Stabilization of Aperiodic Sampled-data Linear Systems with Input Constraints: a Low Complexity Polyhedral Approach

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***Abstract*— The stabilization problem of aperiodic sampled-data linear systems subject to input constraints is dealt with. A state feedback control law is designed to optimize the size of a polyhedral estimate of the region of attraction of the origin (RAO) of the closed-loop system. The control law is derived from the computation of a controlled contractive polytope for the dynamics between two successive sampling instants. The polytope is of low complexity as its number of vertices is fixed a priori. As shown in the numerical example, the polyhedral estimate of the RAO associated with the proposed feedback control is larger than the ones obtained with other approaches in the literature.**

I. Introduction

The use of methods based on polyhedral sets to address the stability analysis and stabilization of dynamic systems is quite appealing [1]. In particular, it is known that a linear uncertain system is robustly stabilizable if and only if there exists a polyhedral control Lyapunov function for it or, equivalently, a polyhedral controlled invariant set [1]. Moreover, polyhedrons form a class of sets particularly suitable for the application of iterative procedures like the one in [2], that converges to the maximal controlled invariant/contractive set for the system. However, the sets obtained by such algorithms become more complex at each iteration, making the obtained solutions intractable in many important cases [1]. In order to circumvent this problem, many approaches exist in the literature for linear systems subject to constraints, as, for instance, [3], [4], [5], [6], [7]. In [3] a procedure that does not rely on iterative computations is developed while in [4] an algorithm based on linear programming that allows to overcome the complexity inherent to the Minkowski set addition is presented. In turn, [5], [6], [7] develop methods to compute polyhedrons of low complexity in order to get conservative but computationally affordable results.

Aperiodic sampled-data systems have been the focus of many recent works, since they allow to model the behavior of networked control systems subject to uncertainties in the communication channel between computer algorithms, actuators and sensors [8]. Many approaches exist to perform the stability analysis of such systems as, for instance, [9], [10], [11] in the linear case and [12], [13], [14] in the presence of input constraints, with the determination of estimates of the region of attraction of the origin (RAO). Some of the existing methods are also suitable for control design, like [12], [15], that, based on quadratic Lyapunov functions and semidefinite programming, provide linear feedback gains for sampled-data systems subject to input saturation.

In this work, we propose a method to design a piecewise linear state feedback control law that guarantees the asymptotic stability of the origin of aperiodic sampled-data linear systems subject to input constraints and leads to a polyhedral estimate of the RAO. Since the complexity – given by the number of vertices – of this estimate is fixed a priori, the resulting control law is of low complexity and suitable for practical use.

As shown in Section II, the method is based on a difference inclusion that models the behavior of the system state between two consecutive sampling instants. In Section III, a controlled contractive polyhedral set of low complexity is computed for this discrete-time model through the solution of an optimization problem with bilinear constraints. This set can be readily used in order to design a state feedback control law for the system. In Section IV, it is proved the equivalence between the asymptotic stability of the origin of the discrete-time system and the asymptotic stability of the origin of the continuous-time one. It is shown that the obtained polyhedron is contained in the RAO of the continuous-time closed-loop system and can therefore be used as an estimate of it. A numerical example is presented in Section V. The paper ends with some conclusions.

Notation. A C-set Ω is a compact and convex set containing the origin in its interior. Given a matrix M, $M_{(i)}$ is its i-th row, $M^{(j)}$ its j-th column, $M_{(ij)}$ its (ij)-entry, M^T its transpose and $\|M\|$ its induced 2-norm. If M is symmetric then $\lambda_{max}(M)$ is its maximum eigenvalue. The operator $\geq$ must be interpreted elementwise when applied to vectors/matrices. $\mathbf{1} \triangleq [1 \dots 1]^T$, $\mathscr{B}_r \triangleq \{x \in \mathbb{R}^n : \ \|x\| \leq r\}$. $\mathscr{V}(V) \triangleq \{x = V\alpha : \alpha \in \mathbb{R}^{n_v}, \alpha \geq 0, \mathbf{1}^T\alpha \leq 1\}, V \in \mathbb{R}^{n \times n_v}$, corresponds to the vertex representation of a compact and convex polyhedron and $\mathscr{P}(H,h) \triangleq \{x \in \mathbb{R}^n : Hx \leq h\}, H \in \mathbb{R}^{n_h \times n}, h \in \mathbb{R}^{n_h}$, corresponds to the hyperplane representation of a closed and convex polyhedron. $\mathbb{N}_m \triangleq \{i \in \mathbb{N} : 1 \leq i \leq m\}$. Given $\Omega \subseteq \mathbb{R}^n, \mathrm{Co}(\Omega)$ is its convex hull and Ω° is its interior.

II. Problem Formulation

Consider the following continuous-time system:

$$\dot{x}(t) = A_p x(t) + B_p u(t) \tag{1}$$

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where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote the state and the input of the plant, respectively, and A_p and B_p are constant matrices of appropriate dimensions. It is assumed that the control input is computed based on the sampled-value of the state at the time instants t_k, $k \in \mathbb{N}$, and satisfies

$$u(t) = f(x(t_k)), \quad \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N}, \tag{2}$$

where $f : \mathbb{R}^n \to \mathscr{U}$, $\mathscr{U} \subset \mathbb{R}^m$ is a polyhedral C-set and $f(0) = 0$.

By convention $t_0 = 0$ and the difference between two successive sampling instants, given by $\delta_k \triangleq t_{k+1} - t_k$, is considered to be lower and upper bounded as follows:

$$0 < \tau_m \le \delta_k \le \tau_M, \ \forall k \in \mathbb{N}. \tag{3}$$

Since δ_k depends on k, this system models an aperiodic sampling strategy. The particular case of periodic sampling corresponds to $\delta_k = \tau_m = \tau_M$ for all $k \in \mathbb{N}$.

The following definition of stability is adapted from [16, Definition 4.1] to the particular case under analysis.

Definition 1: The equilibrium point $x = 0$ of (1)-(2) is

- stable if, for each $\varepsilon > 0$, there is $\beta = \beta(\varepsilon) > 0$ such that

$$\|x(0)\| \le \beta \Rightarrow \|x(t)\| \le \varepsilon, \forall t \ge 0 \tag{4}$$

- asymptotically stable if it is stable and $\beta > 0$ exists such that

$$\|x(0)\| \le \beta \Rightarrow \lim_{t\to\infty} x(t) = 0 \tag{5}$$

where (4) and (5) must hold uniformly for all possible realizations of $\{\delta_k\}_{k\in\mathbb{N}}$ satisfying (3).

Definition 2: Considering that $x = 0$ is asymptotically stable, the *region of attraction of the origin* (RAO) of (1)-(2) is the set of all $x \in \mathbb{R}^n$ such that for $x(0) = x$ it follows that $\lim_{t\to\infty} x(t) = 0$ for all possible realizations of $\{\delta_k\}_{k\in\mathbb{N}}$ satisfying (3).

The objective of this work is to solve the following problem.

Problem 1: Design a state feedback control law $f : \mathbb{R}^n \to \mathscr{U}$ that guarantees the asymptotic stability of the origin and optimizes the size of a polyhedral estimate Ω of the RAO of the resulting closed-loop system.

Denoting $x_k \triangleq x(t_k)$ and $u_k \triangleq f(x_k)$, it follows from the solution of (1) considering (3) that the dynamics between two successive sampling instants can be described by the following difference inclusion:

$$\begin{aligned} x_{k+1} &\in \{A(\delta)x_k + B(\delta)u_k : \ \delta \in \Delta\} \\ &\triangleq G(x_k, u_k), \ u_k \in \mathscr{U}, \end{aligned} \tag{6}$$

where $\Delta \triangleq [\tau_m, \tau_M]$, $A(\delta) \triangleq e^{A_p\delta}$ and $B(\delta) \triangleq \int_0^\delta e^{A_p s} \mathrm{d}s B_p$.

Next, we will first solve Problem 1 in Section III for the discrete-time system (6) (using the analogous of Definitions 1 and 2 in the discrete-time case) and then in Section IV we will show that the same control law and the corresponding estimate of the RAO are also valid for (1)-(2).

A. Basic Concepts

The following definitions will be useful for the development of the results.

Definition 3: The Minkowski function $\Psi_\Omega : \mathbb{R}^n \to \mathbb{R}$ of the C-set $\Omega \subset \mathbb{R}^n$ is given by $\Psi_\Omega(x) \triangleq \min\{\alpha \ge 0 : \ x \in \alpha\Omega\}$. For a compact set $\mathscr{D} \subset \mathbb{R}^n$, $\Psi_\Omega(\mathscr{D}) \triangleq \min\{\alpha \ge 0 : \mathscr{D} \subseteq \alpha\Omega\}$.

This function satisfies the properties stated by the lemma below, adapted from [1, Proposition 3.12].

Lemma 1: $\Psi_\Omega(\cdot)$ is continuous, positive definite, convex, positively homogeneous of order 1, sub-additive (i.e. $\Psi_\Omega(x_1 + x_2) \le \Psi_\Omega(x_1) + \Psi_\Omega(x_2)$) and lower and upper bounded as follows: $m\|x\| \le \Psi_\Omega(x) \le M\|x\|, m, M > 0$ (equivalently $\mathscr{B}_{1/M} \subseteq \Omega \subseteq \mathscr{B}_{1/m}$).

Definition 4: (controlled λ-contractive set) Given $0 \le \lambda < 1$, the C-set $\Omega \subset \mathbb{R}^n$ is said to be controlled λ-contractive for a generic difference inclusion $x_{k+1} \in G(x_k, u_k), u_k \in \mathscr{U}$, if for every $x_k \in \Omega$ there exists $u_k \in \mathscr{U}$ such that $G(x_k, u_k) \subseteq \lambda\Omega$. If $\lambda = 1$, Ω is a controlled (positively) invariant set.

We can also express the definition above using the Minkowski function of Ω by means of the equivalence relation below, which is valid for all $(x_k, u_k) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$G(x_k, u_k) \subseteq \lambda\Omega \ \Leftrightarrow \ \Psi_\Omega(x_{k+1}) \le \lambda, \forall x_{k+1} \in G(x_k, u_k)$$

III. Design of the feedback control law

The design of the state feedback can be divided in two steps. The first one, presented in Section III-A, consists in finding a controlled contractive polyhedral C-set Ω and a corresponding control law for the difference inclusion

$$x_{k+1} \in \{A(\delta)x_k + B(\delta)u_k : \ \delta \in \Delta_J\}, \tag{7}$$

where the input is constrained by $u_k \in \mathscr{U}$ and $\Delta_J \triangleq \{d_j \triangleq \tau_m + (j-1)\tau_J : \ j \in \mathbb{N}_J\}$, $\tau_J \triangleq \frac{\tau_M - \tau_m}{J}$, $J \in \mathbb{N}$. Notice that (7) considers only a finite subset Δ_J of the interval Δ. The second step regards the guarantee that Ω is contractive not only for (7) but also for (6) (not necessarily with the same contraction factor λ). A sufficient condition to ensure this is derived in Section III-B. At last, in Section III-C, it will be shown that the obtained contractive set is included in the RAO of the closed-loop system formed by (6) and the designed state feedback.

A. Computation of a contractive set for system (7)

Given a polyhedral C-set Ω and its vertex representation $\Omega = \mathscr{V}(V), V \in \mathbb{R}^{n\times n_v}$, where the vertices of Ω are represented by columns of V, the following result holds.[1]

Lemma 2: Consider system (7). There exists a state feedback control law $u_k = f(x_k)$ satisfying the constraints and which makes the polyhedral C-set $\Omega = \mathscr{V}(V)$ λ-contractive for the closed-loop system if and only if there exist $U \in \mathbb{R}^{m\times n_v}$ and nonnegative matrices $H_j \in \mathbb{R}^{n_v\times n_v}, j \in \mathbb{N}_J$, such that

$$A(d_j)V + B(d_j)U = VH_j, \quad \forall j \in \mathbb{N}_J \tag{8}$$

$$\mathbf{1}^T H_j \le \lambda \mathbf{1}^T, \quad \forall j \in \mathbb{N}_J \tag{9}$$

$$U^{(i)} \in \mathscr{U}, \quad \forall i \in \mathbb{N}_{n_v} \tag{10}$$

[1] Since Ω has a nonempty interior, $n_v \ge n+1$ by construction.

where $U^{(i)}$ is the i-th column of U.

Proof: The proof is analogous to the one of [1, Proposition 7.26], which deals with linear systems with polytopic uncertainties. ■

The lemma above can be used to obtain a λ-contractive polyhedral C-set Ω for (7). Since V and H_j are variables, constraints (8) are bilinear while (9)-(10) are both linear (since $\mathscr{U}$ is a given polyhedron). In order to optimize the size of $\Omega = \mathscr{V}(V)$, we propose the following optimization problem, where n_v must be fixed *a priori*:

$$\max_{V,U,H_j,\Gamma,L} \sum_{r=1}^{n_r} \Gamma_{(rr)} w_{(r)} \tag{11}$$

$$\text{subject to } (8)-(10)$$

$$R\Gamma = VL,\ \mathbf{1}^T L \leq \mathbf{1}^T,\ L \geq 0 \tag{12}$$

where $L \in \mathbb{R}^{n_v \times n_r}$, $\Gamma \in \mathbb{R}^{n_r \times n_r}$ is diagonal, $w \in \mathbb{R}^{n_r}$ is a vector of positive weights ($w > 0$) for the elements of $\Gamma \geq 0$ and $R \in \mathbb{R}^{n \times n_r}$.

The columns of R, defined *a priori*, are directions along which the polyhedron Ω will be maximized. They can be freely chosen and do not affect the feasibility of the problem, although the estimation size depends on R. Notice that (12) is equivalent to $\Gamma_{(rr)} R^{(r)} \in \mathscr{V}(V) = \Omega, \forall r \in \mathbb{N}_{n_r}$, where $\Gamma_{(rr)}$ is a scaling factor. Consequently, $\mathscr{V}(R\Gamma) \subseteq \Omega$. Thus, the optimization problem maximizes a linear combination of the scaling factors $\Gamma_{(rr)}$, where $w_{(r)}$ are positive weights for each factor, i.e. they weight the maximization of Ω in each one of the directions given by the columns of R. The matrix $R \in \mathbb{R}^{n \times n_r}$ is chosen such that $0 \in \mathscr{V}(R)^{\circ}$ and we introduce in the optimization problem above the following additional constraint: $\Gamma_{(rr)} \geq \eta,\ \forall r \in \mathbb{N}_{n_r}$, where $\eta > 0$ is a numerical tolerance. In this way, we guarantee that $0 \in \Omega^{\circ}$. This property will be used afterwards in the proof of stability.

Since some of the constraints are bilinear, this is a non-linear programming problem and, in principle, there is no guarantee of the global optimality of the solution. On the other hand, it is possible to obtain an initial feasible solution to the constraints using for instance the method presented in [5]. Therefore, the solution of (11) will be at least as good as the one of [5] with respect to the chosen size criterion of the polyhedron. In this work, (11) is solved using the KNITRO toolbox [17].

Remark 1: It should be noticed that, although V is a free variable, the number of columns of V (i.e. the number of vertices of Ω) is defined *a priori*. Hence there is a trade-off regarding the choice of n_v: a large value will result in principle in a larger polytope Ω, but will also increase the numerical complexity of the approach.

Lemma 2 gives a necessary and sufficient condition for the λ-contractivity of the polyhedral C-set $\Omega = \mathscr{V}(V)$, but does not provide the control law $u_k = f(x_k)$. One of the possible choices of construction of $f(\cdot)$ corresponds to the concept of control "at the vertices" [1, Pages 158-159]. Assume without loss of generality that the vertex representation $\Omega = \mathscr{V}(V)$ is minimal (otherwise it is possible to discard the redundant columns of matrix V and the corresponding ones of matrix U). The idea is to interpolate the control value at the vertices as follows:

a) For any pair $(V^{(i)}, U^{(i)})$ of columns of V and U (defined in the statement of Lemma 2), $f(V^{(i)}) = U^{(i)}$;
b) for $x_k \in \Omega$, $f(x_k) = U\alpha$, where $\alpha \in \mathbb{R}^{n_v}$ is such that $x_k = V\alpha, \mathbf{1}^T\alpha = \Psi_{\Omega}(x_k), \alpha \geq 0$.

This control law can be constructed as described next [1].

Firstly, Ω can be partitioned into simplices[2] formed by n vertices and the origin:

$$\Omega^l \triangleq \{x = \bar{V}^l \bar{\alpha} :\ \bar{\alpha} \geq 0,\ \mathbf{1}^T \bar{\alpha} \leq 1,\ \bar{\alpha} \in \mathbb{R}^n\} \tag{13}$$

where $\bar{V}^l$ is a matrix formed by the n columns of V corresponding to the l-th simplex (do not confuse it with $V^{(l)}$). This partition can be obtained using one of the triangulation methods presented in [18, Section 3.1]. We also denote as $\bar{U}^l$ the n columns of U that correspond to the selected columns of V. Each simplex generates a polyhedral cone as follows:

$$C^l \triangleq \{x = \bar{V}^l \bar{\alpha} :\ \bar{\alpha} \geq 0,\ \bar{\alpha} \in \mathbb{R}^n\}. \tag{14}$$

The sets above can be chosen in such a way that:

- Ω^l and C^l have non-empty interiors;
- $\Omega^l \bigcap \Omega^h$ and $C^l \bigcap C^h$ have empty interiors for $l \neq h$;
- $\bigcup_l \Omega^l = \Omega$ and $\bigcup_l C^l = \mathbb{R}^n$.

Then, the piecewise linear control law below is Lipschitz continuous, guarantees the λ-contractivity of Ω and satisfies the constraint $u_k \in \mathscr{U}$ and properties a) and b) above [1]:

$$u_k = f(x_k) \triangleq F^l x_k \triangleq \bar{U}^l (\bar{V}^l)^{-1} x_k, \quad x_k \in \Omega^l, \tag{15}$$

where the inverse of $\bar{V}^l$ exists because Ω^l has a non-empty interior. Notice that (15) satisfies property b), indeed. To see this, assume that simplex 1 (for the other simplices the same considerations apply) Ω^1 is generated by the first n columns $\bar{V}^1$ of $V = [\bar{V}^1\ \tilde{V}]$. Then, if $x_k \in \Omega^1$,

$$x_k = \bar{V}^1 \bar{\alpha} = \begin{bmatrix} \bar{V}^1 & \tilde{V} \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ 0 \end{bmatrix} = V\alpha,$$

where $\bar{\alpha} \in \mathbb{R}^n$ is a nonnegative vector and $\alpha \triangleq [\bar{\alpha}^T\ 0^T]^T \in \mathbb{R}^{n_v}$. It follows that

$$u_k = F^1 x_k = \bar{U}^1 (\bar{V}^1)^{-1} \bar{V}^1 \bar{\alpha} = \bar{U}^1 \bar{\alpha} = \begin{bmatrix} \bar{U}^1 & \tilde{U} \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ 0 \end{bmatrix} = U\alpha.$$

B. Testing contractivity for system (6)

The second step of the method consists in verifying if the control law (15) guarantees the contractivity of Ω for (6), which takes into account all possible values for $\delta_k \in \Delta$ and not only the finite set Δ_J. The following property plays a key role to verify that.

Lemma 3: Given $d, \tau \in \mathbb{R}$, the following identities hold:

$$A(d+\tau) = A(d) + \Phi(\tau) e^{A_p d} A_p \tag{16}$$

$$\begin{aligned} B(d+\tau) &= B(d) + \Phi(\tau) \left(A_p \int_0^d e^{A_p s} \mathrm{d}s B_p + B_p \right) \\ &= B(d) + \Phi(\tau) e^{A_p d} B_p \end{aligned} \tag{17}$$

[2] A simplex (plural: simplices or simplexes) is the simplest kind of polytope with nonempty interior. In $\mathbb{R}^n$ it corresponds to the convex hull of $n+1$ affinely independent points.

where $\Phi(\tau) \triangleq \int_0^{\tau} e^{A_p s} \mathrm{d}s$.

Proof: See the proof of [19, Proposition 1]. ■

Using the lemma above, it follows that

$$A(d+\tau)x + B(d+\tau)u = A(d)x + B(d)u + \\ +\Phi(\tau)e^{A_p d}(A_p x + B_p u) \quad (18)$$

Define now the logarithmic norm of A_p associated with the 2-norm [20]: $\mu(A_p) \triangleq \lambda_{max}\left(\frac{A_p + A_p^T}{2}\right)$. Notice in particular that $\mu(A_p)$ can be negative. The following theorem can now be stated.

Theorem 1: Consider a controlled λ-contractive polyhedral C-set Ω for (7) and the corresponding control law (15). If the constant

$$\bar{c}(\Omega, J) \triangleq c_1(J)c_2c_3(\Omega)c_4(\Omega), \quad (19)$$

where

$$c_1(J) \triangleq \begin{cases} \dfrac{e^{\mu(A_p)\tau_J} - 1}{\mu(A_p)} & \text{if } \mu(A_p) \neq 0, \\ \tau_J & \text{if } \mu(A_p) = 0, \end{cases}$$

$$c_2 \triangleq \max\left(e^{\mu(A_p)\tau_m}, e^{\mu(A_p)\tau_M}\right),\ c_3(\Omega) \triangleq \max_{x\in\Omega} \left\|A_p x + B_p f(x)\right\|,$$

$$c_4(\Omega) \triangleq \Psi_\Omega(\mathscr{B}_1) = \min\{\alpha \geq 0 : \mathscr{B}_1 \subseteq \alpha\Omega\},$$

is such that

$$\nu(\Omega, J) \triangleq \lambda + \bar{c}(\Omega, J) < 1, \quad (20)$$

then the control law (15) guarantees the $\nu(\Omega, J)$-contractivity of Ω for (6).

Proof: We have to show that x_{k+1} given by (6) and (15) satisfies $\Psi_\Omega(x_{k+1}) \leq \nu(\Omega, J), \forall x_k \in \Omega$. Given $x_k \in \Omega, \delta_k \in \Delta$, there exist $d_k \in \Delta_J$ and $\tau_k \in [0, \tau_J]$ such that $\delta_k = d_k + \tau_k$. Then, using (18) it follows that

$$\begin{aligned} x_{k+1} &= A(d_k+\tau_k)x_k + B(d_k+\tau_k)f(x_k) \\ &= \underbrace{A(d_k)x_k + B(d_k)f(x_k)}_{\triangleq y_{k+1}} + \underbrace{\Phi(\tau_k)e^{A_p d_k}(A_p x_k + B_p f(x_k))}_{\triangleq z_{k+1}}. \end{aligned} \quad (21)$$

From the fact that Ω is λ-contractive for (7), $d_k \in \Delta_J$ and $x_k \in \Omega$, it follows that

$$\Psi_\Omega(y_{k+1}) \leq \lambda. \quad (22)$$

Considering now (see [20]) that $\|e^{A_p s}\| \leq e^{\mu(A_p)s}$ for all $s \geq 0$, and since $\tau_k \in [0, \tau_J]$, one obtains:

$$\|\Phi(\tau_k)\| = \left\|\int_0^{\tau_k} e^{A_p s}\mathrm{d}s\right\| \leq \int_0^{\tau_k} \left\|e^{A_p s}\right\| \mathrm{d}s \leq \int_0^{\tau_J} e^{\mu(A_p)s}\mathrm{d}s = c_1(J).$$

Moreover, one has that

$$\left\|e^{A_p d_k}\right\| \leq \max\left(e^{\mu(A_p)\tau_m}, e^{\mu(A_p)\tau_M}\right) = c_2.$$

Using the inequalities above we conclude that

$$\|z_{k+1}\| \leq \|\Phi(\tau_k)\| \left\|e^{A_p d_k}\right\| \left\|A_p x_k + B_p f(x_k)\right\| \leq c_1(J)c_2c_3(\Omega),$$

i.e. $z_{k+1} \in c_1(J)c_2c_3(\Omega)\mathscr{B}_1$. Then, from (21), (22), and the properties in Lemma 1, we get that

$$\begin{aligned} \Psi_\Omega(x_{k+1}) &\leq \Psi_\Omega(y_{k+1}) + \Psi_\Omega(z_{k+1}) \leq \lambda + c_1(J)c_2c_3(\Omega)\Psi_\Omega(\mathscr{B}_1) \\ &= \lambda + \bar{c}(\Omega, J) = \nu(\Omega, J) \underbrace{<}_{(20)} 1. \end{aligned}$$ ■

Notice that $J \in \mathbb{N}$ can be freely chosen. Thus, if (20) is not satisfied for some value of J, we recommend to increment it (as done in [15], [14], for instance) and to recompute the solution of (11).

Using the properties of $f(\cdot)$, the constant $c_3(\Omega)$ can be obtained, in practice, as follows [1]:

$$c_3(\Omega) = \max_i \left\|A_p V^{(i)} + B_p U^{(i)}\right\|.$$

On the other hand, consider a hyperplane representation of Ω, that is, $\Omega = \mathscr{P}(H, h), H \in \mathbb{R}^{n_h \times n}, h \in \mathbb{R}^{n_h}$ (assuming without loss of generality that $h_{(i)} > 0$ for all $i \in \mathbb{N}_{n_h}$). Then, $c_4(\Omega)$ can be computed through

$$\begin{aligned} c_4(\Omega) &= \Psi_\Omega(\mathscr{B}_1) = \min\{\alpha \geq 0 : \mathscr{B}_1 \subseteq \alpha\Omega\} \\ &= \min\{\alpha \geq 0 : Hx \leq \alpha h, \forall x \in \mathscr{B}_1\} = \max_{i \in \mathbb{N}_{n_h}} \frac{\|H_{(i)}\|}{h_{(i)}}. \end{aligned}$$

C. Stability analysis of the discrete-time system

We show next that the ν-contractive polyhedral C-set Ω found for (6) belongs to the RAO of the closed-loop system composed by (6) and (15).

Theorem 2: Given the ν-contractive polyhedral C-set Ω for (6) and the corresponding control law (15), the trajectories of the closed-loop system composed by (6) and (15) satisfy

$$x_k \in \beta\Omega \ \Rightarrow\ x_{k+p} \in \nu^p \beta\Omega, \quad \forall \beta \in [0, 1]. \quad (23)$$

Proof: We will show that (23) holds for $p = 1$ and then the general result will follow by induction. From (15) and the shape of the simplices Ω^l, the control law is positively homogeneous of order 1 inside Ω, i.e.

$$f(\beta x) = \beta f(x),\ \forall x \in \Omega,\ \forall \beta \in [0, 1].$$

Thus, since (6) depends linearly on (x_k, u_k), it follows that the closed-loop system satisfies:

$$G(\beta x, f(\beta x)) = \beta G(x, f(x)),\ \forall x \in \Omega,\ \forall \beta \in [0, 1].$$

Therefore, if $x_k \in \beta\Omega, 0 < \beta \leq 1$ (the case $\beta = 0$ is trivially satisfied), then, defining $\bar{x}_k \triangleq x_k/\beta \in \Omega$:

$$x_{k+1} \in G(x_k, f(x_k)) = G(\beta\bar{x}_k, f(\beta\bar{x}_k)) = \beta G(\bar{x}_k, f(\bar{x}_k)) \subseteq \beta\nu\Omega$$

where the set inclusion follows from the ν-contractivity of Ω and the fact that $\bar{x}_k \in \Omega$. ■

The result above guarantees that the set Ω will remain ν-contractive when scaled down. Since Ω is bounded, it follows that $x_k \overset{k\to\infty}{\to} 0$. Moreover, (23) is equivalent to

$$\Psi_\Omega(x_k) \leq \beta \ \Rightarrow\ \Psi_\Omega(x_{k+p}) \leq \nu^p\beta, \quad \forall \beta \in [0, 1].$$

Thus, $\Psi_\Omega(\cdot)$ is a Lyapunov function inside Ω, which implies that Ω is included in the RAO of the discrete-time closed-loop system (6)-(15).

IV. Stability analysis of the continuous-time system

Consider the continuous-time system composed by (1) and (2) with $f(x(t_k))$ given by (15). From the analytical solution of (1)-(2) in the interval $[t_k, t_{k+1}]$, notice that $\|x(t)\|$ is bounded as shown below:

$$\begin{aligned}\|x(t)\| &\le \|A(t-t_k)\|\|x(t_k)\| + \|B(t-t_k)\|\|u(t_k)\| \\ &\le \left(\|A(t-t_k)\| + \|B(t-t_k)\| \max_l \|F^l\|\right)\|x(t_k)\| \\ &\le \max_{\tau\in[0,\tau_M]} \left(\|A(\tau)\| + \|B(\tau)\| \max_l \|F^l\|\right)\|x(t_k)\| \\ &\triangleq c_A\|x(t_k)\|, \quad \forall t \in [t_k, t_{k+1}],\ \forall k \in \mathbb{N} \end{aligned} \tag{24}$$

where $A(\tau) = e^{A_p\tau}$ and $B(\tau) = \int_0^\tau e^{A_p s}\mathrm{d}sB_p$.

From the previous section (see (23)) we know that the origin of the discrete-time system that describes the behavior of the state $x(t)$ at the sampling instants t_k is asymptotically stable. Combining this property with the bound above, it follows that the origin of the continuous-time closed-loop system (1)-(2) with (15) is also asymptotically stable. More precisely, given $\varepsilon > 0$, there exists $\bar{\beta}(\varepsilon) > 0$ such that $\|x_0\| \le \bar{\beta}$ implies $\|x_k\| \le \varepsilon, \forall k \in \mathbb{N}$ and $x_k \overset{k\to\infty}{\to} 0$. Thus, since $x_k = x(t_k)$ by definition, $\beta(\varepsilon) \triangleq \bar{\beta}(\varepsilon/c_A)$ satisfies conditions (4) and (5) of Definition 1. In particular, if $\|x(0)\| \le \beta$, then $\|x(t_k)\| \le \varepsilon/c_A, \forall k \in \mathbb{N}$. Thus, from (24), it follows that $\|x(t)\| \le c_A(\varepsilon/c_A) = \varepsilon, \forall t \ge 0$. Moreover, from (24) it follows that $x_k \overset{k\to\infty}{\to} 0$ implies $x(t) \overset{t\to\infty}{\to} 0$. Therefore, from (23), we conclude that Ω is included in the RAO of (1)-(2) with $f(\cdot)$ given by (15).

V. Numerical example

Consider system (1) with

$$A_p = \begin{bmatrix}1 & 0\\ 0 & 0.2\end{bmatrix},\ B_p = \begin{bmatrix}1\\1\end{bmatrix} \text{ and } \mathscr{U} = \{u \in \mathbb{R} : \|u\|_\infty \le 1\},$$

where $\Delta = [0.05, 0.1]$. Problem (11) was solved using the KNITRO toolbox [17] and considering

$$n_v = 10,\ \lambda = 0.98,\ J = 20 \text{ and } w = \mathbf{1}.$$

The columns of R, i.e. the directions over which the polyhedron is maximized, correspond to the elements of the following set:

$$\left\{\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} : \theta = \frac{(q-1)\pi}{24}, q \in \mathbb{N}_{48}\right\}.$$

The resulting $\nu(\Omega, J)$-contractive polyhedron Ω is shown in Figure 1, where

$$\nu(\Omega, J) \cong 0.996.$$

This set belongs to the RAO of the closed-loop system (1)-(2) with the piecewise linear control law (15), whose gains are presented in Table I. The corresponding simplices are depicted in Figure 2.

For comparison purposes we also show in Figure 1 the estimates of the RAO of the closed-loop system with the control law

$$u_k = \mathrm{sat}(K_p x_k) \tag{25}$$

Simplex	F^l
1	[-2.7763 1.23]
2	[-2.4569 0.45893]
3	[-9.8438 3.6367]
4	[-1.712 1.2823]
5	[-2.1273 1.3258]
6	[-2.4724 1.2333]
7	[-2.5903 0.91817]
8	[-2.1301 0.31834]
9	[-6.8346 2.3422]
10	[-2.2615 1.4124]

TABLE I

Feedback gains of the piecewise linear control law (15).

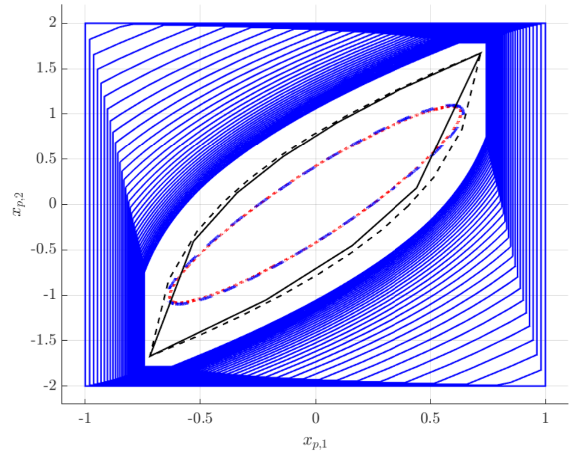

Fig. 1. Estimates of the RAO of the closed-loop system considering (15) and the proposed method (black-continuous for $n_v = 10$ and black-dashed for $n_v = 20$) and considering (25) and the methods in [13] (red-dotted for $K_p = [-10.70\ 4.38]$) and [15] (blue-dashed for $K_p = [-10.70\ 4.38]$). Outer approximations of the maximal λ-contractive C-set in blue-continuous.

obtained with the methods proposed in [13] and [15], where the gain $K_p = [-10.70\ 4.38]$ was obtained with the algorithm of [15]. As it can be seen, the method presented in this work provides a feedback control law of low complexity for which the corresponding estimate of the RAO of the closed-loop system is considerably larger than the estimates obtained in [13] and [15] considering a linear saturated state feedback. If we replace the number of vertices $n_v = 10$ by $n_v = 20$, then the resulting polyhedron encompasses the latter estimates, as it is also shown in Figure 1.

Moreover, Figure 1 depicts a decreasing sequence of outer approximations of the maximal controlled λ-contractive C-set for (7) (with $J = 20$, $\lambda = 0.98$) obtained with the method proposed in [2]. By visual inspection it is possible to have an idea about the conservatism of our method. Notice that the obtained set Ω is an inner approximation (of low complexity) of the maximal controlled λ-contractive C-set for (7). Furthermore, we guarantee that Ω is also contractive for (6) using the result of Theorem 1.

In Figure 3, several trajectories with $x(0)$ at the boundary of Ω and considering (15) with δ_k randomly chosen in the interval Δ are shown. As expected, the convergence of the

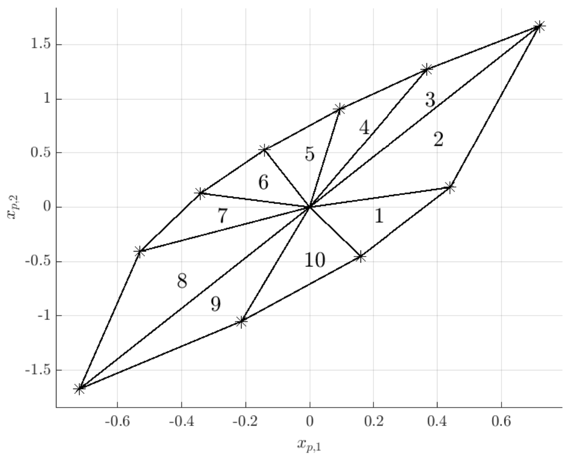

Fig. 2. Partition of Ω in simplices.

trajectories to the origin is ensured showing that Ω is indeed included in its region of attraction.

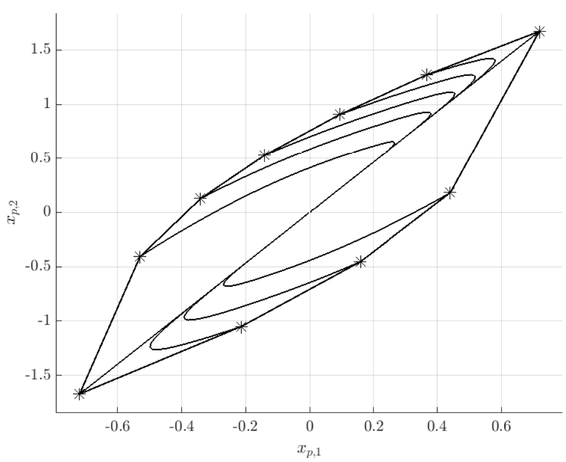

Fig. 3. Trajectories starting at the vertices of Ω.

VI. Conclusions

The problem of stabilization of aperiodic sampled-data linear systems subject to input constraints was tackled using a polyhedral framework. The proposed method allows to find a polytope of fixed complexity, which is controlled contractive for the dynamics of the system between two consecutive sampling instants. From this polytope, it is possible to derive a feedback control law for the system. Among the different existing approaches to do that (e.g. [1]), we chose to construct a piecewise linear control law. It is then shown that the obtained polytope is included in the RAO of the continuous-time plant in closed loop with the computed sampled-data control law. A numerical example shows the efficiency of the resulting feedback control, for which the corresponding estimate of the RAO is larger than other ones obtained in the literature considering linear saturated feedback control laws [13], [15].

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