On Parallel and Sequential Independence in Attributed Graph Rewriting

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Abstract

We use graphs where vertices and arrows are attributed with *sets* of values, and rules that allow to delete data from a graph, to create new vertices or arrows, and to include values in attributes. Rules may be applied simultaneously, yielding a notion of parallelism that generalizes cellular automata in particular by allowing infinite matchings of rules in a graph. This is first used to define a notion of *sequential independence* of a set M of matchings of rules, even when M is infinite. Next, a notion of *parallel independence* of matchings is defined that accounts for the particular treatment of attributes, and it is proven that it characterizes sequential independence. Last, the *effective deletion property*, a condition that ensures that rules can be applied in parallel without conflicts, is proven to generalize parallel independence.

1 Introduction

The notion of parallel independence has been studied mostly in the algebraic approach to graph rewriting, see [1] and the references therein. It basically consists in a condition on concurrent transformations of an object that characterizes the possibility to apply the transformations sequentially in any order such that all such sequences of transformations yield the same result. When two transformations are involved this takes the form of the diamond property and is known as the *Local Church-Rosser Problem.* Parallel independence then allows to define critical pairs, a central notion in Term and Graph Rewriting.

This problem should therefore also be considered in algorithmic approaches to graph rewriting. Indeed, the informal description of parallel independence given above makes perfect sense out of the algebraic approach; it is purely operational. Thus parallel independence necessarily depends on the operational semantics of the rules, that can be defined without resorting to Category Theory. This is the case of the framework described in [5], where graphs are attributed by sets of values (see Section 2). This has been designed so that enough space can always be accommodated for any number of parallel application of rules. Similarly, new vertices and arrows can always be added.

But there is a fundamental difference between the two, that lies in the semantics of the rules described in Section 3: vertices and arrows are always added as *new* objects, but values are added by inclusion in attributes, where they may not be new. This is similar to assignments $x := y$ when x and y have the same value, and is therefore very natural. This is different from E-graphs, an alternate notion of attributed graphs with any number of values (see [3]), and is bound to have an impact on parallel independence.

But we are first faced with the same difficulty as in the algebraic approach, that is to apply transformations meant for the same graph in sequence, hence on already transformed graphs (except for the one considered first). This is solved in Section 4 by taking advantage of the parallel transformation defined in [5] and in Section 3. In Section 5 a definition of parallel independence adapted to this framework is given, and proven to be correct since it is equivalent to sequential independence. Finally, Section 6 is devoted to the *effective deletion property* from [5], a condition that guarantees that the operational semantics of rules is preserved when taken in parallel. It is proved that this property generalizes parallel independence, as it ought to.

2 Attributed Graphs

We consider a fixed many-sorted signature Σ. A *graph* G is a tuple $(V, A, s, t, \mathcal{A}, l)$ where V, A are sets, $\mathcal{A}$ is a Σ-algebra, s, t are the *source* and *target* functions from A to V and l is an *attribution of* G, i.e., a function from $V \cup A$ to $\mathscr{P}(\lfloor \mathcal{A} \rfloor)$ (the carrier set $\lfloor \mathcal{A} \rfloor$ of $\mathcal{A}$ is the disjoint union of the carrier sets of the sorts in $\mathcal{A}$). We assume that V, A and $\lfloor \mathcal{A} \rfloor$ are mutually disjoint, their elements are respectively called *vertices*, *arrows* and *attributes*. Hence vertices and arrows are attributed by *sets* of elements of a Σ-algebra. G is *unlabelled* if $\mathring{G}(x) = \varnothing$ for all $x \in V \cup A$, it is *finite* if the sets V, A and $l(x)$ are finite. The *carrier* of G is the set $\lfloor G \rfloor \stackrel{\text{def}}{=} V \cup A \cup \lfloor \mathcal{A} \rfloor$. When we speak of a graph G without specifying its components, these will be referred to as in $G = (\dot{G}, \vec{G}, \acute{G}, \grave{G}, \mathscr{A}_G, \mathring{G})$.

A graph H is a *subgraph* of G, written $H \lhd G$, if the *underlying graph* $(\dot{H}, \vec{H}, \acute{H}, \grave{H})$ of H is a subgraph of G's underlying graph (in the usual sense), $\mathscr{A}_H = \mathscr{A}_G$ and $\mathring{H}(x) \subseteq \mathring{G}(x)$ for all $x \in \dot{H} \cup \vec{H}$.

A *morphism* α from graph H to graph G is a function from $\lfloor H \rfloor$ to $\lfloor G \rfloor$ such that the restriction of α to $\dot{H} \cup \vec{H}$ is a morphism from H's to G's underlying graphs (that is, $\acute{G} \circ \alpha = \alpha \circ \acute{H}$ and $\grave{G} \circ \alpha = \alpha \circ \grave{H}$, this restriction of α is called the *underlying graph morphism of* α), the restriction of α to $\lfloor \mathscr{A}_H \rfloor$ is a Σ-homomorphism from $\mathscr{A}_H$ to $\mathscr{A}_G$, denoted $\mathring{\alpha}$, and $\mathring{\alpha} \circ \mathring{H}(x) \subseteq \mathring{G} \circ \alpha(x)$ for all $x \in \dot{H} \cup \vec{H}$. This means that α is an isomorphism if and only if α is a bijective morphism *and* α^{-1} is a morphism, hence if and only if the underlying graph morphism of α is an isomorphism, $\mathring{\alpha}$ is a Σ-isomorphism *and* $\mathring{\alpha} \circ \mathring{H} = \mathring{G} \circ \alpha$. For all $F \lhd H$, the *image* $\alpha(F)$ is the smallest subgraph of G w.r.t. the order $\lhd$ such that $\alpha|_{\lfloor F \rfloor}$ is a morphism from F to $\alpha(F)$.

If the underlying graph morphism of α is injective then α is called a *matching*. Note that the Σ-homomorphism $\mathring{\alpha}$ need not be injective.

Given two attributions l and l' of G we define $l \setminus l'$ (resp. $l \cap l'$, $l \cup l'$) as the attribution of G that maps any x to $l(x) \setminus l'(x)$ (resp. $l(x) \cap l'(x)$, $l(x) \cup l'(x)$). If l is an attribution of a subgraph $H \lhd G$, we extend it implicitly to the attribution of G that is identical to l on $\dot{H} \cup \vec{H}$ and maps any other entry to $\emptyset$.

For any sets V, A and attribution l, we say that G *is disjoint from* V, A, l if $\dot{G} \cap V = \varnothing$, $\vec{G} \cap A = \varnothing$ and $\mathring{G}(x) \cap l(x) = \varnothing$ for all $x \in \dot{G} \cup \vec{G}$. We write $G \setminus [V, A, l]$ for the largest subgraph of G (w.r.t. $\lhd$) that is disjoint from G. It is easy to see that this subgraph always exists.

In order to define parallel rewrite relations on graphs, it is convenient to join possibly many different graphs that have a common part, i.e., that are joinable. We start with a simpler notion of joinable functions.

Definition 2.1 (joinable functions)**.** Two functions $f : D \to C$ and $g : D' \to C'$ are *joinable* if $f(x) = g(x)$ for all $x \in D \cap D'$. Then, the *meet* of f and g is the function $f \curlywedge g : D \cap D' \to C \cap C'$ that is the restriction of f (or g) to $D \cap D'$. The *join* $f \curlyvee g$ is the unique function from $D \cup D'$ to $C \cup C'$ such that $f = (f \curlyvee g)|_D$ and $g = (f \curlyvee g)|_{D'}$.

For any set I and any I-indexed family $(f_i : D_i \to C_i)_{i \in I}$ of pairwise joinable functions, let $\curlyvee_{i \in I} f_i$ be the only function from $\bigcup_{i \in I} D_i$ to $\bigcup_{i \in I} C_i$ such that $f_i = \big(\curlyvee_{i \in I} f_i \big)|_{D_i}$ for all $i \in I$.

In particular, functions with disjoint domains are joinable, and every function is joinable with itself: $f \curlyvee f = f \curlywedge f = f$. More generally, any two restrictions $f|_A$ and $f|_B$ of the same function f are joinable, $f|_A \curlywedge f|_B = f|_{A \cap B}$ and $f|_A \curlyvee f|_B = f|_{A \cup B}$.

It is obvious that these operations are commutative. On triples of pairwise joinable functions, they are also associative and distributive over each other.

Definition 2.2 (joinable graphs)**.** Two graphs H and G are *joinable* if $\mathscr{A}_H = \mathscr{A}_G$, $\dot{H} \cap \vec{G} = \vec{H} \cap \dot{G} = \varnothing$, and the functions $\acute{H}$ and $\acute{G}$ (and similarly $\grave{H}$ and $\grave{G}$) are joinable. We can then define the graphs

$$\begin{aligned} H \sqcap G &\stackrel{\text{def}}{=} (\, \dot{H} \cap \dot{G},\ \vec{H} \cap \vec{G},\ \acute{H} \curlywedge \acute{G},\ \grave{H} \curlywedge \grave{G},\ \mathscr{A}_H,\ \mathring{H} \cap \mathring{G} \,), \\ H \sqcup G &\stackrel{\text{def}}{=} (\, \dot{H} \cup \dot{G},\ \vec{H} \cup \vec{G},\ \acute{H} \curlyvee \acute{G},\ \grave{H} \curlyvee \grave{G},\ \mathscr{A}_H,\ \mathring{H} \cup \mathring{G} \,). \end{aligned}$$

Similarly, if $(G_i)_{i \in I}$ is an I-indexed family of graphs (where $I \neq \varnothing$) that are pairwise joinable, hence have the same algebra $\mathcal{A}$ of attributes, then let

$$\bigsqcup_{i \in I} G_i \stackrel{\text{def}}{=} (\, \bigcup_{i \in I} \dot{G}_i,\ \bigcup_{i \in I} \vec{G}_i,\ \curlyvee_{i \in I} \acute{G}_i,\ \curlyvee_{i \in I} \grave{G}_i,\ \mathcal{A},\ \bigcup_{i \in I} \mathring{G}_i \,).$$

It is easy to see that these structures are graphs: the sets of vertices and arrows are disjoint and the adjacency functions have the correct domains and codomains. Note that if H and G are joinable then $H \sqcap G = G \sqcap H \lhd H \lhd H \sqcup G = G \sqcup H$. Similarly, if the G_i's are pairwise joinable then $G_j \lhd \bigsqcup_{i \in I} G_i$ for all $j \in I$. We also see that any two subgraphs of G are joinable, and that $H \lhd G$ iff $H \sqcap G = H$ iff $H \sqcup G = G$. As above, on triples of pairwise joinable graphs, these operations are associative and distributive over each other.

3 Applying Rules in Parallel

In the following, we assume a set $\mathscr{V}$ disjoint from Σ, whose elements are called *variables*. For any finite $X \subseteq \mathscr{V}$, we call (Σ, X)*-graph* a finite graph G such that $\mathscr{A}_G = \mathscr{T}(\Sigma, X)$ (the algebra of Σ-terms over X). We define the set of variables occurring in a (Σ, X)-graph G as $\mathrm{Var}(G) \overset{\text{def}}{=} \bigcup_{x\in\dot{G}\cup\vec{G}} \big(\bigcup_{t\in\mathring{G}(x)} \mathrm{Var}(t)\big)$, where $\mathrm{Var}(t)$ is the set of variables occurring in t.

Definition 3.1 (rules, matchings)**.** A *rule* r is a triple (L, K, R) of (Σ, X)-graphs such that L and R are joinable, $L \sqcap R \lhd K \lhd L$ and $\mathrm{Var}(L) = X$ (see comment below).

A *matching* μ of r *in* a graph G is a matching from L to G such that $\mathring{\mu}(\mathring{L}(x) \setminus \mathring{K}(x)) \cap \mathring{\mu}(\mathring{K}(x)) = \varnothing$ (or equivalently $\mathring{\mu}(\mathring{L}(x) \setminus \mathring{K}(x)) = \mathring{\mu}(\mathring{L}(x)) \setminus \mathring{\mu}(\mathring{K}(x))$) for all $x \in \dot{K} \cup \vec{K}$. We denote $\mathscr{M}(r, G)$ the set of all matchings of r in G (they all have domain $\lfloor L \rfloor$).

We consider finite sets $\mathcal{R}$ of rules such that for all $r, r' \in \mathcal{R}$, if $(L, K, R) = r \neq r' = (L', K', R')$ then $\lfloor L \rfloor \neq \lfloor L' \rfloor$, so that $\mathscr{M}(r, G) \cap \mathscr{M}(r', G) = \varnothing$ for any graph G; we then write $\mathscr{M}(\mathcal{R}, G)$ for $\biguplus_{r\in\mathcal{R}} \mathscr{M}(r, G)$. For any $\mu \in \mathscr{M}(\mathcal{R}, G)$ there is a unique rule $\mathrm{r}_\mu \in \mathcal{R}$ such that $\mu \in \mathscr{M}(\mathrm{r}_\mu, G)$, and its components are denoted $\mathrm{r}_\mu = (\mathrm{L}_\mu, \mathrm{K}_\mu, \mathrm{R}_\mu)$.

Comment: if X were allowed to contain a variable v not occurring in L, then v would freely match any element of $\mathscr{A}_G$ and the set $\mathscr{M}(r, G)$ would contain as many matchings with essentially the same effect. Also note that $\mathrm{Var}(R) \subseteq \mathrm{Var}(L)$, R and K are joinable and $R \sqcap K = L \sqcap R$. The fact that K is not required to be a subgraph of R allows the possible deletion by other rules of data matched by K but not by R. This feature enables a straightforward representation of cellular automata (see [4]).

A rewrite step may involve the creation of new vertices in a graph, corresponding to the vertices of a rule that have no match in the input graph, i.e., those in $\dot{R} \setminus \dot{L}$ (or similarly may create new arrows). These vertices should really be new, not only different from the vertices of the original graph but also different from the vertices created by other transformations (corresponding to other matchings in the graph). This is computationally easy to do but not that easy to formalize in an abstract way. We simply reuse the vertices x from $\dot{R} \setminus \dot{L}$ by *indexing* them with any relevant matching μ, each time yielding a new vertex (x, μ) which is obviously different from any new vertex (x, ν) for any other matching $\nu \neq \mu$, and also from any vertex of G.

Definition 3.2 (graph $G^{\uparrow}_{\mu}$ and matching $\mu{\uparrow}$)**.** For any rule $r = (L, K, R)$, graph G and $\mu \in \mathscr{M}(r, G)$ we define a graph $G^{\uparrow}_{\mu}$ together with a matching $\mu{\uparrow}$ of R in $G^{\uparrow}_{\mu}$. We first define the sets

$$\dot{G}^{\uparrow}_{\mu} \overset{\text{def}}{=} \mu(\dot{R} \cap \dot{K}) \uplus ((\dot{R} \setminus \dot{K}) \times \{\mu\}) \quad \text{and} \quad \vec{G}^{\uparrow}_{\mu} \overset{\text{def}}{=} \mu(\vec{R} \cap \vec{K}) \uplus ((\vec{R} \setminus \vec{K}) \times \{\mu\}).$$

Next we define $\mu{\uparrow}$ by: $\mathring{\mu}{\uparrow} \overset{\text{def}}{=} \mathring{\mu}$ and for all $x \in \dot{R} \cup \vec{R}$, if $x \in \dot{K} \cup \vec{K}$ then $\mu{\uparrow}(x) \overset{\text{def}}{=} \mu(x)$ else $\mu{\uparrow}(x) \overset{\text{def}}{=} (x, \mu)$. Since the restriction of $\mu{\uparrow}$ to $\dot{R} \cup \vec{R}$ is bijective, then $\mu{\uparrow}$ is a matching from R to the graph

$$G^{\uparrow}_{\mu} \overset{\text{def}}{=} (\, \dot{G}^{\uparrow}_{\mu},\, \vec{G}^{\uparrow}_{\mu},\, \mu{\uparrow} \circ \acute{R} \circ \mu{\uparrow}^{-1},\, \mu{\uparrow} \circ \grave{R} \circ \mu{\uparrow}^{-1},\, \mathscr{A}_G,\, \mathring{\mu}{\uparrow} \circ \mathring{R} \circ \mu{\uparrow}^{-1}\,).$$

By construction $\mu{\uparrow}(R) = G^{\uparrow}_{\mu}$, μ and $\mu{\uparrow}$ are joinable and $\mu \curlywedge \mu{\uparrow}$ is a matching from $R \sqcap K$ to $\mu(R \sqcap K)$. It is easy to see that the graph G and the graphs $G^{\uparrow}_{\mu}$ are pairwise joinable.

For any set $M \subseteq \mathscr{M}(\mathcal{R}, G)$ of matchings in a graph G we define below how to transform G by applying simultaneously the rules associated with matches in M.

Definition 3.3 (graph $G\|_M$)**.** For any graph G, set $M \subseteq \mathscr{M}(\mathcal{R}, G)$ and matching $\mu \in \mathscr{M}(\mathcal{R}, G)$, let

$$G\|_M \stackrel{\text{def}}{=} G \setminus [\mathrm{V}_M, \mathrm{A}_M, \ell_M] \sqcup \bigsqcup_{\mu \in M} G_\mu^{\uparrow} \text{ where}$$

$$\mathrm{V}_M \stackrel{\text{def}}{=} \bigcup_{\mu \in M} \mu(\dot{\mathrm{L}}_\mu \setminus \dot{\mathrm{K}}_\mu),\ \mathrm{A}_M \stackrel{\text{def}}{=} \bigcup_{\mu \in M} \mu(\vec{\mathrm{L}}_\mu \setminus \vec{\mathrm{K}}_\mu) \text{ and } \ell_M \stackrel{\text{def}}{=} \bigcup_{\mu \in M} \mathring{\mu} \circ (\mathring{\mathrm{L}}_\mu \setminus \mathring{\mathrm{K}}_\mu) \circ \mu^{-1}.$$

If M is a singleton $\{\mu\}$ we write $G\|_\mu$ for $G\|_M$, V_μ for V_M, etc.

Note that ℓ_M is only defined on the subgraph $\bigsqcup_{\mu \in M} \mu(\mathrm{L}_\mu)$ of G; so ℓ_M is implicitly extended to $\dot{G} \cup \vec{G}$ by mapping other vertices and arrows to $\varnothing$. $G\|_M$ is guaranteed to be a graph since the $\sqcup$ operation is only applied on joinable graphs. Every morphism $\mu\!\uparrow$ is a matching from the right hand side R_μ to $G\|_M$.

The definition of $G\|_M$ bears some similarity with the double pushout diagram (see [3]), where $G \setminus [V, A, l]$ replaces the pushout complement of G (but we are not restricted by the gluing condition) and $\bigsqcup_{\mu \in M} G_\mu^{\uparrow}$ replaces the right pushout. The case where M is a singleton defines the classical semantics of one sequential rewrite step.

Definition 3.4 (sequential rewriting)**.** For any finite set of rules $\mathcal{R}$, we define the relation $\rightarrow_{\mathcal{R}}$ of *sequential rewriting* by stating that, for all graphs G and H,

$$G \rightarrow_{\mathcal{R}} H \text{ iff there exists some } \mu \in \mathscr{M}(\mathcal{R}, G) \text{ such that } H \simeq G\|_\mu.$$

4 Sequential Independence

In the Double-Pushout approach to graph rewriting (see [3]), sequential independence is a property of two consecutive direct transformations, formulated as the existence of two commuting morphisms j_1 and j_2 as shown below.

$$\begin{tikzcd}
L_1 \arrow[d, "\mu_1"] & K_1 \arrow[l] \arrow[r] \arrow[d] & R_1 \arrow[dr] \arrow[drrr, dashed, "j_1"] & & L_2 \arrow[dl, "\mu_2"] \arrow[dlll, dashed, "j_2"'] & K_2 \arrow[l] \arrow[r] \arrow[d] & R_2 \arrow[d] \\
G & D_1 \arrow[l] \arrow[rr] & & H_1 & & D_2 \arrow[ll] \arrow[r] & H_2
\end{tikzcd}$$

It is then proven by the Local Church-Rosser Theorem that the two production rules can be applied in reverse order to G and yield the same result H_2 (we may call this the *swapping property*). Of course, the matchings μ_1 and μ_2 are then replaced by other matchings that are related to μ_1 and μ_2. A drawback of this definition is that it does not account for longer sequences of direct transformations. Indeed, if three consecutive steps are given by (μ_1, μ_2, μ_3), it is possible to swap μ_1 with μ_2 if they are sequential independent, and similarly for μ_2 and μ_3, but this does not imply that μ_1 and μ_3 can be swapped under these hypotheses (because the matchings, and hence the direct transformations, are modified by the swapping operations). We would need to express sequential independence between μ_1 and μ_3, but the definition does not apply since they are not consecutive steps. More elaborate notions of equivalence between sequences of direct transformations are thus required (see [2]).

Because of the specificities of our framework (no pushouts, horizontal morphisms are only canonical injections, and we do not have such a morphism from K to R) we need a different definition of sequential independence. It is natural to think of the swapping property as the definition, but we are faced with another problem. We are dealing with possibly infinite sets of matchings of rules in a graph, and we cannot form a notion of infinite sequences of rewrite steps (because each step may both remove and add data). Yet we do not wish to restrict the notion to finite sets, not simply for the sake of generality but also because it is closely related to parallel independence, a notion that can naturally be defined on infinite sets (see below).

We may however use Definition 3.3 to handle infinite sets of matchings, and thus express sequential independence as the possibility to apply any rule *after* the others (and these can be applied in parallel), yielding the same result as a parallel transformation with the whole set of matchings. Yet this definition would not imply that subsets of a sequential independent set are sequential independent, hence it needs to be stated for all subsets.

Definition 4.1. For any graph G and set $M \subseteq \mathscr{M}(\mathcal{R}, G)$, we say that M is *sequential independent* if for all $M' \subseteq M$ and all $\mu \in M \setminus M'$,

- $\mu(\mathrm{L}_\mu) \lhd G\|_{M'}$, hence there is a is canonical injection j from $\mu(\mathrm{L}_\mu)$ to $G\|_{M'}$,
- there exists an isomorphism α such that $\alpha(G\|_{M' \cup \{\mu\}}) = \big(G\|_{M'}\big)\|_{j \circ \mu}$ and α is the identity on G.

The isomorphism α in Definition 4.1 is necessary to account for the difference between the isomorphic graphs $\mu{\uparrow}(\mathrm{R}_\mu)$ and $(j \circ \mu){\uparrow}(\mathrm{R}_\mu)$.

It is then easy to see (by induction on the cardinality of M) that

Proposition 4.1. *For any graph G and finite set $M \subseteq \mathscr{M}(\mathcal{R}, G)$, if M is sequential independent then*

$$G \to^{\star}_{\mathcal{R}} G\|_M.$$

The converse is obviously not true; one reason is that sequences of rewrite steps cannot generally be swapped.

5 Parallel Independence

In the Double-Pushout approach, parallel independence is a property of two direct transformations of the same object G, formulated as the existence of two commuting morphisms j_1 and j_2 as shown below.

$$\begin{tikzcd}
R_1 \arrow[d] & K_1 \arrow[l] \arrow[r] \arrow[d] & L_1 \arrow[dr, "\mu_1"] \arrow[drrr, dashed, "j_1"] & & L_2 \arrow[dl, "\mu_2"'] \arrow[dlll, dashed, "j_2"'] & K_2 \arrow[l] \arrow[r] \arrow[d] & R_2 \arrow[d] \\
H_1 & D_1 \arrow[l] \arrow[rr] & & G & & D_2 \arrow[ll] \arrow[r] & H_2
\end{tikzcd}$$

This definition can easily be lifted to sets of matchings (or direct transformations) by considering all possible pairs of matchings, with a slight caveat. In this definition the two direct transformations may be identical, thus stating a property of a single transformation that is clearly not shared by all. But Definition 3.3 does not allow to apply any member μ of M more than once (because applying μ any number of times in parallel would jeopardize determinism). For this reason any singleton M shall be considered as parallel independent.

The Local Church-Rosser Theorem mentioned above actually shows that μ_1 and μ_2 are parallel independent iff they correspond to a sequential independent pair (μ_1, μ'_2), where μ_2 and μ'_2 are related. It is the symmetry between μ_1 and μ_2 that entails the swapping property. This is remarquable since parallel independence does not refer to the *results* of the direct transformations involved.

Our goal is therefore to formulate parallel independence in the present framework, in order to obtain an equivalence similar to the Local Church-Rosser Theorem. Considering that the pushout complement D_1 is replaced by the graph $G \setminus [\mathrm{V}_{\mu_1}, \mathrm{A}_{\mu_1}, \ell_{\mu_1}]$, the commuting property of j_2 amounts to $\mu_2(L_2) \lhd G \setminus [\mathrm{V}_{\mu_1}, \mathrm{A}_{\mu_1}, \ell_{\mu_1}]$, that can be more elegantly expressed as $\mu_2(L_2) \sqcap \mu_1(L_1) \lhd \mu_1(K_1)$. This simply means that any graph item that is matched twice cannot be removed.

However, our treatment of attributes makes it possible to recover in the right hand side an attribute that has been deleted in the left hand side (this is of course not possible for vertices or arrows). This possibility should therefore be accounted for in the notion of parallel independence, i.e., an attribute that is matched twice may be deleted provided it is recovered. We also need to consider what it means for an attribute to be matched: it may be the case that an (occurrence of an) attribute is matched by $\nu{\uparrow}$ but not by ν (i.e., it corresponds to an occurrence of a term in the right hand side of a rule but to none in the left hand side). This leads to the following definition.

Definition 5.1. For any graph G and set $M \subseteq \mathscr{M}(\mathcal{R}, G)$, we say that M is *parallel independent* if

$$\mu(\mathrm{L}_\mu) \sqcap (\nu(\mathrm{L}_\nu) \sqcup \nu{\uparrow}(\mathrm{R}_\nu)) \lhd \mu(\mathrm{K}_\mu) \sqcup \mu{\uparrow}(\mathrm{R}_\mu) \text{ for all } \mu, \nu \in M \text{ such that } \mu \neq \nu.$$

This definition may seem strange, but it is easy to see that on unlabelled graphs it amounts to $\nu(\mathrm{L}_\nu) \sqcap \mu(\mathrm{L}_\mu) \lhd \mu(\mathrm{K}_\mu)$ for all $\mu \neq \nu$, i.e., to the standard algebraic notion of parallel independence (translated to the present framework). But the best justification for the definition is the following result.

Theorem 5.1. *For any graph G and set $M \subseteq \mathscr{M}(\mathcal{R}, G)$, M is parallel independent iff M is sequential independent.*

Thus Definition 5.1 arises as a characterization of sequential independence that does not refer to the results of the transformations, and indeed that does not rely on Definition 3.3, though of course it does rely on Definitions 2.2, 3.1 and 3.2.

6 The Effective Deletion Property

We have not yet defined a relation of parallel rewriting as we did for sequential rewriting (Definition 3.4). The reason is that two matchings may conflict as one retains (in $R \sqcap K$) what another removes. The transformation offered by Definition 3.3 performs deletions before unions, which means that these conflicts are resolved by giving priority to retainers over removers. But if the deletion actions of a rule are not executed in a parallel transformation, how can we claim that this rule has been executed (or applied) in parallel with others? Thus, in order to define parallel rewriting with a clear semantics we need to rule out such conflicts.

One possibility is to translate to the present framework the notion of *parallel coherence* that has been devised in order to define algebraic parallel graph transformation (see [4]). This is is a property of two direct transformations of the same object G, formulated as the existence of two commuting morphisms j_1 and j_2 as shown below.

$$
\begin{tikzcd}
R_1 \arrow[dr] & & I_1 \arrow[ll] \arrow[rr] \arrow[dr] \arrow[drrrrrrr, dashed, "j_1"] & & K_1 \arrow[r] \arrow[dl] & L_1 \arrow[dr, "\mu_1"] & & L_2 \arrow[dl, "\mu_2"'] & K_2 \arrow[l] \arrow[dr] & & I_2 \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dlllllll, dashed, "j_2"'] & & R_2 \arrow[dl] \\
& H_1 & & D_1 \arrow[ll] \arrow[rrr] & & & G & & & D_2 \arrow[lll] \arrow[rr] & & H_2 &
\end{tikzcd}
$$

This notion clearly generalizes algebraic parallel independence. In the present framework the object I_2 is replaced by the graph $K_2 \sqcap R_2$, hence the commuting property of j_2 amounts to $\mu_2(K_2 \sqcap R_2) \lhd G \setminus [\mathrm{V}_{\mu_1}, \mathrm{A}_{\mu_1}, \ell_{\mu_1}]$, that can be expressed as $\mu_2(K_2 \sqcap R_2) \sqcap \mu_1(L_1) \lhd \mu_1(K_1)$. This simply means that any graph item that is matched by some $K \sqcap R$ cannot be removed by any rule.

Definition 6.1. For any graph G and set $M \subseteq \mathscr{M}(\mathcal{R}, G)$, we say that M is *parallel coherent* if

$$\mu(\mathrm{L}_\mu) \sqcap \nu(\mathrm{K}_\nu \sqcap \mathrm{R}_\nu) \lhd \mu(\mathrm{K}_\mu) \text{ for all } \mu, \nu \in M.$$

The problem here as above is that deleted attributes can be recovered by the right hand side of rules, and that this possibility is not accounted for in the algebraic definitions, since these do not distinguish between graph items and attributes. This leads to the following definition (see [5]).

Definition 6.2 (effective deletion property, parallel rewriting)**.** For any graph G, a set $M \subseteq \mathscr{M}(\mathcal{R}, G)$ is said to satisfy the *effective deletion property* if $G\|_M$ is disjoint from $\mathrm{V}_M, \mathrm{A}_M, \ell_M \setminus \ell_M^{\uparrow}$, where

$$\ell_M^{\uparrow} \stackrel{\text{def}}{=} \bigcup_{\mu \in M} \mathring{\mu} \circ (\mathring{\mathrm{R}}_\mu \setminus \mathring{\mathrm{K}}_\mu) \circ \mu^{-1}.$$

For any finite set of rules $\mathcal{R}$, we define the relation $\rightrightarrows_{\mathcal{R}}$ of *full parallel rewriting* by stating that, for all graphs G and H,

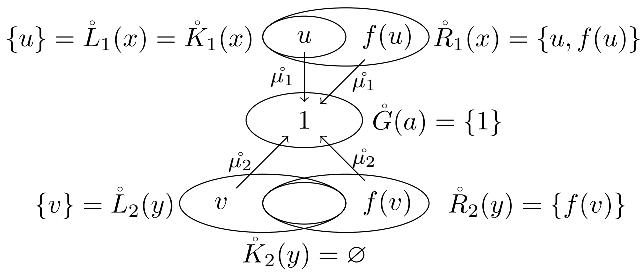
$$G \rightrightarrows_{\mathcal{R}} H \text{ iff } \mathscr{M}(\mathcal{R}, G) \text{ has the effective deletion property and } H \simeq G\|_{\mathscr{M}(\mathcal{R},G)}.$$

It can be shown that $\rightrightarrows_{\mathcal{R}}$ is deterministic up to isomorphism, that is, if $G \rightrightarrows_{\mathcal{R}} H$, $G' \rightrightarrows_{\mathcal{R}} H'$ and $G \simeq G'$ then $H \simeq H'$. In particular, it is possible to represent any cellular automata by a suitable rule r and a class of graphs that correspond to configurations of the automata (every vertex corresponds to a cell), such that $\rightrightarrows_r$ (restricted to such graphs) is the transition function.

This representation of cellular automata satisfies the effective deletion property, but it also satisfies parallel coherence. Hence Definition 6.2 may appear as a weird choice. One motivation behind the present work is to support this definition.

Our first argument in favor of Definition 6.2 is that parallel coherence is not sufficient because it does not generalize parallel independence, as shown by the following example.

Example 6.1. Let us consider rules $r_1 = (L_1, K_1, R_1)$ and $r_2 = (L_2, K_2, R_2)$ where the graphs L_1, K_1 and R_1 have only one vertex x, the graphs L_2, K_2 and R_2 have only one vertex y, and the attributes are as pictured below (u, v are variables and f is a unary function symbol). Let $\mathscr{A}_G$ be the algebra with carrier set $\{1\}$ where f is interpreted as the constant function 1, and let G be the graph that has a unique vertex a with attribute $\{1\}$.



There are exactly two matchings of $\{r_1, r_2\}$ in G: μ_1 and μ_2 defined by $\mu_1(x) = a$, $\mathring{\mu}_1(u) = 1$, $\mu_2(y) = a$ and $\mathring{\mu}_2(v) = 1$. Let $M = \{\mu_1, \mu_2\}$, we see that M is not parallel coherent since $\mu_1(R_1 \sqcap K_1) \sqcap \mu_2(L_2) = G$ is not a subgraph of $\mu_2(K_2)$. However, we see that M is sequential independent since the matchings can be applied sequentially in any order, yielding the same graph G.

Our second argument is that the effective deletion property is sufficient because it does generalize both parallel coherence and parallel independence.

Theorem 6.2. *For any graph G and set $M \subseteq \mathscr{M}(\mathcal{R}, G)$ if M is parallel independent or parallel coherent then M has the effective deletion property.*

Hence effective deletion encompasses both a general algebraic notion translated to the present (non algebraic) framework, and a notion specific to this framework but that relies on an objective fact, that is Theorem 5.1. This does not mean that no other property is possible (especially a less general one) and that Definition 6.2 cannot be questioned. It is still a matter of choice, but there is evidence that this is a reasonable one.

We also see that

Corollary. *If $\mathscr{M}(\mathcal{R}, G)$ is finite and parallel independent then $G \rightarrow^{\star}_{\mathcal{R}} G\|_M$ and $G \rightrightarrows_{\mathcal{R}} G\|_M$.*

Hence in this case full parallel rewriting deterministically chooses one among the graphs reachable from G by sequential rewriting.

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Appendix: Proofs

Lemma 6.3. *For every rule $r = (L, K, R)$, graph G and $\mu \in \mathscr{M}(r, G)$, the graphs G and $G^{\uparrow}_{\mu}$ are joinable, $\mu(R \sqcap K) \lhd G \sqcap G^{\uparrow}_{\mu}$ and $G \sqcap G^{\uparrow}_{\mu}$ has the same underlying graph as $\mu(R \sqcap K)$.*

Proof. It is obvious that $\dot{G} \cap \vec{G}^{\uparrow}_{\mu} = \vec{G} \cap \dot{G}^{\uparrow}_{\mu} = \varnothing$ and $\vec{G} \cap \vec{G}^{\uparrow}_{\mu} = \mu(\vec{R} \cap \vec{K})$, hence for all $g \in \vec{G} \cap \vec{G}^{\uparrow}_{\mu}$ there is a $f \in \vec{R} \cap \vec{K}$ such that $g = \mu(f) = \mu{\uparrow}(f)$, hence

$$\acute{G}^{\uparrow}_{\mu}(g) = \acute{G}^{\uparrow}_{\mu} \circ \mu{\uparrow}(f) = \mu{\uparrow} \circ \acute{R}(f) = \mu \circ \acute{K}(f) = \acute{G} \circ \mu(f) = \acute{G}(g)$$

so that $\acute{G}^{\uparrow}_{\mu}$ and $\acute{G}$ are joinable and similarly for $\grave{G}^{\uparrow}_{\mu}$ and $\grave{G}$, hence $G^{\uparrow}_{\mu}$ and G are joinable.

We have $\mu(R \sqcap K) \lhd \mu(K) \lhd G$ and

$$\mu(R \sqcap K) = \mu{\uparrow}(R \sqcap K) \lhd \mu{\uparrow}(R) = G^{\uparrow}_{\mu},$$

hence $\mu(R \sqcap K) \lhd G \sqcap G^{\uparrow}_{\mu}$. Besides, for all $y \in \dot{G} \cap \dot{G}^{\uparrow}_{\mu} = \mu(\dot{R} \cap \dot{K})$ there exists a $x \in \dot{R} \cap \dot{K}$ such that $\mu(x) = y$, hence $\dot{G} \cap \dot{G}^{\uparrow}_{\mu} \subseteq \mu(\dot{R} \cap \dot{K})$ and similarly $\vec{G} \cap \vec{G}^{\uparrow}_{\mu} \subseteq \mu(\vec{R} \cap \vec{K})$, hence the graphs $G \sqcap G^{\uparrow}_{\mu}$ and $\mu(R \sqcap K)$ have the same underlying graph. □

Lemma 6.4. *For all I-indexed families $(G_i)_{i \in I}$ of pairwise joinable graphs, for all sets V, A and all attributions l,*

$$\Big(\bigsqcup_{i \in I} G_i\Big) \setminus [V, A, l] = \bigsqcup_{i \in I} G_i \setminus [V, A, l].$$

Proof. Since $G_j \lhd \bigsqcup_{i \in I} G_i$ for all $j \in I$ then $G_j \setminus [V, A, l] \lhd \big(\bigsqcup_{i \in I} G_i\big) \setminus [V, A, l]$, hence $\bigsqcup_{j \in I} G_j \setminus [V, A, l] \lhd \big(\bigsqcup_{i \in I} G_i\big) \setminus [V, A, l]$.

Conversely, let $H \lhd \bigsqcup_{i \in I} G_i$ such that H is disjoint from V, A, l. For all $f \in \vec{H}$ and all $a \in \mathring{H}(x)$ there exists an $i \in I$ such that $f \in \vec{G}_i$ and $a \in \mathring{G}_i(f)$. Let $x = \acute{H}(f)$ and $y = \grave{H}(f)$, so that f is an arrow from x to y. Obviously $f \notin A$, $x, y \notin V$ and $a \notin l(f)$. Since $x, y \in \dot{G}_i$, then the graph with vertices x, y with attributes $\varnothing$ and arrow f with attribute $\{a\}$ is a subgraph of G_i disjoint from V, A, l, hence is a subgraph of $G_i \setminus [V, A, l]$ and therefore of $\bigsqcup_{j \in I} G_j \setminus [V, A, l]$. Similarly, for all $x \in \dot{H}$ and all $a \in \mathring{H}(x)$ the graph with vertex x attributed by $\{a\}$ is a subgraph of $\bigsqcup_{j \in I} G_j \setminus [V, A, l]$. Since H is the union of all such graphs then $H \lhd \bigsqcup_{j \in I} G_j \setminus [V, A, l]$, and this holds for $H = \big(\bigsqcup_{i \in I} G_i\big) \setminus [V, A, l]$. □

Proof of Theorem 6.2. Let $H = G\|_M$. Since $\mathrm{V}_M \subseteq \dot{G}$ then by Lemma 6.3 we have

$$\begin{aligned}
\dot{H} \cap \mathrm{V}_M &= \bigcup_{\nu \in M} \nu(\dot{\mathrm{R}}_\nu \cap \dot{\mathrm{K}}_\nu) \cap \mathrm{V}_M \\
&= \bigcup_{\mu, \nu \in M} \nu(\dot{\mathrm{R}}_\nu \cap \dot{\mathrm{K}}_\nu) \cap \mu(\dot{\mathrm{L}}_\mu) \setminus \mu(\dot{\mathrm{K}}_\mu) \\
&\subseteq \bigcup_{\mu \neq \nu \in M} \nu(\dot{\mathrm{L}}_\nu) \cap \mu(\dot{\mathrm{L}}_\mu) \setminus \mu(\dot{\mathrm{K}}_\mu),
\end{aligned}$$

since $\nu(\dot{\mathrm{R}}_\nu \cap \dot{\mathrm{K}}_\nu) \subseteq \nu(\dot{\mathrm{K}}_\nu) \subseteq \nu(\dot{\mathrm{L}}_\nu)$.

Since M is parallel independent then $\mu(\mathrm{L}_\mu) \sqcap (\nu(\mathrm{L}_\nu) \sqcup G^{\uparrow}_{\nu}) \lhd \mu(\mathrm{K}_\mu) \sqcup G^{\uparrow}_{\mu}$ for all $\mu \neq \nu$, hence $\mu(\mathrm{L}_\mu) \sqcap \nu(\mathrm{L}_\nu) \lhd \mu(\mathrm{K}_\mu) \sqcup (G^{\uparrow}_{\mu} \sqcap G)$ and again by Lemma 6.3 $\mu(\dot{\mathrm{L}}_\mu) \cap \nu(\dot{\mathrm{L}}_\nu) \subseteq \mu(\dot{\mathrm{K}}_\mu) \cup \mu(\dot{\mathrm{R}}_\mu \cap \dot{\mathrm{K}}_\mu) = \mu(\dot{\mathrm{K}}_\mu)$. Hence $\dot{H} \cap \mathrm{V}_M = \varnothing$ and similarly $\vec{H} \cap \mathrm{A}_M = \varnothing$.

In order to prove that H is disjoint from V_M, A_M, $\ell_M \setminus \ell^{\uparrow}_M$, there only remains to prove that $\mathring{H}(x) \cap \ell_M(x) \setminus \ell^{\uparrow}_M(x) = \varnothing$ for all $x \in \dot{H} \cup \vec{H}$. This is true if $x \notin \dot{G} \cup \vec{G}$ since then $\ell_M(x) = \varnothing$, hence we assume that $x \in \dot{G} \cup \vec{G}$, so that $\mathring{H}(x) \cap \ell_M(x) = \bigcup_{\mu \in M} \mathring{G}^{\uparrow}_{\mu} \cap \ell_M(x) = \bigcup_{\mu \in M} \mathring{\mu} \circ \mathring{\mathrm{R}}_\mu \circ \mu^{-1}(x) \cap \ell_M(x)$ and we need to prove that $\mathring{\mu} \circ \mathring{\mathrm{R}}_\mu \circ \mu^{-1}(x) \cap \ell_M(x) \setminus \ell^{\uparrow}_M(x) = \varnothing$ for all $\mu \in M$, or equivalently

$$\bigcup_{\nu \in M} \mathring{\mu} \circ \mathring{\mathrm{R}}_\mu \circ \mu^{-1}(x) \cap \mathring{\nu} \circ \mathring{\mathrm{L}}_\nu \circ \nu^{-1}(x) \setminus \mathring{\nu} \circ \mathring{\mathrm{K}}_\nu \circ \nu^{-1}(x) \subseteq \ell^{\uparrow}_M(x).$$

We first see that for any sets A and B we have $\mathring{\mu}(A) \setminus \mathring{\mu}(A \cap B) \subseteq \mathring{\mu}(A \setminus B)$, hence

$$\mathring{\mu} \circ \mathring{\mathrm{R}}_\mu \circ \mu^{-1}(x) \setminus \mathring{\mu} \circ (\mathring{\mathrm{R}}_\mu \cap \mathring{\mathrm{K}}_\mu) \circ \mu^{-1}(x) \subseteq \mathring{\mu} \circ (\mathring{\mathrm{R}}_\mu \setminus \mathring{\mathrm{K}}_\mu) \circ \mu^{-1}(x) \subseteq \ell^{\uparrow}_M(x).$$

Next, for all $\nu \in M$ such that $\nu \neq \mu$, since M is parallel independent then

$$\begin{aligned}
&\mathring{\mu} \circ (\mathring{\mathrm{R}}_\mu \cap \mathring{\mathrm{K}}_\mu) \circ \mu^{-1}(x) \cap \mathring{\nu} \circ \mathring{\mathrm{L}}_\nu \circ \nu^{-1}(x) \\
&\subseteq \mathring{\mu} \circ \mathring{\mathrm{L}}_\mu \circ \mu^{-1}(x) \cap \mathring{\nu} \circ \mathring{\mathrm{L}}_\nu \circ \nu^{-1}(x) \\
&\subseteq \mathring{\nu} \circ \mathring{\mathrm{K}}_\nu \circ \nu^{-1}(x) \cup \mathring{\nu} \circ \mathring{\mathrm{R}}_\nu \circ \nu^{-1}(x) \\
&\subseteq \mathring{\nu} \circ \mathring{\mathrm{K}}_\nu \circ \nu^{-1}(x) \cup \mathring{\nu} \circ (\mathring{\mathrm{R}}_\nu \setminus \mathring{\mathrm{K}}_\nu) \circ \nu^{-1}(x) \\
&\subseteq \mathring{\nu} \circ \mathring{\mathrm{K}}_\nu \circ \nu^{-1}(x) \cup \ell^{\uparrow}_M(x).
\end{aligned}$$

Then, we use the fact that $A = (A \cap B) \cup (A \setminus B)$ to deduce that

$$\begin{aligned}
&\mathring{\mu} \circ \mathring{\mathrm{R}}_\mu \circ \mu^{-1}(x) \cap \mathring{\nu} \circ \mathring{\mathrm{L}}_\nu \circ \nu^{-1}(x) \\
&= \big(\mathring{\mu} \circ (\mathring{\mathrm{R}}_\mu \cap \mathring{\mathrm{K}}_\mu) \circ \mu^{-1}(x) \cap \mathring{\nu} \circ \mathring{\mathrm{L}}_\nu \circ \nu^{-1}(x)\big) \\
&\qquad \cup \big(\mathring{\nu} \circ \mathring{\mathrm{L}}_\nu \circ \nu^{-1}(x) \cap \mathring{\mu} \circ \mathring{\mathrm{R}}_\mu \circ \mu^{-1}(x) \setminus \mathring{\mu} \circ (\mathring{\mathrm{R}}_\mu \cap \mathring{\mathrm{K}}_\mu) \circ \mu^{-1}(x)\big) \\
&\subseteq \big(\mathring{\nu} \circ \mathring{\mathrm{K}}_\nu \circ \nu^{-1}(x) \cup \ell^{\uparrow}_M(x)\big) \\
&\qquad \cup \big(\mathring{\mu} \circ \mathring{\mathrm{R}}_\mu \circ \mu^{-1}(x) \setminus \mathring{\mu} \circ (\mathring{\mathrm{R}}_\mu \cap \mathring{\mathrm{K}}_\mu) \circ \mu^{-1}(x)\big) \\
&\subseteq \mathring{\nu} \circ \mathring{\mathrm{K}}_\nu \circ \nu^{-1}(x) \cup \ell^{\uparrow}_M(x).
\end{aligned}$$

We notice that this is also true when $\nu = \mu$ since $\mathrm{L}_\mu \sqcap \mathrm{R}_\mu \lhd \mathrm{K}_\mu$, hence

$$\mathring{\mu} \circ \mathring{\mathrm{R}}_\mu \circ \mu^{-1}(x) \cap \mathring{\nu} \circ \mathring{\mathrm{L}}_\nu \circ \nu^{-1}(x) \setminus \mathring{\nu} \circ \mathring{\mathrm{K}}_\nu \circ \nu^{-1}(x) \subseteq \ell^{\uparrow}_M(x)$$

for all $\nu \in M$. □

Proof of Theorem 5.1. Only if part. For all $M' \subseteq M$ and $\mu \in M \setminus M'$, let $R = \bigsqcup_{\nu \in M'} G^{\uparrow}_{\nu}$ so that $G\|_{M'} = G \setminus [\mathrm{V}_{M'}, \mathrm{A}_{M'}, \ell_{M'}] \sqcup R$. For all $\nu \in M'$ we have $\mu(\mathrm{L}_\mu) \sqcap \nu(\mathrm{L}_\nu) \lhd \nu(\mathrm{K}_\nu) \sqcup G^{\uparrow}_{\nu}$ and $\mu(\mathrm{L}_\mu) \sqcap \nu(\mathrm{L}_\nu) \lhd G$, hence by Lemma 6.3

$$\mu(\dot{\mathrm{L}}_\mu) \cap \nu(\dot{\mathrm{L}}_\nu) \subseteq \nu(\dot{\mathrm{K}}_\nu) \cup \nu(\dot{\mathrm{R}}_\nu \cap \dot{\mathrm{K}}_\nu) = \nu(\dot{\mathrm{K}}_\nu)$$

or equivalently $\mu(\dot{\mathrm{L}}_\mu) \cap \nu(\dot{\mathrm{L}}_\nu) \setminus \nu(\dot{\mathrm{K}}_\nu) = \varnothing$. Thus

$$\mu(\dot{\mathrm{L}}_\mu) \cap \mathrm{V}_{M'} = \bigcup_{\nu \in M'} \mu(\dot{\mathrm{L}}_\mu) \cap \nu(\dot{\mathrm{L}}_\nu) \setminus \nu(\dot{\mathrm{K}}_\nu) = \varnothing$$

and therefore $\mu(\dot{\mathrm{L}}_\mu) \subseteq \dot{G}\|_{M'}$. Similarly we get $\mu(\vec{\mathrm{L}}_\mu) \subseteq \vec{G}\|_{M'}$. Then, for all $x \in \mu(\dot{\mathrm{L}}_\mu) \cup \mu(\vec{\mathrm{L}}_\mu)$, we have

$$\mathring{\mu} \circ \mathring{\mathrm{L}}_\mu \circ \mu^{-1}(x) \cap \mathring{\nu} \circ \mathring{\mathrm{L}}_\nu \circ \nu^{-1}(x) \subseteq \mathring{\nu} \circ \mathring{\mathrm{K}}_\nu \circ \nu^{-1}(x) \cup \mathring{G}_\nu^{\uparrow}(x)$$

hence

$$\mathring{\mu} \circ \mathring{\mathrm{L}}_\mu \circ \mu^{-1}(x) \cap \mathring{\nu} \circ \mathring{\mathrm{L}}_\nu \circ \nu^{-1}(x) \setminus \mathring{\nu} \circ \mathring{\mathrm{K}}_\nu \circ \nu^{-1}(x) \subseteq \mathring{G}_\nu^{\uparrow}(x) \subseteq \mathring{R}(x).$$

Thus

$$\mathring{\mu} \circ \mathring{\mathrm{L}}_\mu \circ \mu^{-1}(x) \cap \ell_{M'}(x) = \bigcup_{\nu \in M'} \mathring{\mu} \circ \mathring{\mathrm{L}}_\mu \circ \mu^{-1}(x) \cap \mathring{\nu} \circ \mathring{\mathrm{L}}_\nu \circ \nu^{-1}(x) \setminus \mathring{\nu} \circ \mathring{\mathrm{K}}_\nu \circ \nu^{-1}(x) \subseteq \mathring{R}(x)$$

and then

$$\mathring{\mu} \circ \mathring{\mathrm{L}}_\mu \circ \mu^{-1}(x) \subseteq \mathring{\mu} \circ \mathring{\mathrm{L}}_\mu \circ \mu^{-1}(x) \setminus \ell_{M'}(x) \cup \mathring{R}(x) \subseteq \mathring{G}\|_{M'}.$$

Therefore, $\mu(\mathrm{L}_\mu) \lhd G\|_{M'}$.

Let j be the canonical injection from $\mu(\mathrm{L}_\mu)$ to $G\|_{M'}$ and $\mu' = j \circ \mu$, so that $\mu' \in \mathscr{M}(\mathrm{r}_\mu, G\|_{M'})$, $\mu'(\mathrm{L}_\mu) = \mu(\mathrm{L}_\mu)$ and $\mu'(\mathrm{K}_\mu) = \mu(\mathrm{K}_\mu)$, hence $\mathrm{V}_{\mu'} = \mathrm{V}_\mu$, $\mathrm{A}_{\mu'} = \mathrm{A}_\mu$ and $\ell_{\mu'} = \ell_\mu$. Let $H = G \sqcup R \sqcup \mu{\uparrow}(\mathrm{R}_\mu)$ and $H' = G \sqcup R \sqcup \mu'{\uparrow}(\mathrm{R}_\mu)$. Note that $G\|_{M' \cup \{\mu\}} \lhd H$, and also that $\mathrm{R}_{\mu'} = \mathrm{R}_\mu$ hence $\mu'{\uparrow}(\mathrm{R}_\mu) = (G\|_{M'})_{\mu'}^{\uparrow}$ and (using Lemma 6.4)

$$\begin{aligned}
(G\|_{M'})\|_{\mu'} &= \big(G \setminus [\mathrm{V}_{M'}, \mathrm{A}_{M'}, \ell_{M'}] \sqcup \bigsqcup_{\nu \in M'} G_\nu^{\uparrow}\big) \setminus [\mathrm{V}_{\mu'}, \mathrm{A}_{\mu'}, \ell_{\mu'}] \sqcup \mu'{\uparrow}(\mathrm{R}_\mu) \\
&= G \setminus [\mathrm{V}_M, \mathrm{A}_M, \ell_M] \sqcup \bigsqcup_{\nu \in M'} G_\nu^{\uparrow} \setminus [\mathrm{V}_\mu, \mathrm{A}_\mu, \ell_\mu] \sqcup \mu'{\uparrow}(\mathrm{R}_\mu) \\
&\lhd H'.
\end{aligned}$$

By Theorem 6.2 M has the effective deletion property, i.e., $G\|_M$ is disjoint from V_M, A_M, $\ell_M \setminus \ell_M^{\uparrow}$ hence in particular $G_\nu^{\uparrow}$ is disjoint from V_μ, A_μ, $\ell_\mu \setminus \ell_M^{\uparrow}$ for all $\nu \in M'$, so that

$$G_\nu^{\uparrow} \setminus [\mathrm{V}_\mu, \mathrm{A}_\mu, \ell_\mu] = G_\nu^{\uparrow} \setminus [\mathrm{V}_\mu \setminus \mathrm{V}_\mu, \mathrm{A}_\mu \setminus \mathrm{A}_\mu, \ell_\mu \setminus (\ell_\mu \setminus \ell_M^{\uparrow})] = G_\nu^{\uparrow} \setminus [\varnothing, \varnothing, \ell_\mu \cap \ell_M^{\uparrow}].$$

For all $x \in \dot{G}_\nu^{\uparrow} \cup \vec{G}_\nu^{\uparrow}$, if $x \notin \dot{G} \cup \vec{G}$ then $\ell_\mu(x) = \varnothing$, otherwise $\mathring{G}_\mu^{\uparrow}(x) = \mathring{\mu} \circ \mathring{\mathrm{R}}_\mu \circ \mu^{-1}(x) = \mathring{\mu}' \circ \mathring{\mathrm{R}}_{\mu'} \circ \mu'^{-1}(x)$. Since $\mu(\mathrm{L}_\mu) \sqcap G_\nu^{\uparrow} \lhd \mu(\mathrm{K}_\mu) \sqcup G_\mu^{\uparrow}$ we have

$$\mathring{G}_\nu^{\uparrow}(x) \cap \mathring{\mu} \circ \mathring{\mathrm{L}}_\mu \circ \mu^{-1}(x) \subseteq \mathring{\mu} \circ \mathring{\mathrm{K}}_\mu \circ \mu^{-1}(x) \cup \mathring{G}_\mu^{\uparrow}(x)$$

or equivalently $\mathring{G}_\nu^{\uparrow}(x) \cap \ell_\mu(x) \subseteq \mathring{G}_\mu^{\uparrow}(x)$, and we therefore have

$$\mathring{G}_\nu^{\uparrow}(x) \cap \ell_\mu(x) \cap \ell_M^{\uparrow}(x) \subseteq \mathring{\mu}' \circ \mathring{\mathrm{R}}_{\mu'} \circ \mu'^{-1}(x).$$

We thus see that $G^{\uparrow}_{\nu} \setminus [\mathrm{V}_{\mu}, \mathrm{A}_{\mu}, \ell_{\mu}]$ has all the vertices and arrows of $G^{\uparrow}_{\nu}$, and the attributes that are removed are all in the graph $\mu'{\uparrow}(\mathrm{R}_{\mu})$, hence

$$G^{\uparrow}_{\nu} \setminus [\mathrm{V}_{\mu}, \mathrm{A}_{\mu}, \ell_{\mu}] \sqcup \mu'{\uparrow}(\mathrm{R}_{\mu}) = G^{\uparrow}_{\nu} \sqcup \mu'{\uparrow}(\mathrm{R}_{\mu})$$

and therefore $(G\|_{M'})\|_{\mu'} = G \setminus [\mathrm{V}_M, \mathrm{A}_M, \ell_M] \sqcup R \sqcup \mu'{\uparrow}(\mathrm{R}_{\mu})$. It is then easy to build an isomorphism $\alpha : H \to H'$ such that $\alpha(G\|_{M' \cup \{\mu\}}) = \big(G\|_{M'}\big)\|_{\mu'}$ and $\alpha|_{\lfloor G \rfloor} = 1_G$.

If part. For all $\mu, \nu \in M$ such that $\mu \neq \nu$, we have $\nu(\mathrm{L}_{\nu}) \lhd G\|_{\mu} = G \setminus [\mathrm{V}_{\mu}, \mathrm{A}_{\mu}, \ell_{\mu}] \sqcup G^{\uparrow}_{\mu}$. Since $\mu(\mathrm{K}_{\mu}) \lhd \mu(\mathrm{L}_{\mu}) \lhd G$, then

$$\begin{aligned}\nu(\mathrm{L}_{\nu}) \sqcap \mu(\mathrm{L}_{\mu}) \lhd G\|_{\mu} \sqcap \mu(\mathrm{L}_{\mu}) &= \mu(\mathrm{L}_{\mu}) \setminus [\mathrm{V}_{\mu}, \mathrm{A}_{\mu}, \ell_{\mu}] \sqcup (G^{\uparrow}_{\mu} \sqcap \mu(\mathrm{L}_{\mu})) \\ &= \mu(\mathrm{K}_{\mu}) \sqcup (G^{\uparrow}_{\mu} \sqcap \mu(\mathrm{L}_{\mu})) \\ &\lhd \mu(\mathrm{K}_{\mu}) \sqcup G^{\uparrow}_{\mu}.\end{aligned}$$

Besides, there is an isomorphism α such that $\alpha(G\|_M) = \big(G\|_{\nu}\big)\|_{\mu'}$ and $\alpha|_{\lfloor G \rfloor} = 1_G$, where $M = \{\mu, \nu\}$ and $\mu' = j \circ \mu \in \mathscr{M}(\mathrm{r}_{\mu}, G\|_{M'})$, hence $\mathrm{V}_{\mu'} = \mathrm{V}_{\mu}$, $\mathrm{A}_{\mu'} = \mathrm{A}_{\mu}$ and $\ell_{\mu'} = \ell_{\mu}$. Let $H = G\|_M \sqcap \mu(\mathrm{L}_{\mu})$ and $H' = \big(G\|_{\nu}\big)\|_{\mu'} \sqcap \mu(\mathrm{L}_{\mu})$, since $\mu(\mathrm{L}_{\mu}) \lhd G$ then $H = H'$. We see that

$$H = \mu(\mathrm{K}_{\mu}) \setminus [\mathrm{V}_{\nu}, \mathrm{A}_{\nu}, \ell_{\nu}] \sqcup (G^{\uparrow}_{\nu} \sqcap \mu(\mathrm{L}_{\mu})) \sqcup (G^{\uparrow}_{\mu} \sqcap \mu(\mathrm{L}_{\mu}))$$

and similarly (using Lemma 6.4) that

$$\begin{aligned}H' &= \mu(\mathrm{K}_{\mu}) \setminus [\mathrm{V}_{\nu}, \mathrm{A}_{\nu}, \ell_{\nu}] \sqcup (G^{\uparrow}_{\nu} \setminus [\mathrm{V}_{\mu}, \mathrm{A}_{\mu}, \ell_{\mu}] \sqcap \mu(\mathrm{L}_{\mu})) \sqcup (\mu'{\uparrow}(\mathrm{R}_{\mu}) \sqcap \mu(\mathrm{L}_{\mu})) \\ &= \mu(\mathrm{K}_{\mu}) \setminus [\mathrm{V}_{\nu}, \mathrm{A}_{\nu}, \ell_{\nu}] \sqcup (G^{\uparrow}_{\nu} \sqcap \mu(\mathrm{K}_{\mu})) \sqcup (\mu'{\uparrow}(\mathrm{R}_{\mu}) \sqcap \mu(\mathrm{L}_{\mu})).\end{aligned}$$

We therefore have $\dot{H} = \dot{H}'$. By Lemma 6.3 we have $\dot{G}^{\uparrow}_{\nu} \cap \mu(\dot{\mathrm{L}}_{\mu}) = \nu(\dot{\mathrm{R}}_{\nu} \cap \dot{\mathrm{K}}_{\nu}) \cap \mu(\dot{\mathrm{L}}_{\mu})$ and $\mu'{\uparrow}(\dot{\mathrm{R}}_{\mu}) \cap \mu(\dot{\mathrm{L}}_{\mu}) = \dot{G}^{\uparrow}_{\mu} \cap \mu(\dot{\mathrm{L}}_{\mu}) = \mu(\dot{\mathrm{R}}_{\mu} \cap \dot{\mathrm{K}}_{\mu}) \subseteq \mu(\dot{\mathrm{K}}_{\mu})$. Hence $\dot{H}' \setminus \mu(\dot{\mathrm{K}}_{\mu}) = \varnothing$ and $\dot{H} \setminus \mu(\dot{\mathrm{K}}_{\mu}) = \dot{G}^{\uparrow}_{\nu} \cap \mu(\dot{\mathrm{L}}_{\mu}) \setminus \mu(\dot{\mathrm{K}}_{\mu})$. Thus $\dot{G}^{\uparrow}_{\nu} \cap \mu(\dot{\mathrm{L}}_{\mu}) \subseteq \mu(\dot{\mathrm{K}}_{\mu})$. Similarly, we get $\vec{G}^{\uparrow}_{\nu} \cap \mu(\vec{\mathrm{L}}_{\mu}) \subseteq \mu(\vec{\mathrm{K}}_{\mu})$.

For all $x \in \dot{H} \cup \vec{H}$ we have $\mathring{G}^{\uparrow}_{\mu}(x) = \mathring{\mu} \circ \mathring{\mathrm{R}}_{\mu} \circ \mu^{-1}(x) = \mathring{\mu}' \circ \mathring{\mathrm{R}}_{\mu} \circ \mu'^{-1}(x)$, hence obviously $\mathring{H}' \setminus (\mathring{\mu} \circ \mathring{\mathrm{K}}_{\mu} \circ \mu^{-1}(x) \cup \mathring{G}^{\uparrow}_{\mu}(x)) = \varnothing$ and $\mathring{H}' \setminus (\mathring{\mu} \circ \mathring{\mathrm{K}}_{\mu} \circ \mu^{-1}(x) \cup \mathring{G}^{\uparrow}_{\mu}(x)) = \mathring{G}^{\uparrow}_{\nu}(x) \cap \mathring{\mu} \circ \mathring{\mathrm{L}}_{\mu} \circ \mu^{-1}(x) \setminus (\mathring{\mu} \circ \mathring{\mathrm{K}}_{\mu} \circ \mu^{-1}(x) \cup \mathring{G}^{\uparrow}_{\mu}(x))$. Thus $\mathring{G}^{\uparrow}_{\nu}(x) \cap \mathring{\mu} \circ \mathring{\mathrm{L}}_{\mu} \circ \mu^{-1}(x) \subseteq \mathring{\mu} \circ \mathring{\mathrm{K}}_{\mu} \circ \mu^{-1}(x) \cup \mathring{G}^{\uparrow}_{\mu}(x)$.

We conclude that $G^{\uparrow}_{\nu} \sqcap \mu(\mathrm{L}_{\mu}) \lhd \mu(\mathrm{K}_{\mu}) \sqcup G^{\uparrow}_{\mu}$. $\square$