}}(x) = a(x)_1 = z.$$

Thus $\mathcal{A}$ is the algebra that traditionally represents A considered as a graph (more precisely, as a graph G such that $\mathsf{M}G \simeq A$).

But the important point is that not only the monograph A but also its typing $f : A \to T$ can be uniquely reconstructed from any partitioned $\mathsf{S}T$-algebra $\mathcal{A}$, and that this is also true on morphisms. In other words, A_T is not just a functor, it is an isomorphism of categories.

Theorem 6.6. *For every monograph T, $\mathsf{A}_T : \mathbf{MonoGr}/T \xrightarrow{\cong} \mathsf{S}T\text{-}\mathbf{PAlg}$.*

Corollary 6.7. *For every monadic signature Σ there exists a monograph T such that $\Sigma\text{-}\mathbf{PAlg} \simeq \mathbf{MonoGr}/T$.*

Proof. By Lemma 5.4 there exists T such that $\Sigma \simeq \mathsf{S}T$, hence $\mathbf{MonoGr}/T \simeq \mathsf{S}T\text{-}\mathbf{PAlg} \simeq \Sigma\text{-}\mathbf{PAlg}$ by Lemma 6.2. $\qquad\square$

We thus see that the categories of partitioned monadic algebras are isomorphic to the slice categories of monographs. Note that in the case of graphs, the partitioned Σ_g-algebras correspond to those graphs whose sets of vertices and edges are disjoint. This is a common restriction for graphs but not for Σ-algebras. By Lemma 6.3 we obtain a similar result for the categories $\Sigma\text{-}\mathbf{Alg}$.

Corollary 6.8. *For every monograph T, $\mathbf{MonoGr}/T \cong \mathsf{S}T\text{-}\mathbf{Alg}$, and for every monadic signature Σ there is a monograph T such that $\Sigma\text{-}\mathbf{Alg} \cong \mathbf{MonoGr}/T$.*

The first half of Corollary 6.8 also yields that $\mathbf{Graphs}/G \cong \mathsf{S}\circ\mathsf{M}(G)\text{-}\mathbf{Alg}$ for all graphs G, i.e., typed graphs are equivalent to algebras of monadic signatures (but not every monadic signature is isomorphic to some $\mathsf{S}\circ\mathsf{M}(G)$). Signatures are sometimes called *types* (see, e.g., [2, Chapter 9]), which leads to the following reading of Corollary 6.8:

> *algebras of monadic many-sorted types are essentially typed monographs.*

This may seem strange since monographs may be typed in many different ways, while algebras are defined with only one type (but this type may correspond to many different type monographs). We now illustrate the corresponding equivalence between E-graphs and a category of typed monographs.

Example 6.9. The signature Σ_e of E-graphs from [4] has six operators $\mathtt{src_g}$, $\mathtt{tgt_g}$, $\mathtt{src_n}$, $\mathtt{tgt_n}$, $\mathtt{src_e}$, $\mathtt{tgt_e}$ and five sorts in

$$\mathrm{S_e} = \{\mathtt{edges_g}, \mathtt{edges_n}, \mathtt{edges_e}, \mathtt{nodes_g}, \mathtt{nodes_v}\},$$

and is defined by

$$\begin{aligned}
\Sigma_e : \mathtt{src_g}, \mathtt{tgt_g} &\mapsto (\mathtt{edges_g}, \mathtt{nodes_g})\\
\mathtt{src_n} &\mapsto (\mathtt{edges_n}, \mathtt{nodes_g})\\
\mathtt{tgt_n} &\mapsto (\mathtt{edges_n}, \mathtt{nodes_v})\\
\mathtt{src_e} &\mapsto (\mathtt{edges_e}, \mathtt{edges_g})\\
\mathtt{tgt_e} &\mapsto (\mathtt{edges_e}, \mathtt{nodes_v})
\end{aligned}$$

hence $O_{\mathtt{edges_g}} = \{\mathtt{src_g}, \mathtt{tgt_g}\}$, $O_{\mathtt{edges_n}} = \{\mathtt{src_n}, \mathtt{tgt_n}\}$, $O_{\mathtt{edges_e}} = \{\mathtt{src_e}, \mathtt{tgt_e}\}$ and $O_{\mathtt{nodes_g}} = O_{\mathtt{nodes_v}} = \varnothing$. There are four possible monographs $T = (\mathrm{S_e}, t)$, given by

$$\begin{aligned}
t(\mathtt{nodes_g}) &= t(\mathtt{nodes_v}) = \varepsilon\\
t(\mathtt{edges_g}) &= \mathtt{nodes_g}\,\mathtt{nodes_g}\\
t(\mathtt{edges_n}) &= \mathtt{nodes_g}\,\mathtt{nodes_v} \text{ or } \mathtt{nodes_v}\,\mathtt{nodes_g}\\
t(\mathtt{edges_e}) &= \mathtt{edges_g}\,\mathtt{nodes_v} \text{ or } \mathtt{nodes_v}\,\mathtt{edges_g}.
\end{aligned}$$

These four monographs are depicted below.

$$T_1 \qquad\qquad T_2 \qquad\qquad T_3 \qquad\qquad T_4$$

Note that, by Theorem 6.6, the categories $\mathbf{MonoGr}/T_i$ for $1 \leqslant i \leqslant 4$ are isomorphic, even though the T_i's are not. The type indicated by the syntax (and consistent with the figures in [4]) is T_1. An example of a monograph A typed by T_1 is
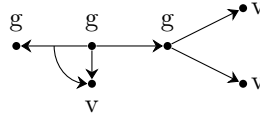
where g stands for $\mathtt{nodes_g}$ and v for $\mathtt{nodes_v}$. The types of the other edges can easily be deduced, yielding a unique typing morphism $f : A \to T_1$. This drawing can also be read as an E-graph, i.e., as a Σ_{e}-algebra $\mathcal{E}$ by determining the carrier sets of the five sorts and the functions interpreting the six operator names of Σ_{e}. This is a tedious task that corresponds to the transformation performed by the functor A_{T_1}, since $\mathcal{E} \simeq \mathsf{A}_{T_1} f$ as the reader can check.

Conversely, drawing an E-graph $\mathcal{E}$ corresponds to computing the typed monograph $\mathsf{A}_{T_1}^{-1} \mathcal{E}$ and then drawing it. Note that the operator names of Σ_{e} do not appear in the drawing of $\mathcal{E}$ but the sorts do, exactly as in $\mathsf{A}_{T_1}^{-1} \mathcal{E}$.

We see that E-graphs can be defined simply by drawing T_1, just as typed graphs can be defined by drawing their type graph. Of course, it is not always easy to draw a monograph, and it can be more convenient to give it explicitly as in Example 5.6, or indeed in the following example where the type monograph is infinite, though standard and with only one node. This example illustrates how a monograph may be typed in many different ways.

Example 6.10. The signature Σ_{h} of hypergraphs (see [7, Example 3.4]) is defined by the set of sorts $\mathrm{S_h} = \{\mathtt{V}\} \cup \{\mathtt{H}_{n,m} \mid n, m \in \omega\}$ and the $n + m$ operators

$$\Sigma_{\mathrm{h}} : \mathtt{src}_i^{n,m}, \mathtt{tgt}_j^{n,m} \mapsto (\mathtt{H}_{n,m}, \mathtt{V}) \text{ for all } 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m.$$

For any hypergraph $\mathcal{H}$ (i.e., any Σ_{h}-algebra) and $n, m \in \omega$, let us call (n, m)-hyperedges the elements of the set $\mathcal{H}_{\mathtt{H}_{n,m}}$; these are the hyperedges with n sources and m targets. The corresponding type monograph $\mathrm{T_h} = (\mathrm{S_h}, \mathrm{t_h})$ is defined by
  $\mathrm{t_h}(\mathtt{V}) = \varepsilon$ vertices are nodes
  $\mathrm{t_h}(\mathtt{H}_{n,m}) = \mathtt{V}^{n+m}$ (n, m)-hyperedges are edges joining $n + m$ vertices
for all $n, m \in \omega$. Hypergraphs are therefore isomorphic to monographs typed by $\mathrm{T_h}$, i.e., every edge is typed by some $\mathtt{H}_{n,m}$ (or $\mathtt{V}$ if it is a node). An edge of length 2 can therefore be typed either by $\mathtt{H}_{2,0}$, $\mathtt{H}_{1,1}$ or $\mathtt{H}_{0,2}$ and thus represent either a $(2, 0)$-, a $(1, 1)$- or a $(0, 2)$-hyperedge.

7   Some Properties of MonoGr

Depending on the signature Σ, working with Σ-algebras can be cumbersome. For instance, the proofs given in [4, Chapter 11] have to deal with the five carrier sets and six operators of E-graphs. With only one carrier set and one

map, monographs allow simpler notations and proofs. This can be illustrated by establishing a few fundamental properties of **MonoGr** (see e.g. [5] or [1] for the following notions from Category Theory that are not defined in Section 2).

By Corollary 6.8 it would be surprising if **MonoGr** had a terminal object, since such a monograph would be a type for all monographs, hence the corresponding signature would be in a sense universal. A more direct argument is given below.

Definition 7.1 (monographs M_α). *For every ordinal $\alpha > 0$ let a_α be the functional relation that to every $\lambda < \alpha$ associates the unique $\{0\}$-sequence of length λ. Let $M_\alpha \stackrel{\text{def}}{=} (\alpha, a_\alpha)$.*

It is clear that M_α is a standard α-monograph, since $a_\alpha \subseteq \alpha \times \alpha^{<\alpha}$ and $a_\alpha(0) = \varepsilon$, i.e., 0 is a node of M_α.

Lemma 7.2. *For all ordinals $\alpha > 0$, β and every β-monograph B, if there is a morphism $f : M_\alpha \to B$ then $\alpha \leqslant \beta$.*

Proof. α is the grade of M_α, since for any $\lambda < \alpha$ there is an edge of length λ, that is $\ell_{a_\alpha}(\lambda) = \lambda$, hence $a_\alpha(\lambda) \notin \alpha^{<\lambda}$, and therefore M_α is not a λ-monograph. By the existence of f the grade α of M_α is less than the grade of B, hence $\alpha \leqslant \beta$. $\qquad\square$

Theorem 7.3. MonoGr, SMonoGr *and* **FMonoGr** *have no terminal object.*

Proof. Suppose that B is a terminal monograph, then there is an ordinal β such that B is a β-monograph, and there is a morphism from $M_{\beta+1}$ to B. By Lemma 7.2 this implies that $\beta + 1 \leqslant \beta$, a contradiction. This still holds if B is standard since $M_{\beta+1}$ is standard. And it also holds if B is a finite ω-monograph, since then β can be chosen finite, and then $M_{\beta+1}$ is also a finite ω-monograph. $\qquad\square$

This of course is a major difference between **MonoGr** and **Graphs**, but also with the category **Sets** (whose terminal object is 1). Other properties derive directly from those of **Sets**. This can be illustrated by using the standard construction of pullbacks in **Sets** for building pullbacks of monographs.

Lemma 7.4. *Let B, C, D be α-monographs and $f : B \to D$, $g : C \to D$ be morphisms, then there exists an α-monograph A and morphisms $g' : A \to B$, $f' : A \to C$ such that (A, f', g') is a pullback of (f, g, D) in* **MonoGr**.

Proof. Let $E = \{(y, z) \in E_b \times E_c \mid f(y) = g(z)\}$, $g' = \pi_1|_E$, $f' = \pi_2|_E$ and $A = (E, a)$, where a maps every $x \in E$ to $\langle b \circ g'(x), c \circ f'(x) \rangle$.

We first prove that A is an α-monograph, i.e., that $a(x) \in E^{<\alpha}$ for all $x \in E$. Let $y = g'(x)$ and $z = f'(x)$ (so that $x = (y, z)$), then $\ell_b(y) = \ell_d(f(y)) = \ell_d(g(z)) = \ell_c(z)$, i.e., $b \circ g'(x)$ and $c \circ f'(x)$ have the same length $\lambda < \alpha$. Then, for all $\iota < \lambda$,

$$f((b \circ g'(x))_\iota) = (f^{<\alpha} \circ b(y))_\iota = (d \circ f(y))_\iota$$
$$= (d \circ g(z))_\iota = (g^{<\alpha} \circ c(z))_\iota = g((c \circ f'(x))_\iota),$$

hence $a(x)_\iota = ((b \circ g'(x))_\iota, (c \circ f'(x))_\iota) \in E$, so that $a(x) \in E^\lambda \subseteq E^{<\alpha}$.

It is obvious that $g'^{<\alpha} \circ a(x) = b \circ g'(x)$ and $f'^{<\alpha} \circ a(x) = c \circ f'(x)$, hence $g' : A \to B$ and $f' : A \to C$ are morphisms. There remains to prove that (A, g', f') is a pullback of (f, g, D).

Let A' be a monograph and $g'' : A' \to B$, $f'' : A' \to C$ be morphisms such that $f \circ g'' = g \circ f''$. Since (E, f', g') is a pullback of (f, g, E_d) in **Sets** (see [4]) then there exists a unique function h from $E_{a'}$ to E such that $g'' = g' \circ h$ and $f'' = f' \circ h$. Then, for all $x \in E_{a'}$,

$$
\begin{aligned}
a \circ h(x) &= \langle b \circ g' \circ h(x), c \circ f' \circ h(x) \rangle \\
&= \langle b \circ g''(x), c \circ f''(x) \rangle \\
&= \langle g''^{<\alpha} \circ a'(x), f''^{<\alpha} \circ a'(x) \rangle \\
&= \langle g'^{<\alpha} \circ h^{<\alpha} \circ a'(x), f'^{<\alpha} \circ h^{<\alpha} \circ a'(x) \rangle \\
&= \langle \pi_1 \circ (h^{<\alpha} \circ a'(x)), \pi_2 \circ (h^{<\alpha} \circ a'(x)) \rangle \\
&= h^{<\alpha} \circ a'(x)
\end{aligned}
$$

hence $h : A' \to A$ is a morphism in **MonoGr**. $\qquad\square$

Theorem 7.5. *The categories* **MonoGr**, **SMonoGr**, **FMonoGr**, O-**MonoGr** *and* O-**SMonoGr** *have pullbacks for every set O of ordinals.*

Proof. By Lemma 7.4 and since A is finite (resp. standard, resp. an O-monograph) if so are B and C. $\qquad\square$

Corollary 7.6. *The monomorphisms in* **MonoGr** *are the injective morphisms.*

Proof. Assume $f : B \to D$ is a monomorphism and let $C = B$, $g = f$ and (A, f', g') be the pullback of (f, g, D) defined in the proof of Lemma 7.4, then $f \circ g' = f \circ f'$ hence $\pi_1|_{E_a} = g' = f' = \pi_2|_{E_a}$. For all $x, y \in E_b$, if $f(x) = f(y)$ then $(x, y) \in E_a$ and $x = g'(x, y) = f'(x, y) = y$, hence f is injective. The converse is obvious. $\qquad\square$

Another way of stating this last property is that the forgetful functor from **MonoGr** to **Sets** reflects monomorphisms (and since it is faithful it obviously preserves them as well).

Similarly the standard construction of pushouts in **Sets** can be used to build pushouts in **MonoGr** (see [3]). Since $(\varnothing, \varnothing)$ is an initial object in **MonoGr** (as in all the considered subcategories), we get the following property.

Theorem 7.7. *The categories* **MonoGr**, **SMonoGr**, **FMonoGr**, O-**MonoGr** *and* O-**SMonoGr** *are finitely co-complete for every set O of ordinals.*

Note that the categories mentioned in Theorem 7.3 are not finitely complete since limits of empty diagrams are terminal objects.

As above, the construction of pushouts can be used to establish that the epimorphisms in **MonoGr** are exactly the surjective morphisms (i.e., the forgetful functor also reflects epimorphisms). But then it is easy to see that the

isomorphisms in **MonoGr** are exactly the bijective morphisms: if $f : A \to B$ and $g : B \to A$ are such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, then f is bijective since the underlying functions of id_A and id_B are $\mathrm{Id}_{\mathrm{E}_a}$ and $\mathrm{Id}_{\mathrm{E}_b}$. Hence **MonoGr** is obviously *balanced*, i.e., its isomorphisms are exactly the morphisms that are both mono and epimorphisms.

Since the forgetful functor reflects both mono and epimorphisms, this obviously means that it reflects isomorphisms. This has an interesting consequence.

Lemma 7.8. *The forgetful functor from* **MonoGr** *preserves and reflects pullbacks and pushouts.*

Proof. We first see that the forgetful functor preserves pullbacks. Let (f, g, D) be a sink in **MonoGr** and (A', f'', g'') be a pullback of (f, g, D). Let (A, f', g') be the pullback of (f, g, D) constructed in the proof of Lemma 7.4, then there is an isomorphism $i : A' \to A$ such that $f'' = f' \circ i$ and $g'' = g' \circ i$. Since (E_a, f', g') is a pullback of (f, g, E_d) in **Sets**, then so is $(\mathrm{E}_{a'}, f'', g'')$.

Since the forgetful functor is faithful, reflects isomorphisms and preserves pullbacks, then [5, Theorem 24.7] it also reflects them. The case for pushouts is similar. □

This means that properties of pullbacks and pushouts in **Sets** can easily be transfered to **MonoGr**. Of particular importance in the Double Pushout approach to Algebraic Graph Transformation is the notion of *adhesive* category from [6].

Definition 7.9. *A pushout square* (A, B, C, D) *is a* van Kampen square *if for any commutative cube*
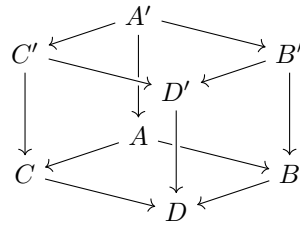
where the back faces (A', A, B', B) *and* (A', A, C', C) *are pullbacks, it is the case that the top face* (A', B', C', D') *is a pushout iff the front faces* (B', B, D', D) *and* (C', C, D', D) *are both pullbacks.*

A category has pushouts along monomorphisms *if all sources* (A, f, g) *have pushouts whenever* f *or* g *is a monomorphism.*

A category is adhesive *[6] if it has pullbacks, pushouts along monomorphisms and all such pushouts are van Kampen squares.*

Theorem 7.10. *The categories* **MonoGr**, **SMonoGr**, **FMonoGr**, O-**MonoGr** *and* O-**SMonoGr** *are adhesive for every set* O *of ordinals.*

Proof. In any of these categories a commutative cube built on a pushout along a monomorphism as bottom face and with pullbacks as back faces, has an underlying cube in **Sets** that has the same properties by Lemma 7.8 and Corollary 7.6. Since **Sets** is an adhesive category (see [6]) the underlying bottom face is a van Kampen square, hence such is the bottom face of the initial cube by Lemma 7.8. We conclude with Theorems 7.5 and 7.7. □

Note that their slice categories are therefore also adhesive by [4, Theorem 4.15]. Other properties can be found in [3], in particular a gluing condition that characterizes the existence of a pushout complement, and a construction of a product for some pairs of monographs.

8 Conclusion

Monographs generalize standard notions of directed graphs in the following ways.

1. Nodes are considered as a special kind of edges, namely the edges of length zero.
2. Edges can be adjacent to any edges and not only to nodes. This means that a monograph may contain edges but no nodes. This also means that edges can loop on themselves.
3. Edges can have any length, including infinite ones. Similarly (but this is not a special feature) monographs can be infinite.
4. Edges are well-ordered, hence they generalize directed arrows.

The prefix of "monograph" is justified by this unified view of nodes as edges and indiscriminate adjacency that provide formal conciseness; the suffix is justified by the correspondence (up to isomorphism) between finite ω-monographs and their drawings.

Monographs are universal with respect to monadic many sorted signatures (or graph structures) and the corresponding algebras, in the sense that monographs are equivalent to graph structures extended with suitable ordering conventions on their operator names, and that categories of typed monographs are equivalent to the corresponding categories of algebras. Since many standard or exotic notions of directed graphs can be represented as monadic algebras, they can also be represented as typed monographs.

Drawing a typed monograph appears to be a much easier task than drawing an algebra whose signature is a graph structure. This suggests that working directly with typed monographs rather than monadic algebras would be closer to graphical intuition as well as formally simpler.

Future work includes determining further properties of categories of monographs, especially in relation with the Double-Pushout approach to Algebraic Graph Transformation, studying submonographs and partial morphisms [9], attributed monographs as a way of generalizing the notion of attributed graphs based on E-graphs (see [4, Definition 8.4], see also [8]), and their relations with the Single-Pushout approach [7].

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