On Monographs, Monadic Many-Sorted Algebras and Graph Structures

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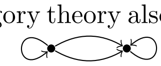
Abstract

A simple notion of *monograph* is proposed that generalizes the standard notion of graph and can be drawn consistently with them. It is shown that monadic many-sorted signatures can be represented by monographs, and that the corresponding algebras are isomorphic to the monographs typed by the corresponding signature monograph. Monographs therefore provide a simple unifying framework for working with monadic algebras. The simplicity of monographs is illustrated by deducing some of their categorial properties from those of sets.

1 Introduction

Many different notions of graphs are used in mathematics and computer science: simple graphs, oriented graphs, multigraphs, hypergraphs, etc. One favourite notion in the context of graph rewriting is that also known as *quivers*, i.e., structures of the form (N, E, s, t) where N, E are sets and s, t are functions from E (edges) to N (nodes), identifying the source and target tips of every edge (or arrow). We may identify two or three reasons for this: the need to represent data structures with pointers, the fact that the category of quivers is isomorphic to the category of Σ_{g}-algebras, where Σ_{g} is the signature with two sorts `nodes` and `edges` and two function symbols `src` and `tgt` of type `edges` $\rightarrow$ `nodes`, and possibly also the frequent use of category theory, since (small) categories are quivers endowed with a partial binary operation on E. In conformity with this tradition, by *graph* we mean quiver throughout this paper.

Category theory also provides a representation of graphs as functors from the category  (where the loops are identities) to the category **Sets**, so that the category **Graphs** is isomorphic to a functor category [3]. Such representations of graphs are useful for deducing in a simple way some properties of the category **Graphs**. This last isomorphism is actually derived from the previous one, by representing Σ_{g} as a category.

But to conveniently represent elaborate data structures it is often necessary to enrich the structure of graphs with other objects: nodes or edges may be labelled with elements from a fixed set, or with the elements of some algebra, or graphs may be typed by another graph (i.e., a graph comes with a morphism from itself to this other graph, considered as its type). An interesting example can be found in [3] with the notion of E-graphs, since some of these new objects are also considered as edges or nodes. More precisely, an E-graph is an algebra whose signature Σ_{e} can be represented by the following graph:

$$\begin{tikzcd}
& \mathtt{edges}_{\mathrm{g}} \arrow[rr, bend left, "\mathtt{src}_{\mathrm{g}}"] \arrow[rr, bend right, "\mathtt{tgt}_{\mathrm{g}}"'] & & \mathtt{nodes}_{\mathrm{g}} & \\
\mathtt{edges}_{\mathrm{e}} \arrow[ur, "\mathtt{src}_{\mathrm{e}}"] \arrow[drr, "\mathtt{tgt}_{\mathrm{e}}"] & & & & \mathtt{edges}_{\mathrm{n}} \arrow[ul, "\mathtt{src}_{\mathrm{n}}"'] \arrow[dll, "\mathtt{tgt}_{\mathrm{n}}"'] \\
& & \mathtt{nodes}_{\mathrm{v}} & &
\end{tikzcd}$$

The names given to the sorts and function symbols help to understand the structure of the Σ_{e}-algebras: the $\mathtt{edges}_{\mathrm{g}}$ relate the $\mathtt{nodes}_{\mathrm{g}}$ among themselves, the $\mathtt{edges}_{\mathrm{n}}$ relate the $\mathtt{nodes}_{\mathrm{g}}$ to the $\mathtt{nodes}_{\mathrm{v}}$, and the $\mathtt{edges}_{\mathrm{e}}$ relate the $\mathtt{edges}_{\mathrm{g}}$ to the $\mathtt{nodes}_{\mathrm{v}}$. These extra edges allow to attach as many values (elements of $\mathtt{nodes}_{\mathrm{v}}$) to edges and nodes of the inner graph. But then we see that in E-graphs some edges can be adjacent to other edges. This is non standard, but we may still accept such structures as some form of graph, if only because we understand how they can be drawn.

Hence the way of generalizing the notion of graphs seems to involve a generalization of the signature of graphs considered as algebras. This path has been followed by Michael Löwe in [5], where a *graph structure* is defined as a monadic many-sorted signature. Indeed in the examples above, and in many examples provided in [5], all function symbols have arity 1 and can therefore be considered as edges from their domain to their range sort. Is this the reason why they are called graph structures? But the example above shows that, if Σ_{e}-algebras are interpreted as graphs of some form, these are very different from the graph Σ_{e}. Besides, it is not convenient that our understanding of such structures should be based on syntax, i.e., on the particular names given to objects in the signature.

Furthermore, it is difficult to see how the algebras of some very simple monadic signatures can be interpreted as graphs of any form. Take for instance Σ_{g} and reverse the target function to $\mathtt{tgt} : \mathtt{nodes} \to \mathtt{edges}$. Then there is a symmetry between the sorts $\mathtt{nodes}$ and $\mathtt{edges}$, which means that in an algebra of this signature nodes and edges would be objects of the same nature. Is this still a graph? Can we draw it? Worse still, if the two sorts are collapsed into one, does it mean that an edge can be adjacent to itself at both ends?

We may address these problems by restricting graph structures to some class of monadic signatures whose algebras are guaranteed to behave in an orthodox way, say by exhibiting clearly separated edges and nodes. But this could be prone to arbitrariness, and it would still present another drawback: that the notion of graph structure does not easily give rise to a category. Indeed, it is difficult to define morphisms between algebras of different signatures, if only because they can have any number of carrier sets.

The approach adopted here is rather to reject any *structural* distinction between nodes and edges, to gather them all in a *single* carrier set and to rely on a *unique* function to distinguish them. For this reason, the resulting structures are called *monographs*. The definitions of monographs and their morphisms, given in Section 2, are thus quite simple although for reasons that will only be made clear in Section 4 we have to resort to ordinals.

Monographs are not algebras, they may not contain nodes, they may contain self-loops, but they can be drawn as proposed in Section 3, provided of course that they are finite (in a strong sense). In particular, such drawings will correspond to the standard way of drawing a graph, for those monographs that can be identified with graphs.

The relationship between monographs and graph structures (monadic signatures) is explored in Section 4. This gives rise to an isomorphism-dense embedding of monographs into many-sorted monadic signatures, that will expose a fundamental difference between them. This result is used in Section 5 to exhibit isomorphisms between the categories of (partitioned) algebras of all graph structures and all slice categories of monographs, i.e., the categories of typed monographs. In this sense monographs provide a complete representation of graph structures.

As a result of their simplicity the category of monographs and some of its subcategories can easily be shown to share a number of properties with **Graphs**, as illustrated in Section 6.

2 Definitions and Notations

For any sets A, B, any relation $r \subseteq A \times B$ and any subset $X \subseteq A$, we write $r[X]$ for the set $\{y \in B \mid x \in X \wedge (x,y) \in r\}$. For any $x \in A$, by abuse of notation we write $r[x]$ for $r[\{x\}]$. If r is functional we write $r(x)$ for the unique element of $r[x]$. We write r^{-1} for the relation $\{(y,x) \mid (x,y) \in r\} \subseteq B \times A$.

A *function* $f : A \to B$ is a morphism in the category **Sets**; it therefore consists of a domain A, a codomain B and a functional relation $r \subseteq A \times B$. The domain A must be the set $\{x \mid (x,y) \in r\}$, but the codomain B may be any superset of the image $r[A] = \{y \mid (x,y) \in r\}$. Functions may therefore be composed only if the domain of the left operand is equal to the codomain of the right operand. However, the standard notations for functions will also be used with functional relations. In particular, they may be composed whenever the domain of the left operand includes the image of the right operand. When a functional relation is composed with a function, it is always the underlying functional relation that is meant.

More generally, any object and its obvious underlying object will be written similarly, i.e., the forgetful functor will be omitted, whenever the ambiguity can easily be lifted from the context. Category theoretic concepts and notations will be consistent with [1], unless stated otherwise. In particular, id_A denotes the identity morphism of the object A in any category, except in **Sets** where it is denoted Id_A (the identity function of A) as a way of reminding the reader that A is a set. In **Sets** the standard product $\times$, projections π_1 and π_2 and coproduct $+$ are used. For functional relations f, g with the same domain A, let $(f,g)(x) \stackrel{\text{def}}{=} (f(x), g(x))$ for all $x \in A$; if $f : A \to B$ and $g : A \to C$ are functions then $(f,g) : A \to B \times C$ is the unique function such that $\pi_1 \circ (f,g) = f$ and $\pi_2 \circ (f,g) = g$.

Isomorphism between objects in a category, or between categories, is denoted by the symbol $\simeq$. For any two categories $\boldsymbol{A}$ and $\boldsymbol{B}$, a functor $F : \boldsymbol{A} \to \boldsymbol{B}$ is faithful (resp. full) if F is injective (resp. surjective) from the set of $\boldsymbol{A}$-morphisms from X to Y to the set of $\boldsymbol{B}$-morphisms from FX to FY, for all $\boldsymbol{A}$-objects X and Y. If F is faithful and injective on objects, then it is an embedding. F is isomorphism-dense if for every $\boldsymbol{B}$-object Y there exists an $\boldsymbol{A}$-object X such that $FX \simeq Y$. Categories $\boldsymbol{A}$ and $\boldsymbol{B}$ are equivalent, written $\boldsymbol{A} \approxeq \boldsymbol{B}$, if there is a full, faithful and isomorphism-dense functor from one to the other.

For any object T of $\boldsymbol{A}$, the *slice category* $\boldsymbol{A}/T$ has as objects the morphisms of codomain T of $\boldsymbol{A}$, as morphisms from object $f : A \to T$ to object $g : B \to T$ the morphisms $k : A \to B$ of $\boldsymbol{A}$ such that $g \circ k = f$, and the composition of morphisms in $\boldsymbol{A}/T$ is defined as the composition of the underlying morphisms in $\boldsymbol{A}$ [3]. It is easy to see that id_T is a terminal object of $\boldsymbol{A}/T$.

An ordinal is a set α such that every element of α is a subset of α, and such that the restriction of the membership relation $\in$ to α is a strict well-ordering of α (every non empty subset of α has a minimal element). Every member of an ordinal is an ordinal, and we write $\lambda < \alpha$ for $\lambda \in \alpha$. For any two ordinals α, β we have either $\alpha < \beta$, $\alpha = \beta$ or $\alpha > \beta$. Every ordinal α has a successor $\alpha \cup \{\alpha\}$, written $\alpha + 1$. For these properties and others we refer to [8]. Natural numbers n are identified with finite ordinals, so that $n = \{0, 1, \ldots, n-1\}$.

Definition 2.1 (sequences of ordinal length)**.** *For any set E and ordinal λ, an* E-sequence s of length λ *is an element of E^λ, i.e., a function $s : \lambda \to E$. For any $s \in E^\lambda$ and $\iota < \lambda$, the image of ι by s is written s_ι. If λ is finite and non zero then s can be described as $s = s_0 \cdots s_{\lambda-1}$. For any $x \in E$ we write $x \mid s$ and say that x* occurs in s *if there exists $\iota < \lambda$ such that $s_\iota = x$.*

For any ordinal α, let

$$E^{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\lambda<\alpha} E^\lambda.$$

For any set F and function $f : E \to F$, let $f^{<\alpha} : E^{<\alpha} \to F^{<\alpha}$ be the function defined by $f^{<\alpha}(s) \stackrel{\text{def}}{=} f \circ s$ for all $s \in E^{<\alpha}$.

Note that $E^0 = \{\varnothing\}$, i.e., $\varnothing$ is the only sequence of length 0, and that for any $s \in E^{<\alpha}$ and any ordinal $\beta \geqslant \alpha$, we have $s \in E^{<\beta}$ and $f^{<\beta}(s) = f^{<\alpha}(s)$. It is obvious that, if $f : E \to F$ and $g : F \to G$ then $(g \circ f)^{<\alpha} = g^{<\alpha} \circ f^{<\alpha}$. If s and s' are respectively E- and F-sequences both of length λ, then (s, s') is an $(E \times F)$-sequence of length λ, and then $\pi_1^{<\alpha}(s, s') = \pi_1 \circ (s, s') = s$ and similarly $\pi_2^{<\alpha}(s, s') = s'$ for all $\alpha > \lambda$. If $f : E \to F$ and $g : E \to G$ then $(f, g)^{<\alpha}(s) = (f, g) \circ s = (f \circ s, g \circ s) = (f^{<\alpha}(s), g^{<\alpha}(s))$ for all $s \in E^{<\alpha}$, hence $(f, g)^{<\alpha} = (f^{<\alpha}, g^{<\alpha})$.

Obviously $E^{<\omega}$ can be identified with the carrier set of the free monoid E^*, but in the sequel we have no use of any monoid structure.

Definition 2.2 (monographs)**.** *For any ordinal α, an* α-monograph *is a pair (E, a) where E is a set whose elements are called* edges *of A, and $a \subseteq E \times E^{<\alpha}$ is a functional relation, called the* map *of A. A pair $A = (E, a)$ is a* monograph *if it is an α-monograph for some ordinal α; we then say that α is an ordinal* for A. *The* grade *of A is the smallest ordinal for A. Monographs will usually be denoted by upper-case letters (A, B, ...), their functional relation by the corresponding lower-case letter (a, b, ...) and their set of edges E_a, E_b, ...*

The length $\ell_a(x)$ *of an edge $x \in \mathrm{E}_a$ is the length of $a(x)$, i.e., the unique ordinal λ such that $a(x) \in E^\lambda$. The* trace of A *is the set* $\mathrm{tr}(A) \stackrel{\text{def}}{=} \ell_a[\mathrm{E}_a]$. *For any set O of ordinals, an* O-monograph *A is a monograph such that* $\mathrm{tr}(A) \subseteq O$.

An edge x is adjacent to $y \in \mathrm{E}_a$ *if $y \mid a(x)$. A* self-loop *is an edge x that is adjacent only to x, i.e., such that $a(x)$ is a $\{x\}$-sequence. A* node *is an edge* *of length 0, and the set of nodes of A is written* N_a*. A is* standard *if $a(x)$ is a* N_a*-sequence for all $x \in \mathrm{E}_a$.*

A morphism *from monograph $A = (\mathrm{E}_a, a)$ to monograph $B = (\mathrm{E}_b, b)$ is a function $f : \mathrm{E}_a \to \mathrm{E}_b$ such that $f^{<\alpha} \circ a = b \circ f$, where α is any*[1] *ordinal for A.*

It is obvious that A is an α-monograph iff $\mathrm{tr}(A) \subseteq \alpha$, hence the notion of O-monograph generalizes that of α-monograph without ambiguity (even though ordinals are sets of ordinals). It is also easy to see that there exists an ordinal for any two monographs, and indeed for any set of monographs (e.g. the sum of their grades).

Note that a monograph A is essentially defined by its map a, since the set E_a is the domain of a. But a is only a set of pairs and not a function, there is no codomain to artificially separate monographs that have the same map. This means in particular that any α-monograph is a β-monograph for all $\beta \geqslant \alpha$. But A is not defined by its adjacency relation $y \mid a(x)$ on edges, since the sequences $a(x)$ may not be uniquely determined by this relation. Also note that the adjacency relation may not be symmetric: a node is never adjacent to any edge, while many edges may be adjacent to a node.

We easily see that the length of edges are preserved by morphisms: if f is a morphism from A to B then for all $x \in \mathrm{E}_a$, $\ell_b(f(x))$ is the length of the E_b-sequence $b \circ f(x) = f^{<\alpha} \circ a(x)$, which is the length of the E_a-sequence $a(x)$, i.e., $\ell_a(x) = \ell_b(f(x))$. Hence $\mathrm{tr}(A) \subseteq \mathrm{tr}(B)$, and the equality holds if f is surjective. This also means that the grade of B is at least that of A, hence that every ordinal for B is an ordinal for A. We also see that

$$f^{-1}[\mathrm{N}_b] = \{x \in \mathrm{E}_a \mid \ell_b(f(x)) = 0\} = \mathrm{N}_a$$

and hence, if $b \circ f(x)$ is a N_b-sequence then $a(x)$ is a N_a-sequence, so that A is standard whenever B is standard.

Given morphisms f from A to B and g from B to C, we see that $g \circ f$ is a morphism from A to C by letting α be an ordinal for B, so that

$$(g \circ f)^{<\alpha} \circ a = g^{<\alpha} \circ f^{<\alpha} \circ a = g^{<\alpha} \circ b \circ f = c \circ g \circ f.$$

Besides, the identity function $\mathrm{Id}_{\mathrm{E}_a}$ of E_a is obviously a morphism from A to A.

Definition 2.3. *Let* **MonoGr** *be the category of monographs and their morphisms. Let* **StdMonoGr** *be its full subcategory of standard monographs. For any set O of ordinals, let O-***MonoGr** *(resp. O-***StdMonoGr***) be the full subcategory of O-monographs (resp. standard O-monographs).*

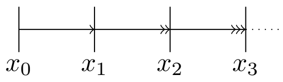
A monograph A is finite *if E_a is finite. Let* **FMonoGr** *be the full subcategory of finite ω-monographs.*

3 Drawing Monographs

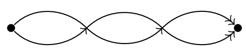
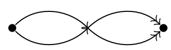
Obviously we may endeavour to draw a monograph A only if E_a is finite and if its edges have finite lengths, i.e., if A is a finite ω-monograph. We can easily identify any graph $G = (N, E, s, t)$ as the standard $\{0, 2\}$-monograph $(N + E, g)$ where $g(x) = \varnothing$ for all $x \in N$ and $g(e) = s(e)t(e)$ for all $e \in E$. If we require that

[1]Imposing the grade of A for α here would be a useless constraint.

such monographs should be drawn as their corresponding graphs, then a node should be represented by a bullet $\bullet$ and an edge of length 2 by an arrow joining its two adjacent nodes. But the adjacent edges may not be nodes and there might be more than 2 of them, hence we adopt the following convention: an edge e of length at least 2 is represented as a sequence of connected arrows with an increasing number of tips

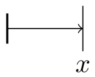


(where $a(e) = x_0x_1x_2x_3\cdots$) and such that any arrow should enter x_i at the same angle as the next arrow leaves x_i. This is important when x_i is a node since several adjacent edges may traverse the corresponding bullet, and they should not be confused. For the sake of clarity we will also represent symmetric adjacencies by a pair of crossings rather than a single one, e.g., if $a(e) = xe'y$ and $a(e') = xey$, where x and y are nodes, the drawing may be

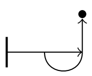
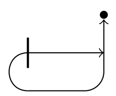
 but not 

As is the case of graphs, monographs may not be planar and drawing them may require crossing edges that are not adjacent; in this case no arrow tip is present at the intersection and no confusion is possible with the adjacency crossings. However, it may seem preferable in such cases to erase one arrow in the proximity of the other, as in ╳.

There remains to represent the edges of length 1. Since $a(e) = x$ is standardly written $a : e \mapsto x$, the edge e will be drawn as



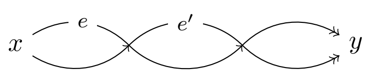
In order to avoid confusions there should be only one arrow out of the thick dash, e.g., if $a(e) = e'$ and $a(e') = ex$ where x is a node, the drawing may be

 but not 

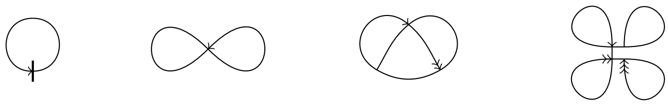
since this last drawing may be interpreted as the monograph $a(e) = x$ and $a(e') = ee$, that is not isomorphic to the intended monograph.

Possible drawings and names for the self-loops of length 1 to 4 are given in Figure 1. The Clover can easily be generalized to greater lengths.

It is sometimes necessary to name the edges in a drawing. We may then adopt the convention used for drawing diagrams in a category: the bullets are replaced by the names of the corresponding nodes, and arrows are interrupted to write their name at a place free from intersection, as in

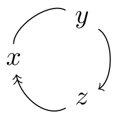


Note that no confusion is possible between the names of nodes and those of other edges, e.g., in



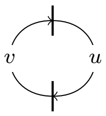
the Fixpoint the Snake[2] the Pretzel the Clover

Figure 1: The self-loops

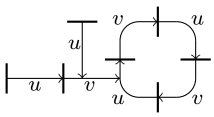


it is clear that x and z are nodes and y is an edge of length 3.

We may also draw typed monographs, i.e., monographs A equipped with a morphism f from A to a monograph T, considered as a type. Then every edge $e \in \mathrm{E}_a$ has a type $f(e)$ that can be written at the proximity of e. For instance, let T be the monograph



then a monograph typed by T is drawn with labels u and v as in



Of course, knowing that f is a morphism sometimes allows to deduce the type of an edge, possibly from the types of adjacent edges. In the present case, indicating a single type would have been enough to deduce all the others.

These figures have been produced with the TikZ package [9].

4 Monadic Signatures as Monographs

As mentioned in Section 1, graph structures, i.e., monadic many-sorted signatures, can be represented as graphs. More precisely, there is an obvious isomorphism between the category **Graphs** and the category of monadic signatures defined below[3].

Definition 4.1 (monadic signatures)**.** *A* (monadic) signature *is a function* $\Sigma : F \to S \times S$*; the elements of its domain* F*, that may be written* Σ_{f}*, are called*

[2]It is believed that the symbol ∞ represents the mythical snake Ananta Shesha (*ananta* is sanskrit for *endless*).

[3]For the sake of simplicity, we do not allow the overloading of function symbols as in [7], which would be irrelevant anyway since we wish to abstract the syntax away, hence to consider signatures only up to isomorphisms.

function symbols *and the elements of* S*, that may be written* Σ_{s}*, are called* sorts*.* Σ *is* finite *if both* F *and* S *are finite. Let* $\Sigma_{\mathrm{d}} \stackrel{\text{def}}{=} \pi_1 \circ \Sigma$ *and* $\Sigma_{\mathrm{r}} \stackrel{\text{def}}{=} \pi_2 \circ \Sigma$*, then* $\Sigma_{\mathrm{d}}(f)$ *and* $\Sigma_{\mathrm{r}}(f)$ *are respectively the* domain *and* range sorts *of* $f \in F$*.*

A morphism m *from signature* Σ *to signature* Σ' *is a pair* $m = (m_{\mathrm{f}}, m_{\mathrm{s}})$ *of functions, where* $m_{\mathrm{f}} : \Sigma_{\mathrm{f}} \to \Sigma'_{\mathrm{f}}$ *and* $m_{\mathrm{s}} : \Sigma_{\mathrm{s}} \to \Sigma'_{\mathrm{s}}$*, such that*

$$\Sigma' \circ m_{\mathrm{f}} = (m_{\mathrm{s}} \times m_{\mathrm{s}}) \circ \Sigma.$$

Let $\mathrm{id}_\Sigma \stackrel{\text{def}}{=} (\mathrm{Id}_{\Sigma_{\mathrm{f}}}, \mathrm{Id}_{\Sigma_{\mathrm{s}}})$ *and given two morphisms* $m : \Sigma \to \Sigma'$ *and* $n : \Sigma' \to \Sigma''$*, let* $n \circ m \stackrel{\text{def}}{=} (n_{\mathrm{f}} \circ m_{\mathrm{f}}, n_{\mathrm{s}} \circ m_{\mathrm{s}})$*; then* $\mathrm{id}_\Sigma : \Sigma \to \Sigma$ *and* $n \circ m : \Sigma \to \Sigma''$ *are morphisms. Let* **MonSig** *be the category of monadic signatures and their morphisms, and* **FMonSig** *its full subcategory of finite signatures.*

The obvious isomorphism from **MonSig** to **Graphs** maps every monadic signature $\Sigma : F \to S \times S$ to the graph $(S, F, \Sigma_{\mathrm{d}}, \Sigma_{\mathrm{r}})$. But we have seen in Section 1 on E-graphs that this representation of the monadic signature Σ_{e} bears no relation with the expected graphical representations of E-graphs. It would be more natural to represent Σ_{e} as an E-graph, and possibly any monadic signature Σ as a monograph.

Since the image $\Sigma(F)$ is a subset of $S \times S$, it can be viewed as a binary relation on S, hence there exists a monograph with S as set of edges whose adjacency relation is exactly $\Sigma(F)$. However, this monograph may not be unique since, as mentioned in Section 2, a monograph is not generally determined by its adjacency relation. Similarly, the orientation of edges in E-graphs is not determined by the signature Σ_{e}, it is only a convention given by the particular names of its function symbols.

For this reason it is more convenient to define a function from monographs to monadic signatures: any monograph determines a unique adjacency relation that can then be interpreted as a signature.

Definition 4.2 (functor S)**.** *To every monograph* $T = (\mathrm{E}_t, t)$ *we associate the set*

$$\mathrm{F}_t \stackrel{\text{def}}{=} \{(e, \iota) \mid e \in \mathrm{E}_t \wedge \iota < \ell_t(e)\}$$

of function symbols, and the signature $\mathsf{S}T : \mathrm{F}_t \to \mathrm{E}_t \times \mathrm{E}_t$ *defined by*

$$\mathsf{S}T(e, \iota) \stackrel{\text{def}}{=} (e, t(e)_\iota) \text{ for all } (e, \iota) \in \mathrm{F}_t.$$

To every morphism of monographs $f : T \to U$ *we associate the morphism* $\mathsf{S}f : \mathsf{S}T \to \mathsf{S}U$ *defined by*

- $(\mathsf{S}f)_{\mathrm{f}}(e, \iota) \stackrel{\text{def}}{=} (f(e), \iota) \in \mathrm{F}_u$ *for all* $(e, \iota) \in \mathrm{F}_t$*, and*
- $(\mathsf{S}f)_{\mathrm{s}} \stackrel{\text{def}}{=} f$ *(as a function from* E_t *to* E_u*).*

Note that the signature $\mathsf{S}T$ is finite iff T is a finite ω-monograph.

Lemma 4.3. S *is an embedding from* **MonoGr** *to* **MonSig**.

Proof. We first prove that S is a functor. For every monograph T we have $(\mathsf{S}\mathrm{id}_T)_{\mathrm{f}} = \mathrm{Id}_{\mathrm{F}_t}$ and $(\mathsf{S}\mathrm{id}_T)_{\mathrm{s}} = \mathrm{Id}_{\mathrm{E}_t}$, hence $\mathsf{S}\mathrm{id}_T = (\mathrm{Id}_{\mathrm{F}_t}, \mathrm{Id}_{\mathrm{E}_t}) = \mathrm{id}_{\mathsf{S}T}$. For every morphisms $f : T \to U$ in **MonoGr**, we first see that $\mathsf{S}f$ is a morphism in **MonSig** as claimed in Definition 4.1. Indeed, for all $(e, \iota) \in \mathrm{F}_t$ we have $\mathsf{S}U \circ (\mathsf{S}f)_{\mathrm{f}}(e, \iota) = \mathsf{S}U(f(e), \iota) = (f(e), (u \circ f(e))_\iota)$ and

$$\big((\mathsf{S}f)_{\mathrm{s}} \times (\mathsf{S}f)_{\mathrm{s}}\big) \circ \mathsf{S}T(e, \iota) = (f \times f) \circ (e, t(e)_\iota) = (f(e), f(t(e)_\iota)).$$

Let α be an ordinal for T and U, then $f(t(e)_\iota) = (f^{<\alpha} \circ t(e))_\iota = (u \circ f(e))_\iota$, hence $\mathsf{S}U \circ (\mathsf{S}f)_{\mathrm{f}} = \big((\mathsf{S}f)_{\mathrm{s}} \times (\mathsf{S}f)_{\mathrm{s}}\big) \circ \mathsf{S}T$, and therefore $\mathsf{S}f : \mathsf{S}T \to \mathsf{S}U$.

Then, for every morphism $g : U \to V$ in **MonoGr**, we have $(\mathsf{S}(g \circ f))_{\mathrm{s}} = g \circ f = (\mathsf{S}g)_{\mathrm{s}} \circ (\mathsf{S}f)_{\mathrm{s}}$ and for every $(e, \iota) \in \mathrm{F}_t$ we have

$$(\mathsf{S}(g \circ f))_{\mathrm{f}}(e, \iota) = (g \circ f(e), \iota) = (\mathsf{S}g)_{\mathrm{f}}(f(e), \iota) = (\mathsf{S}g)_{\mathrm{f}} \circ (\mathsf{S}f)_{\mathrm{f}}(e, \iota),$$

hence $\mathsf{S}(g \circ f) = \mathsf{S}g \circ \mathsf{S}f$, and S is therefore a functor from **MonoGr** to **MonSig**.

We now prove that S is injective on objects. Let T, U be monographs such that $\mathsf{S}T = \mathsf{S}U$, then $\mathrm{E}_t = \mathrm{E}_u$ and $\mathrm{F}_t = \mathrm{F}_u$, so that $\ell_t(e) = \ell_u(e)$ for all $e \in \mathrm{E}_t$. We also have $\mathsf{S}T(e, \iota) = \mathsf{S}U(e, \iota)$ for all $(e, \iota) \in \mathrm{F}_t$, hence $t(e)_\iota = u(e)_\iota$ for all $\iota < \ell_t(e)$, so that $t = u$ and therefore $T = U$.

Finally, S is faithful since for all $f, g : T \to U$ such that $\mathsf{S}f = \mathsf{S}g$ we have $f = (\mathsf{S}f)_{\mathrm{s}} = (\mathsf{S}g)_{\mathrm{s}} = g$. □

The next lemma uses the Axiom of Choice through its equivalent formulation known as the Numeration Theorem [8].

Lemma 4.4. S *is isomorphism-dense: for every monadic signature Σ there exists a monograph T such that $\mathsf{S}T \simeq \Sigma$.*

Proof. Let $\Sigma : F \to S \times S$ be a monadic signature and for every $s \in S$ let $O_s = \{f \in F \mid \Sigma_{\mathrm{d}}(f) = s\}$. By the Numeration Theorem there exists an ordinal λ_s equipollent to O_s, i.e., such that there exists a bijection $\sigma_s : \lambda_s \to O_s$. Let $t(s)$ be the S-sequence of length λ_s defined by $t(s)_\iota = \Sigma_{\mathrm{r}} \circ \sigma_s(\iota)$ for all $\iota \in \lambda_s$ (so that $\ell_t(s) = \lambda_s$), and let T be the monograph (S, t).

Let $m_{\mathrm{s}} = \mathrm{Id}_S$, let m_{f} be the function that to every $(s, \iota) \in \mathrm{F}_t$ maps $\sigma_s(\iota) \in F$, and let $m = (m_{\mathrm{f}}, m_{\mathrm{s}})$. We first see that m is a morphism from $\mathsf{S}T$ to Σ since for all $(s, \iota) \in (\mathsf{S}T)_{\mathrm{f}} = \mathrm{F}_t$ we have

$$(m_{\mathrm{s}} \times m_{\mathrm{s}}) \circ \mathsf{S}T(s, \iota) = (\mathrm{Id}_S \times \mathrm{Id}_S)(s, t(s)_\iota) = (s, \Sigma_{\mathrm{r}} \circ \sigma_s(\iota)),$$

but $\sigma_s(\iota) \in O_s$ hence $\Sigma_{\mathrm{d}}(\sigma_s(\iota)) = s$, and therefore

$$(s, \Sigma_{\mathrm{r}} \circ \sigma_s(\iota)) = (\Sigma_{\mathrm{d}} \circ \sigma_s(\iota), \Sigma_{\mathrm{r}} \circ \sigma_s(\iota)) = \Sigma \circ \sigma_s(\iota) = \Sigma \circ m_{\mathrm{f}}(s, \iota).$$

We now prove that m is an isomorphism, i.e., that m_{s} and m_{f} are bijective. For any $(s, \iota), (s', \kappa) \in \mathrm{F}_t$ such that $m_{\mathrm{f}}(s, \iota) = m_{\mathrm{f}}(s', \kappa)$, then $\sigma_s(\iota) = \sigma_{s'}(\kappa)$ hence $s = \Sigma_{\mathrm{d}} \circ \sigma_s(\iota) = \Sigma_{\mathrm{d}} \circ \sigma_{s'}(\kappa) = s'$ and therefore $\iota = \kappa$ since σ_s is injective. For any $f \in \Sigma$, let $s = \Sigma_{\mathrm{d}}(f)$, so that $f \in O_s$, and let $\iota = \sigma_s^{-1}(f)$, then $(s, \iota) \in \mathrm{F}_t$ (since $\iota < \lambda_s = \ell_t(s)$) and $m_{\mathrm{f}}(s, \iota) = \sigma_s(\iota) = f$. Hence m_{f} is bijective, and so is m_{s}, which yields $\mathsf{S}T \simeq \Sigma$. □

The reason why monographs require edges of ordinal length now becomes apparent: the length of an edge s is the cardinality of O_s, i.e., the number of function symbols whose domain sort is s, and no restriction on this cardinality is ascribed to signatures. In finite signatures this cardinal is obviously finite, which trivially yields the following consequence.

Corollary 4.5. S *is an isomorphism-dense embedding from* **FMonoGr** *to* **FMonSig**.

We now show on an example that the functor S is not full, hence is not an equivalence between the categories **MonoGr** and **MonSig**.

Example 4.6. *The monadic signature* Σ_{g} *has two function symbols* src*,* tgt*, two sorts in* $\mathrm{S}_{\mathrm{g}} = \{\mathit{nodes}, \mathit{edges}\}$ *and is defined by:*

$$\Sigma_{\mathrm{g}} : \mathit{src}, \mathit{tgt} \mapsto (\mathit{edges}, \mathit{nodes}).$$

Then $O_{\mathit{nodes}} = \varnothing$ *and* $O_{\mathit{edges}} = \{\mathit{src}, \mathit{tgt}\}$ *has 2 elements. Let* $\sigma : 2 \to O_{\mathit{edges}}$ *be the bijection defined by*

$$\sigma : 0 \mapsto \mathit{src}, 1 \mapsto \mathit{tgt}$$

and t *be the map defined by*

$$t(\mathit{nodes}) = \varnothing,\ t(\mathit{edges}) = \mathit{nodes}\,\mathit{nodes}$$

then $\mathrm{T}_{\mathrm{g}} = (\mathrm{S}_{\mathrm{g}}, t)$ *is a monograph. The signature* $\mathsf{S}\mathrm{T}_{\mathrm{g}}$ *has the same sorts as* Σ_{g}*, two function symbols* $(\mathit{edges}, 0)$*,* $(\mathit{edges}, 1)$ *and is defined by*

$$\mathsf{S}\mathrm{T}_{\mathrm{g}} : (\mathit{edges}, 0), (\mathit{edges}, 1) \mapsto (\mathit{edges}, \mathit{nodes}).$$

Hence $\mathsf{S}\mathrm{T}_{\mathrm{g}}$ *is indeed isomorphic to* Σ_{g}*. However, the only automorphism of* T_{g} *is* $\mathrm{id}_{\mathrm{T}_{\mathrm{g}}}$*, while* Σ_{g} *has a non trivial automorphism* $m = ((\mathit{src}\ \mathit{tgt}), \mathrm{Id}_{\mathrm{S}_{\mathrm{g}}})$ *(in cycle notation), hence* S *is not surjective on morphisms.*

This automorphism reflects the fact that Σ_{g} *does not define an order between its function symbols* src *and* tgt*. The orientation of edges as arrows from* src *to* tgt *is only a matter of convention that is reflected in the choice of* σ *above. This contrasts with monographs, where the edges are inherently oriented by the ordinals in their length. In the translation from* **MonoGr** *to* **MonSig***, the orientations of edges are necessarily lost. Note however that in this example, since* src *and* tgt *have the same range sort, the other obvious choice for* σ *yields the same monograph* T_{g}*.*

We therefore see that in most cases there are many distinct, non isomorphic monographs that faithfully represent a single signature, depending on the chosen orientations of their edges. Monographs carry more information than signatures, but the additional information is precisely the kind of information that has to be provided by means of syntax when a monadic signature is intended as a graph structure. By observing the examples given in [5, Section 3.1], we see that this syntactic information mostly consists in an order on function symbols, given either by indices taken in $\mathbb{N}$ or by calling them “source” and “target”.

We also observe in Examples 3.1 to 3.6 a separation of sorts into domain and range sorts. It is easy to see that a monograph T is standard iff the signature $\mathsf{S}T$ is separated, i.e., no sort occurs both as a domain and a range sort. Thus the range sorts are the nodes of T and the domain sorts are edges of diverse lengths that relate nodes. Only Example 3.7, defining the notion of ALR-graph, is non standard and requires a more detailed examination.

Example 4.7. *Let* Σ_{a} *be the monadic signature defined by the set of sorts* $\mathrm{S_a} = \{\mathtt{V}, \mathtt{E}, \mathtt{V\text{-}Ass}, \mathtt{E\text{-}Ass}, \mathtt{Graph}, \mathtt{Morphism}\}$ *and the following function symbols:*

$$\begin{array}{lrl}
\Sigma_{\mathrm{a}}: & \mathtt{s}, \mathtt{t} & \mapsto (\mathtt{E}, \mathtt{V}) \\
 & \mathtt{s}_{\mathtt{V}}, \mathtt{t}_{\mathtt{V}} & \mapsto (\mathtt{V\text{-}Ass}, \mathtt{V}) \\
 & \mathtt{s}_{\mathtt{E}}, \mathtt{t}_{\mathtt{E}} & \mapsto (\mathtt{E\text{-}Ass}, \mathtt{E}) \\
 & \mathtt{s}_{\mathtt{G}}, \mathtt{t}_{\mathtt{G}} & \mapsto (\mathtt{Morphism}, \mathtt{Graph}) \\
 & \mathtt{abstract}_{\mathtt{V}} & \mapsto (\mathtt{V}, \mathtt{Graph}) \\
 & \mathtt{abstract}_{\mathtt{E}} & \mapsto (\mathtt{E}, \mathtt{Graph}) \\
 & \mathtt{abstract}_{\mathtt{V\text{-}Ass}} & \mapsto (\mathtt{V\text{-}Ass}, \mathtt{Morphism}) \\
 & \mathtt{abstract}_{\mathtt{E\text{-}Ass}} & \mapsto (\mathtt{E\text{-}Ass}, \mathtt{Morphism})
\end{array}$$

An ALR-graph is a Σ_{a}*-algebra. It is not very clear how such structures can be considered as graphs, especially because there is no conventional way of ordering the "abstract" function symbols w.r.t. sources and targets. Textual explanations are provided in [5] to help the reader's understanding of ALR-graphs. The explanations given below on the corresponding monograph (where abstract function symbols are placed between sources and targets) are much simpler and almost superfluous. The set of edges is of course* $\mathrm{S_a}$*, and the map* $\mathrm{t_a}$ *is defined by:*

$\mathrm{t_a}(\mathtt{Graph}) = \varnothing$	*graphs are represented by nodes*
$\mathrm{t_a}(\mathtt{V}) = \mathtt{Graph}$	*to every vertex is associated a graph*
$\mathrm{t_a}(\mathtt{E}) = \mathtt{V}\,\mathtt{Graph}\,\mathtt{V}$	*an edge joins two vertices through a graph*
$\mathrm{t_a}(\mathtt{Morphism}) = \mathtt{Graph}\,\mathtt{Graph}$	*a morphism joins two graphs*
$\mathrm{t_a}(\mathtt{V\text{-}Ass}) = \mathtt{V}\,\mathtt{Morphism}\,\mathtt{V}$	*a vertex association joins two vertices through a morphism*
$\mathrm{t_a}(\mathtt{E\text{-}Ass}) = \mathtt{E}\,\mathtt{Morphism}\,\mathtt{E}$	*an edge association joins two edges through a morphism.*

We thus see that specifying a monadic signature by a monograph may yield a better understanding of the structure of the corresponding algebras, at least if these are meant as graph structures. The next section shows that this is always possible.

5 Monadic Algebras as Typed Monographs

Now that graph structures have been embedded in monographs, we may investigate the relation that the corresponding algebras bear with these monographs. We first need a definition of Σ-algebras and Σ-homomorphisms that, for the sake of simplicity, are restricted to monadic signatures. We will also use the standard reduct functors (see [10]) adapted to Definition 4.1.

Definition 5.1 (Σ-algebras and functor R_m)**.** *For any monadic signature* $\Sigma : F \to S \times S$*, a* Σ-algebra $\mathcal{A}$ *is a tuple* $((\mathcal{A}_s)_{s\in S}, (f^{\mathcal{A}})_{f\in F})$ *where* $(\mathcal{A}_s)_{s\in S}$ *is an* S*-indexed family of sets and* $f^{\mathcal{A}} : \mathcal{A}_{\Sigma_{\mathrm{d}}(f)} \to \mathcal{A}_{\Sigma_{\mathrm{r}}(f)}$ *is a function for all* $f \in F$*.* $\mathcal{A}$ *is* partitioned *if* $s \neq s' \Rightarrow \mathcal{A}_s \cap \mathcal{A}_{s'} = \varnothing$ *for all* $s, s' \in S$*.*

A Σ-homomorphism $h : \mathcal{A} \to \mathcal{B}$ *from a* Σ*-algebra* $\mathcal{A}$ *to a* Σ*-algebra* $\mathcal{B}$ *is an* S*-indexed family of functions* $(h_s)_{s\in S}$ *where* $h_s : \mathcal{A}_s \to \mathcal{B}_s$ *for all* $s \in S$*, such that*

$$f^{\mathcal{B}} \circ h_{\Sigma_{\mathrm{d}}(f)} = h_{\Sigma_{\mathrm{r}}(f)} \circ f^{\mathcal{A}}$$

*for all $f \in F$. Let $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ be the Σ-homomorphism $(\mathrm{Id}_{\mathcal{A}_s})_{s\in S}$, and for any Σ-homomorphism $h : \mathcal{A} \to \mathcal{B}$ and $k : \mathcal{B} \to \mathcal{C}$, let $k \circ h : \mathcal{A} \to \mathcal{C}$ be the Σ-homomorphism $(k_s \circ h_s)_{s\in S}$. Let Σ-**Alg** be the category of Σ-algebras with Σ-homomorphisms as their morphisms, and Σ-**PAlg** be its full subcategory of partitioned algebras.*

Given another signature $\Sigma' : F' \to S' \times S'$ and a morphism $m : \Sigma' \to \Sigma$ in **MonSig**, *the m-*reduct functor $\mathsf{R}_m : \Sigma\text{-}\mathbf{Alg} \to \Sigma'\text{-}\mathbf{Alg}$ *is defined by*

- $\mathsf{R}_m\mathcal{A} \overset{\text{def}}{=} ((\mathcal{A}_{m_{\mathrm{s}}(s')})_{s'\in S'}, (m_{\mathrm{f}}(f')^{\mathcal{A}})_{f'\in F'})$ *for every Σ-algebra $\mathcal{A}$, and*
- $\mathsf{R}_m h \overset{\text{def}}{=} (h_{m_{\mathrm{s}}(s')})_{s'\in S'}$ *for every Σ-homomorphism h.*

Following Example 4.6, we notice that $\mathrm{T_g}$ =  is the terminal graph. Besides, it is easy to see that the monographs A such that there exists a morphism from A to $\mathrm{T_g}$ are exactly the standard $\{0, 2\}$-monographs, that have been identified to graphs in Section 3. But it is obvious that for any standard $\{0, 2\}$-monograph A there is a unique morphism from A to $\mathrm{T_g}$ (the nodes of A are mapped to the node of $\mathrm{T_g}$ and its edges of length 2 are mapped to the loop of $\mathrm{T_g}$). In other words, graphs can be identified to the objects of the slice category $\mathbf{MonoGr}/\mathrm{T_g}$, i.e., the monographs typed by $\mathrm{T_g}$. But they can also be identified with the Σ_{g}-algebras, and hence to the ST_{g}-algebras according to the following obvious isomorphisms.

Lemma 5.2. *If $\Sigma \simeq \Sigma'$ then Σ-**Alg** $\simeq \Sigma'$-**Alg** and Σ-**PAlg** $\simeq \Sigma'$-**PAlg**.*

Proof. If there are inverse morphisms $m : \Sigma' \to \Sigma$ and $m' : \Sigma \to \Sigma'$ then there are inverse functors $\mathsf{R}_m : \Sigma\text{-}\mathbf{Alg} \to \Sigma'\text{-}\mathbf{Alg}$ and $\mathsf{R}_{m'} : \Sigma'\text{-}\mathbf{Alg} \to \Sigma\text{-}\mathbf{Alg}$ since $\mathsf{R}_{m'} \circ \mathsf{R}_m = \mathsf{R}_{m\circ m'} = \mathsf{R}_{\mathrm{id}_\Sigma}$ is the identity functor of Σ-**Alg**, and symmetrically $\mathsf{R}_m \circ \mathsf{R}_{m'}$ is the identity functor of Σ'-**Alg**. For every partitioned Σ-algebra $\mathcal{A}$ the Σ'-algebra $\mathsf{R}_m\mathcal{A}$ is partitioned since m_{s} is injective, hence the restrictions of R_m and $\mathsf{R}_{m'}$ to partitioned algebras are also inverse to each other. □

Note that Σ-**Alg** is not isomorphic to Σ-**PAlg** since many distinct algebras may be Σ-isomorphic to the same partitioned algebra. There is however a trivial equivalence between these categories.

Lemma 5.3. *For every signature Σ, Σ-**PAlg** $\approxeq \Sigma$-**Alg***

Proof. Given $\Sigma : F \to S \times S$ and a Σ-algebra $\mathcal{A}$, we consider the Σ-algebra $\mathcal{A}'$ defined by $\mathcal{A}'_s = \mathcal{A}_s \times \{s\}$ for all $s \in S$, and $f^{\mathcal{A}'}$ is the function from $\mathcal{A}'_{\Sigma_{\mathrm{d}}(f)}$ to $\mathcal{A}'_{\Sigma_{\mathrm{r}}(f)}$ that maps every $(x, \Sigma_{\mathrm{d}}(f))$ to $(f^{\mathcal{A}}(x), \Sigma_{\mathrm{r}}(f))$, for all $f \in F$. It is then obvious that $\mathcal{A}'$ is partitioned and that the projection on the first coordinate is an isomorphism from $\mathcal{A}'$ to $\mathcal{A}$. □

To summarize, we can say that the monographs typed by $\mathrm{T_g}$ can be identified with the ST_{g}-algebras. We are now going to generalize this fact to all monographs but, of course, we need to provide a precise meaning to this identification. We first establish an isomorphism of categories through the following functor.

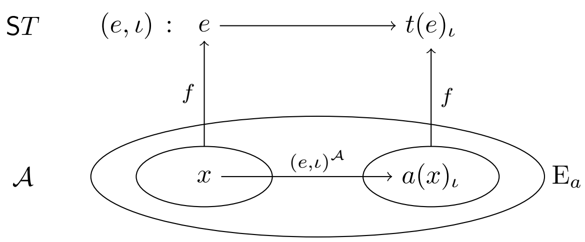
Definition 5.4 (functor A_T). *Given a monograph T, we define the function A_T that maps every object $f : A \to T$ of $\mathbf{MonoGr}/T$ to the partitioned $\mathsf{S}T$-algebra $\mathsf{A}_T f$ defined by*

- $(\mathsf{A}_T f)_e \stackrel{\text{def}}{=} f^{-1}[e]$ *for all* $e \in \mathrm{E}_t$*, and*
- $(e,\iota)^{\mathsf{A}_T f}(x) \stackrel{\text{def}}{=} a(x)_\iota$ *for all* $x \in f^{-1}[e]$ *and* $(e,\iota) \in \mathrm{F}_t$*.*

Besides, A_T *also maps every morphism* $k : f \to g$ *of* $\mathbf{MonoGr}/T$*, where* $f : A \to T$ *and* $g : B \to T$*, to the* $\mathsf{S}T$*-homomorphism* $\mathsf{A}_T k$ *from* $\mathsf{A}_T f$ *to* $\mathsf{A}_T g$ *defined by*

$$(\mathsf{A}_T k)_e \stackrel{\text{def}}{=} k|_{f^{-1}[e]} \text{ for all } e \in \mathrm{E}_t.$$

The $\mathsf{S}T$-algebra $\mathcal{A} = \mathsf{A}_T f$ can be pictured as follows.



Of course, there remains to prove that $\mathcal{A}$ is indeed an algebra, i.e., that $(e,\iota)^{\mathcal{A}}$ is a function from $f^{-1}[e]$ to $f^{-1}[t(e)_\iota]$. This is part of proving that A_T is an isomorphism from the slice category of monographs typed by T to the category of partitioned $\mathsf{S}T$-algebras.

Theorem 5.5. *For every monograph* T*,* $\mathbf{MonoGr}/T \simeq \mathsf{S}T$*-*$\mathbf{PAlg}$*.*

Proof. Let $\Sigma = \mathsf{S}T$, so that for all $(e,\iota) \in \mathrm{F}_t$ we have $\Sigma_{\mathrm{d}}(e,\iota) = e \in \mathrm{E}_t$ and $\Sigma_{\mathrm{r}}(e,\iota) = t(e)_\iota \in \mathrm{E}_t$ (see Definition 4.2).

We first prove that A_T maps objects of $\mathbf{MonoGr}/T$ to objects of Σ-$\mathbf{PAlg}$. For any $f : A \to T$, let α be an ordinal for A and $\mathcal{A} = \mathsf{A}_T f$. For every $(e,\iota) \in \mathrm{F}_t$ and every $x \in f^{-1}[e] = \mathcal{A}_{\Sigma_{\mathrm{d}}(e,\iota)}$, we have

$$f((e,\iota)^{\mathcal{A}}(x)) = f(a(x)_\iota) = (f^{<\alpha} \circ a(x))_\iota = (t \circ f(x))_\iota = t(e)_\iota = \Sigma_{\mathrm{r}}(e,\iota),$$

hence $(e,\iota)^{\mathcal{A}}$ is a function from $\mathcal{A}_{\Sigma_{\mathrm{d}}(e,\iota)}$ to $f^{-1}[\Sigma_{\mathrm{r}}(e,\iota)] = \mathcal{A}_{\Sigma_{\mathrm{r}}(e,\iota)}$, so that $\mathcal{A}$ is indeed a Σ-algebra. Besides, $\mathcal{A}$ is obviously partitioned.

We next prove that A_T maps morphisms of $\mathbf{MonoGr}/T$ to morphisms of Σ-$\mathbf{PAlg}$ with suitable domains and codomains. For any morphism $k : f \to g$ of $\mathbf{MonoGr}/T$, where $f : A \to T$ and $g : B \to T$, let $\mathcal{A} = \mathsf{A}_T f$, $\mathcal{B} = \mathsf{A}_T g$ and α an ordinal for A and B. By definition k is a morphism from A to B such that $g \circ k = f$, hence for all $e \in \mathrm{E}_t$ and $x \in \mathcal{A}_e = f^{-1}[e]$ we have

$$g((\mathsf{A}_T k)_e(x)) = g \circ k|_{f^{-1}[e]}(x) = f(x) = e,$$

hence $(\mathsf{A}_T k)_e$ is a function from $\mathcal{A}_e$ to $g^{-1}[e] = \mathcal{B}_e$. Then, for all $(e,\iota) \in \mathrm{F}_t$ and for all $x \in \mathcal{A}_e = \mathcal{A}_{\Sigma_{\mathrm{d}}(e,\iota)}$ we have

$$\begin{aligned}
(e,\iota)^{\mathcal{B}} \circ (\mathsf{A}_T k)_e(x) &= (e,\iota)^{\mathcal{B}} \circ k(x) \\
&= (b \circ k(x))_\iota \\
&= (k^{<\alpha} \circ a(x))_\iota \\
&= k(a(x)_\iota) \\
&= k \circ (e,\iota)^{\mathcal{A}}(x) \\
&= (\mathsf{A}_T k)_{\Sigma_{\mathrm{r}}(e,\iota)} \circ (e,\iota)^{\mathcal{A}}(x),
\end{aligned}$$

hence $\mathsf{A}_T k$ is a Σ-homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

We then prove that identities and morphism composition are preserved by A_T. For every object $f : A \to T$ of $\mathbf{MonoGr}/T$, its identity $\mathrm{id}_f : f \to f$ is $\mathrm{id}_A : A \to A$, hence for all $e \in \mathrm{E}_t$ we have $(\mathsf{A}_T \mathrm{id}_f)_e = \mathrm{id}_A|_{f^{-1}[e]} = \mathrm{Id}_{\mathcal{A}_e}$, where $\mathcal{A} = \mathsf{A}_T f$, hence $\mathsf{A}_T \mathrm{id}_f = \mathrm{id}_{\mathcal{A}}$.

For any morphisms $k : f \to g$ and $l : g \to h$ of $\mathbf{MonoGr}/T$, where $f : A \to T$, $g : B \to T$ and $h : C \to T$, we have for all $e \in \mathrm{E}_t$ that

$$(\mathsf{A}_T l)_e \circ (\mathsf{A}_T k)_e = l|_{g^{-1}[e]} \circ k|_{f^{-1}[e]} = (l \circ k)|_{f^{-1}[e]} = (\mathsf{A}_T(l \circ k))_e,$$

hence $\mathsf{A}_T(l \circ k) = \mathsf{A}_T l \circ \mathsf{A}_T k$. Thus A_T is indeed a functor from $\mathbf{MonoGr}/T$ to $\mathrm{S}T$-$\mathbf{PAlg}$, and we next see that it is an isomorphism.

A_T is injective on objects. Let $f : A \to T$, $g : B \to T$, $\mathcal{A} = \mathsf{A}_T f$ and $\mathcal{B} = \mathsf{A}_T g$ such that $\mathcal{A} = \mathcal{B}$, then $f^{-1}[e] = \mathcal{A}_e = \mathcal{B}_e = g^{-1}[e]$ for all $e \in \mathrm{E}_t$, hence $\mathrm{E}_a = \bigcup_{e \in \mathrm{E}_t} f^{-1}[e] = \bigcup_{e \in \mathrm{E}_t} g^{-1}[e] = \mathrm{E}_b$ and $f = g$ as functions from E_a to E_t. We also have, for all $(e, \iota) \in \mathrm{F}_t$ and all $x \in f^{-1}[e]$, that $a(x)_\iota = (e, \iota)^{\mathcal{A}}(x) = (e, \iota)^{\mathcal{B}}(x) = b(x)_\iota$. This is true for all $\iota \in \ell_t(e) = \ell_a(x) = \ell_b(x)$, hence $a(x) = b(x)$ for all $x \in \mathrm{E}_a$, hence $A = B$ and therefore $f = g$ as objects of $\mathbf{MonoGr}/T$.

A_T is surjective on objects. Let $\mathcal{A}$ be any partitioned Σ-algebra, α be an ordinal for T and $E = \bigcup_{e \in \mathrm{E}_t} \mathcal{A}_e$. Let $f : E \to \mathrm{E}_t$ be the function that to any $x \in E$ maps the unique $e \in \mathrm{E}_t$ such that $x \in \mathcal{A}_e$. For all $x \in E$, let $a(x)$ be the E-sequence of length $\ell_t(e)$, where $e = f(x)$, defined by, for all $\iota < \ell_t(e)$, $a(x)_\iota = (e, \iota)^{\mathcal{A}}(x) \in \mathcal{A}_{\Sigma_{\mathrm{r}}(e,\iota)} \subseteq E$. Since $\ell_t(e) < \alpha$ then $A = (E, a)$ is an α-monograph. For all $x \in E$ and all $\iota < \ell_a(x) = \ell_t(e)$ where $e = f(x)$, we have

$$(f^{<\alpha} \circ a(x))_\iota = f(a(x)_\iota) = f((e, \iota)^{\mathcal{A}}(x)) = \Sigma_{\mathrm{r}}(e, \iota) = t(e)_\iota = (t \circ f(x))_\iota,$$

hence $f^{<\alpha} \circ a = t \circ f$, which proves that $f : A \to T$ is an object of $\mathbf{MonoGr}/T$. Then for all $e \in \mathrm{E}_t$, we have $(\mathsf{A}_T f)_e = f^{-1}[e] = \mathcal{A}_e$ by definition of f. Furthermore, for all $(e, \iota) \in \mathrm{F}_t$ and all $x \in f^{-1}[e]$, we have $(e, \iota)^{\mathsf{A}_T f}(x) = a(x)_\iota = (e, \iota)^{\mathcal{A}}(x)$ by definition of a, hence $\mathsf{A}_T f = \mathcal{A}$.

It is obvious that A_T is injective on morphisms, hence there only remains to prove that it is surjective on morphisms. Let $f : A \to T$, $g : B \to T$, $\mathcal{A} = \mathsf{A}_T f$, $\mathcal{B} = \mathsf{A}_T g$ and h be any Σ-homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Then for all $e \in \mathrm{E}_t$ there is a function $h_e : \mathcal{A}_e \to \mathcal{B}_e$ such that $(e, \iota)^{\mathcal{B}} \circ h_{\Sigma_{\mathrm{d}}(e,\iota)} = h_{\Sigma_{\mathrm{r}}(e,\iota)} \circ (e, \iota)^{\mathcal{A}}$ for all $(e, \iota) \in \mathrm{F}_t$. Let $k : \mathrm{E}_a \to \mathrm{E}_b$ be the function that to every $x \in \mathrm{E}_a$ maps $h_{f(x)}(x) \in \mathcal{B}_{f(x)} = g^{-1}[f(x)] \subseteq \mathrm{E}_b$, so that $g \circ k(x) = f(x)$. Then, for all $x \in \mathrm{E}_a$ and all $\iota < \ell_a(x)$, let $e = f(x)$, we have

$$\begin{aligned}
(k^{<\alpha} \circ a(x))_\iota &= k(a(x)_\iota) \\
&= h_{f(a(x)_\iota)}(a(x)_\iota) \\
&= h_{(f^{<\alpha} \circ a(x))_\iota}((e, \iota)^{\mathcal{A}}(x)) \\
&= h_{t(e)_\iota} \circ (e, \iota)^{\mathcal{A}}(x) \\
&= h_{\Sigma_{\mathrm{r}}(e,\iota)} \circ (e, \iota)^{\mathcal{A}}(x) \\
&= (e, \iota)^{\mathcal{B}} \circ h_{\Sigma_{\mathrm{d}}(e,\iota)}(x) \\
&= b(h_e(x))_\iota \\
&= (b \circ k(x))_\iota,
\end{aligned}$$

hence $k : A \to B$ is a morphism such that $g \circ k = f$ and therefore $k : f \to g$ is a morphism of $\mathbf{MonoGr}/T$. We finally see that $(\mathsf{A}_T k)_e = k|_{f^{-1}[e]} = h_e$ for all $e \in \mathrm{E}_t$, hence $\mathsf{A}_T k = h$. ☐

Corollary 5.6. *For every monadic signature Σ, there exists a monograph T such that $\mathbf{MonoGr}/T \simeq \Sigma\text{-}\mathbf{PAlg}$.*

Proof. By Lemma 4.4 there exists T such that $\Sigma \simeq \mathsf{S}T$, hence $\mathbf{MonoGr}/T \simeq \mathsf{S}T\text{-}\mathbf{PAlg} \simeq \Sigma\text{-}\mathbf{PAlg}$ by Lemma 5.2. ☐

We thus see that the categories of partitioned monadic algebras are isomorphic to the slice categories of monographs. By Corollary 4.5, the partitioned algebras of finite monadic signatures are isomorphic to the slice categories of monographs typed by finite ω-monographs. Note that in the case of graphs, the partitioned Σ_{g}-algebras correspond to those graphs whose sets of vertices and edges are disjoint. This is a common restriction for graphs but not for Σ-algebras. We can easily extend the previous result to the categories $\Sigma\text{-}\mathbf{Alg}$.

Corollary 5.7. *For every monograph T, $\mathbf{MonoGr}/T \approxeq \mathsf{S}T\text{-}\mathbf{Alg}$. For every monadic signature Σ, there exists a monograph T such that $\mathbf{MonoGr}/T \approxeq \Sigma\text{-}\mathbf{Alg}$.*

Proof. By Lemma 5.3. ☐

Signatures are sometimes called *types* (see, e.g., [2, Chapter 9]), which leads to the following reading of Corollary 5.7: *the categories of algebras of monadic types are equivalent to the categories of typed monographs.*

Example 5.8. *The signature Σ_{e} of E-graphs from [3] has six function symbols $\mathtt{src}_{\mathrm{g}}$, $\mathtt{tgt}_{\mathrm{g}}$, $\mathtt{src}_{\mathrm{n}}$, $\mathtt{tgt}_{\mathrm{n}}$, $\mathtt{src}_{\mathrm{e}}$, $\mathtt{tgt}_{\mathrm{e}}$ and five sorts in*

$$\mathrm{S}_{\mathrm{e}} = \{\mathtt{edges}_{\mathrm{g}}, \mathtt{edges}_{\mathrm{n}}, \mathtt{edges}_{\mathrm{e}}, \mathtt{nodes}_{\mathrm{g}}, \mathtt{nodes}_{\mathrm{v}}\},$$

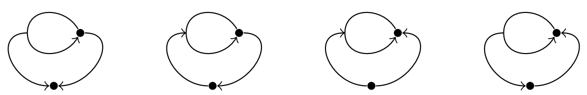
and is defined by

$$\begin{array}{rlll} \Sigma_{\mathrm{e}} : & \mathtt{src}_{\mathrm{g}}, \mathtt{tgt}_{\mathrm{g}} & \mapsto & (\mathtt{edges}_{\mathrm{g}}, \mathtt{nodes}_{\mathrm{g}}) \\ & \mathtt{src}_{\mathrm{n}} & \mapsto & (\mathtt{edges}_{\mathrm{n}}, \mathtt{nodes}_{\mathrm{g}}) \\ & \mathtt{tgt}_{\mathrm{n}} & \mapsto & (\mathtt{edges}_{\mathrm{n}}, \mathtt{nodes}_{\mathrm{v}}) \\ & \mathtt{src}_{\mathrm{e}} & \mapsto & (\mathtt{edges}_{\mathrm{e}}, \mathtt{edges}_{\mathrm{g}}) \\ & \mathtt{tgt}_{\mathrm{e}} & \mapsto & (\mathtt{edges}_{\mathrm{e}}, \mathtt{nodes}_{\mathrm{v}}) \end{array}$$

hence $O_{\mathtt{edges}_{\mathrm{g}}} = \{\mathtt{src}_{\mathrm{g}}, \mathtt{tgt}_{\mathrm{g}}\}$, $O_{\mathtt{edges}_{\mathrm{n}}} = \{\mathtt{src}_{\mathrm{n}}, \mathtt{tgt}_{\mathrm{n}}\}$, $O_{\mathtt{edges}_{\mathrm{g}}} = \{\mathtt{src}_{\mathrm{e}}, \mathtt{tgt}_{\mathrm{e}}\}$ and $O_{\mathtt{nodes}_{\mathrm{g}}} = O_{\mathtt{nodes}_{\mathrm{v}}} = \varnothing$. There are four possible monographs $T = (\mathrm{S}_{\mathrm{e}}, t)$, given by $t(\mathtt{nodes}_{\mathrm{g}}) = t(\mathtt{nodes}_{\mathrm{v}}) = \varnothing$, $t(\mathtt{edges}_{\mathrm{g}}) = \mathtt{nodes}_{\mathrm{g}}\,\mathtt{nodes}_{\mathrm{g}}$ and

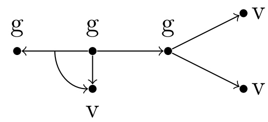
$$\begin{array}{l} t(\mathtt{edges}_{\mathrm{n}}) = \mathtt{nodes}_{\mathrm{g}}\,\mathtt{nodes}_{\mathrm{v}} \textit{ or } \mathtt{nodes}_{\mathrm{v}}\,\mathtt{nodes}_{\mathrm{g}} \\ t(\mathtt{edges}_{\mathrm{e}}) = \mathtt{edges}_{\mathrm{g}}\,\mathtt{nodes}_{\mathrm{v}} \textit{ or } \mathtt{nodes}_{\mathrm{v}}\,\mathtt{edges}_{\mathrm{g}}. \end{array}$$

These four monographs are depicted below.



T_1 T_2 T_3 T_4

Note that, by Theorem 5.5, the categories **MonoGr**$/T_i$ *for* $1 \leqslant i \leqslant 4$ *are isomorphic, even though the* T_i*'s are not. The type indicated by the syntax (and consistent with the figures in [3]) is* T_1*. An example of a monograph* A *typed by* T_1 *is*



where g *stands for* $\texttt{nodes}_{\mathrm{g}}$ *and* v *for* $\texttt{nodes}_{\mathrm{v}}$*. The types of the other edges can easily be deduced, yielding a unique typing morphism* $f : A \to T_1$*. We leave it to the reader to check that* A *is consistent with the drawing of the E-graph* $\mathsf{A}_{T_1} f$*.*

We also see from Example 4.6 that **MonoGr**$/\mathrm{T_g} \simeq \Sigma_{\mathrm{g}}$-**PAlg** $\approxeq$ **Graphs**, i.e., graphs are equivalent to monographs typed by $\mathrm{T_g}$.

6 Categorial Properties of Monographs

We next see that the category **MonoGr** share many properties with **Graphs**. Some properties are preserved in slice categories and through equivalence of categories, hence can be trivially transfered to categories of monadic algebras. More importantly, the proofs are easier to carry out on monographs then, say, on E-graphs since these have five carrier sets and six operations, see [3].

We first see that all the categories of Definition 2.3 have as initial object the empty monograph $(\varnothing, \varnothing)$. Other properties are listed below.

6.1 Pullbacks and monomorphisms

Lemma 6.1. *Let* B, C, D *be* α*-monographs and* $f : B \to D$, $g : C \to D$ *be morphisms, then there exists an* α*-monograph* A *and morphisms* $g' : A \to B$, $f' : A \to C$ *such that*

1. $\mathrm{tr}(A) \subseteq \mathrm{tr}(B) \cap \mathrm{tr}(C)$,
2. *if* D *is standard then so is* A,
3. *if* B *and* C *are finite then so is* A.
4. *the square*

$$\begin{tikzcd} A \arrow[r, "g'"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

is a pullback in **MonoGr**.

Proof. We use the standard construction of pullbacks in **Sets**: let

$$E = \bigcup_{e \in \mathrm{E}_d} f^{-1}[e] \times g^{-1}[e] \subseteq \mathrm{E}_b \times \mathrm{E}_c,$$

$g' = \pi_1|_E$ and $f' = \pi_2|_E$, then

$$
\begin{array}{ccc}
E & \xrightarrow{g'} & \mathrm{E}_b \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathrm{E}_c & \xrightarrow{g} & \mathrm{E}_d
\end{array}
$$

is a pullback in **Sets** [3].

For every $x \in E$, let $y = g'(x)$ and $z = f'(x)$ (so that $x = (y, z)$), then there exists an $e \in \mathrm{E}_d$ such that $f(y) = e = g(z)$, hence $\ell_b(y) = \ell_d(e) = \ell_c(z)$, i.e., $b \circ g'(x)$ and $c \circ f'(x)$ have the same length λ. Then, for all $\iota < \lambda$,

$$
\begin{aligned}
f(b \circ g'(x)_\iota) &= f^{<\alpha} \circ b \circ g'(x)_\iota \\
&= d \circ f \circ g'(x)_\iota \\
&= d \circ g \circ f'(x)_\iota \\
&= g^{<\alpha} \circ c \circ f'(x)_\iota \\
&= g(c \circ f'(x)_\iota)
\end{aligned}
$$

hence $(b \circ g'(x)_\iota, c \circ f'(x)_\iota) \in E$. Let $a(x) = (b \circ g'(x), c \circ f'(x))$ and $A = (E, a)$, then $a(x) \in E^{<\alpha}$ hence A is an α-monograph. It is obvious that $g'^{<\alpha} \circ a(x) = b \circ g'(x)$ and $f'^{<\alpha} \circ a(x) = c \circ f'(x)$, hence $g' : A \to B$ and $f' : A \to C$ are morphisms.

1. $\mathrm{tr}(A) \subseteq \mathrm{tr}(B)$ by virtue of morphism g', and $\mathrm{tr}(A) \subseteq \mathrm{tr}(C)$ by f'.

2. A is standard if D is standard by virtue of morphism $f \circ g'$.

3. If E_b and E_c are finite then so is $E \subseteq \mathrm{E}_b \times \mathrm{E}_c$.

4. Let A' be a monograph and $g'' : A' \to B$, $f'' : A' \to C$ be morphisms such that $f \circ g'' = g \circ f''$, then there exists a unique function h from $\mathrm{E}_{a'}$ to E such that $g'' = g' \circ h$ and $f'' = f' \circ h$. Then, for all $x \in \mathrm{E}_{a'}$,

$$
\begin{aligned}
a \circ h(x) &= (b \circ g' \circ h(x), c \circ f' \circ h(x)) \\
&= (b \circ g''(x), c \circ f''(x)) \\
&= (g''^{<\alpha} \circ a'(x), f''^{<\alpha} \circ a'(x)) \\
&= (g'^{<\alpha} \circ h^{<\alpha} \circ a'(x), f'^{<\alpha} \circ h^{<\alpha} \circ a'(x)) \\
&= h^{<\alpha} \circ a'(x)
\end{aligned}
$$

hence $h : A' \to A$ is a morphism in **MonoGr**, which proves that (A, g', f') is a pullback of (f, g, D).

□

Theorem 6.2. *The categories* **MonoGr**, **StdMonoGr**, **FMonoGr**, O-**MonoGr** *and* O-**StdMonoGr** *have pullbacks for every set* O *of ordinals.*

Proof. Trivial by Lemma 6.1 since, if $\mathrm{tr}(B) \subseteq O$ and $\mathrm{tr}(C) \subseteq O$ then $\mathrm{tr}(A) \subseteq \mathrm{tr}(B) \cap \mathrm{tr}(C) \subseteq O$. □

Corollary 6.3. *The monomorphisms in* **MonoGr** *are the injective morphisms.*

Proof. Assume $f : B \to D$ is a monomorphism and let $C = B$, $g = f$ and (A, f', g') be the pullback of (f, g, D) defined in the proof of Lemma 6.1, then $f \circ g' = f \circ f'$ hence $\pi_1|_{\mathrm{E}_a} = g' = f' = \pi_2|_{\mathrm{E}_a}$. For all $x, y \in \mathrm{E}_b$, if $f(x) = f(y)$ then $(x, y) \in \mathrm{E}_a$ and $x = g'(x, y) = f'(x, y) = y$, hence f is injective. The converse is obvious. □

6.2 Pushouts and epimorphisms

Lemma 6.4. *For any ordinal α, α-monographs A, B, C and morphisms $f : A \to B$ and $g : A \to C$, there exist an α-monograph D and morphisms $f' : C \to D$ and $g' : B \to D$ such that*

1. $\mathrm{tr}(D) = \mathrm{tr}(B) \cup \mathrm{tr}(C)$,
2. *if B and C are standard then so is D,*
3. *if B and C are finite then so is D,*
4. *the square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ C & \xrightarrow{f'} & D \end{array}$$

is a pushout in **MonoGr**.

Proof. We use the standard construction of pushouts in **Sets**: let $\sim$ be the smallest equivalence relation on the direct sum $\mathrm{E}_b + \mathrm{E}_c$ such that $f(x) \sim g(x)$ for all $x \in \mathrm{E}_a$, and f' (resp. g') be the canonical surjection from E_c (resp. E_b) to the quotient $E = (\mathrm{E}_b + \mathrm{E}_c)/\sim$, then

$$\begin{array}{ccc} \mathrm{E}_a & \xrightarrow{f} & \mathrm{E}_b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \mathrm{E}_c & \xrightarrow{f'} & E \end{array}$$

is a pushout in **Sets** [3].

For all $(y, z) \in \mathrm{E}_b \times \mathrm{E}_c$ such that $g'(y) = f'(z)$, i.e., the class of y modulo $\sim$ is the same as the class of z, hence $y \sim z$ and there exists a $n \in \mathbb{N}$ and a sequence $x_1, \ldots, x_{2n+1}$ of elements of E_a such that $y = f(x_1)$, $z = g(x_{2n+1})$ and

$$\begin{cases} g(x_{2i-1}) = g(x_{2i}) \\ f(x_{2i}) = f(x_{2i+1}) \end{cases}$$

for all $1 \leqslant i \leqslant n$. Since $b \circ f = f^{<\alpha} \circ a$ and $c \circ g = g^{<\alpha} \circ a$, this entails that

$$\begin{cases} g^{<\alpha} \circ a(x_{2i-1}) = g^{<\alpha} \circ a(x_{2i}) \\ f^{<\alpha} \circ a(x_{2i}) = f^{<\alpha} \circ a(x_{2i+1}). \end{cases}$$

The commuting property $g' \circ f = f' \circ g$ in **Sets** yields $g'^{<\alpha} \circ f^{<\alpha} = f'^{<\alpha} \circ g^{<\alpha}$, thus $f'^{<\alpha} \circ g^{<\alpha} \circ a(x_{2i-1}) = g'^{<\alpha} \circ f^{<\alpha} \circ a(x_{2i}) = f'^{<\alpha} \circ g^{<\alpha} \circ a(x_{2i+1})$ and hence $f'^{<\alpha} \circ g^{<\alpha} \circ a(x_1) = f'^{<\alpha} \circ g^{<\alpha} \circ a(x_{2n+1})$ by a trivial induction. We conclude that

$$\begin{aligned} g'^{<\alpha} \circ b(y) &= g'^{<\alpha} \circ f^{<\alpha} \circ a(x_1) \\ &= f'^{<\alpha} \circ g^{<\alpha} \circ a(x_1) \\ &= f'^{<\alpha} \circ g^{<\alpha} \circ a(x_{2n+1}) \\ &= f'^{<\alpha} \circ c(z). \end{aligned}$$

We can now build a functional relation $d \subseteq E \times E^{<\alpha}$ in the following way: every equivalence class $e \in E$ contains either an element $y \in \mathrm{E}_b$, and then $e = g'(y)$ and we let $d(e) = g'^{<\alpha} \circ b(y)$, or an element $z \in \mathrm{E}_c$, and then $e = f'(z)$ and we let $d(e) = f'^{<\alpha} \circ c(z)$; this relation is functional since $d(e)$ does not depend on the choice of y or z. Let $D = (E, d)$, then D is an α-monograph and $g' : B \to D$, $f' : C \to D$ are morphisms since $d \circ g' = g'^{<\alpha} \circ b$, $d \circ f' = f'^{<\alpha} \circ c$ by definition of d.

1. Since f' and g' are morphisms then $\mathrm{tr}(B) \subseteq \mathrm{tr}(D)$ and $\mathrm{tr}(C) \subseteq \mathrm{tr}(D)$. Conversely, for every $e \in E$ there is either a $y \in \mathrm{E}_b$ such that $e = g'(y)$, hence $\ell_d(e) = \ell_b(y) \in \mathrm{tr}(B)$, or there is a $z \in \mathrm{E}_c$ such that $e = f'(z)$, hence $\ell_d(e) = \ell_c(z) \in \mathrm{tr}(C)$. Hence $\mathrm{tr}(D) = \mathrm{tr}(B) \cup \mathrm{tr}(C)$.

2. For all $e \in D$, if $e = g'(y)$ for some $y \in \mathrm{E}_b$, then $b(y) \in \mathrm{N}_b^{<\alpha}$ since B is standard, hence $d(e) = g'^{<\alpha} \circ b(y) \in \mathrm{N}_d^{<\alpha}$ since $\mathrm{N}_d = g'(\mathrm{N}_b)$. Otherwise $e = f'(z)$ for some $z \in \mathrm{E}_c$ and we get the same result, hence D is standard.

3. If E_b and E_c are finite then E is finite.

4. Let D' be a monograph and $g'' : B \to D'$ and $f'' : C \to D'$ be morphisms such that $f \circ g'' = g \circ f''$. Since (g', f', E) is the pushout of (E_a, f, g) then there exists a unique function h from E to $\mathrm{E}_{d'}$ such that $g'' = h \circ g'$ and $f'' = h \circ f'$. For $e \in E$, if $e = g'(y)$ for some $y \in \mathrm{E}_b$ then

$$h^{<\alpha} \circ d(e) = h^{<\alpha} \circ g'^{<\alpha} \circ b(y) = g''^{<\alpha} \circ b(y) = d' \circ g''(y) = d' \circ h(e),$$

and similarly if $e = f'(z)$ for some $z \in \mathrm{E}_c$, hence $h^{<\alpha} \circ d = d' \circ h$, i.e., $h : D \to D'$ is a morphism in **MonoGr**, which proves that (g', f', D) is a pushout of (A, f, g).

□

Together with the existence of an initial object this implies that monographs have coproducts and that all finite diagrams have colimits.

Theorem 6.5. *The categories* **MonoGr***,* **StdMonoGr***,* **FMonoGr***, O-***MonoGr** *and O-***StdMonoGr** *are finitely co-complete for every set O of ordinals.*

Proof. Trivial by Lemma 6.4, as above, and by [1, Theorem 12.4]. □

Corollary 6.6. *The epimorphisms in* **MonoGr** *are the surjective morphisms.*

Proof. Assume $f : A \to B$ is an epimorphism and let $C = B$, $g = f$ and (f', g', D) be the pushout of (A, f, g) defined in the proof of Lemma 6.4, then for all $(y, z) \in \mathrm{E}_b \times \mathrm{E}_c$ such that $g'(y) = f'(z)$, there exists a $x_1 \in \mathrm{E}_a$ such that $y = f(x_1)$; this is true in particular if $z = y$. But $f' \circ f = g' \circ f$ hence $f' = g'$ and therefore $g'(y) = f'(y)$, thence the existence of x_1 for any y; this proves that f is surjective. The converse is obvious. □

6.3 Adhesivity

It is easy to see that the isomorphisms in **MonoGr** are exactly the bijective morphisms: if $f : A \to B$ and $g : B \to A$ are such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, then f is bijective since the underlying functions of id_A and id_B are $\mathrm{Id}_{\mathrm{E}_a}$ and $\mathrm{Id}_{\mathrm{E}_b}$.

It is well known (see [1]) that pushouts (and similarly pullbacks) are essentially unique in the sense that the pushouts of a given source (A, f, g) only differ by an isomorphism. Another general property of pushouts (see the notion of *epi-sink* in [1, 11.7]) can be expressed in the category **Sets** as follows: if (f', g', E) is a pushout of (A, f, g) then any $e \in E$ has either a preimage y by f' or a preimage z by g' (f' and g' are said to be *jointly surjective*, see [3, 2.17]).

Lemma 6.7. *Any square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "(1)"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

of α-monographs is a pushout (resp. pullback) in **MonoGr** *iff the underlying square*

$$\begin{tikzcd} \mathrm{E}_a \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "(2)"] & \mathrm{E}_b \arrow[d, "h"] \\ \mathrm{E}_c \arrow[r, "k"] & \mathrm{E}_d \end{tikzcd}$$

is a pushout (resp. pullback) in **Sets**.

Proof. Let (f', g', D') be the pushout of (A, f, g) constructed in the proof of Lemma 6.4, so that $(f', g', \mathrm{E}_{d'})$ is a pushout of (E_a, f, g) in **Sets**. If (1) is a pushout then there is an isomorphism $i : D \to D'$ such that $f' = i \circ k$ and $g' = i \circ h$, but i is bijective from E_d to $\mathrm{E}_{d'}$, hence is an isomorphism in **Sets**, hence (2) is a pushout.

Conversely, if (2) is a pushout then there is a bijection $j : \mathrm{E}_d \to \mathrm{E}_{d'}$ such that $f' = j \circ k$ and $g' = j \circ h$. Since $h : B \to D$ and $g' : B \to D'$ are morphisms in **MonoGr** then

$$j^{<\alpha} \circ d \circ h = j^{<\alpha} \circ h^{<\alpha} \circ b = g'^{<\alpha} \circ b = d' \circ g' = d' \circ j \circ h$$

and similarly $j^{<\alpha} \circ d \circ k = d' \circ j \circ k$. Since (2) is a pushout then h and k are jointly surjective, hence $j^{<\alpha} \circ d = d' \circ j$. Hence $j : D \to D'$ is an isomorphism in **MonoGr** and (1) is therefore a pushout.

The proof for pullbacks is similar. □

Definition 6.8. *A pushout square (A, B, C, D) is a* van Kampen square *if for any commutative cube*

$$\begin{tikzcd}
& A' \arrow[dl] \arrow[rr] \arrow[dd] & & B' \arrow[dl] \arrow[dd] \\
C' \arrow[rr] \arrow[dd] & & D' \arrow[dd] & \\
& A \arrow[dl] \arrow[rr] & & B \arrow[dl] \\
C \arrow[rr] & & D &
\end{tikzcd}$$

where the back faces (A', A, B', B) and (A', A, C', C) are pullbacks, it is the case that the top face (A', B', C', D') is a pushout iff the front faces (B', B, D', D) and (C', C, D', D) are both pullbacks.

A category has pushouts along monomorphisms *if all sources (A, f, g) such that f or g is a monomorphism have a pushout.*

A category is adhesive *[4] if it has pullbacks, pushouts along monomorphisms and all such pushouts are van Kampen squares.*

Theorem 6.9. *The categories* **MonoGr***,* **StdMonoGr***,* **FMonoGr***, O-***MonoGr** *and O-***StdMonoGr** *are adhesive for every set O of ordinals.*

Proof. In any of these categories a commutative cube built on a pushout along a monomorphism as bottom face and with pullbacks as back faces, has an underlying cube in **Sets** that has the same properties by Lemma 6.7 and Corollary 6.3. Since **Sets** is an adhesive category (see [4]) the underlying bottom face is a van Kampen square, hence such is the bottom face of the initial cube by Lemma 6.7. We conclude with Theorems 6.2 and 6.5. ☐

6.4 Pushout complements

Definition 6.10. *A* pushout complement *of morphisms $f : A \to B$ and $g' : B \to D$ is an object C and a pair of morphisms $f' : A \to C$ and $g : C \to D$ such that*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\
C \arrow[r, "f'"'] & D
\end{tikzcd}$$

is a pushout square.

This notion is central in the Double-Pushout approach to algebraic graph transformation [6, 3], where it is necessary to find a pushout complement to a graph (or an object in a category) in order to perform a rule-based transformation of this object. If a pushout complement exists in **MonoGr** then it is essentially unique since **MonoGr** is adhesive (see [4, Lemma 4.5]). But pushout complements may not exist, hence it is important to be able to test wether this is the case or not. In the category **Graphs** this test is known as the *gluing condition* (see [3, 3.9]). Before a similar test can be established for **MonoGr**, a remark on pushouts is necessary.

Since every pushout of a source (A, f, g) is isomorphic to the pushout built in Lemma 6.4, it is clear that any property of this particular construction that is stable by bijective morphisms is true of all pushouts (g', f', D) of (A, f, g).

The gluing condition provided in Theorem 6.11 below is divided in two parts. The first one, close to the condition on *identification points* in **Graphs**, ensures the existence of a pushout complement E_c in **Sets**. The second one, close to the condition on *dangling points*, ensures the existence of a suitable map c for E_c.

Theorem 6.11 (gluing condition)**.** *The morphisms $f : A \to B$ and $g' : B \to D$ have a pushout complement in* **MonoGr** *iff*

(1) *for all $y, y' \in \mathrm{E}_b$, if $g'(y) = g'(y')$ then $y = y'$ or $y \in f(\mathrm{E}_a)$, and*

(2) *for all $y \in \mathrm{E}_b$ and $e \in \mathrm{E}_d$, if $g'(y) \mid d(e)$ then $e \in g'(\mathrm{E}_b)$ or $y \in f(\mathrm{E}_a)$.*

Proof. Let α be an ordinal for D.

Only if part. Let $g : A \to C$ and $f' : C \to D$ be a pushout complement of f and g'. If D is the pushout constructed in the proof of Lemma 6.4, then property (1) is obvious since $g'(y) = g'(y')$ entails $y \sim y'$, and by the definition of $\sim$; it is then easy to see that it remains true if D is only isomorphic to this construction. Similarly we have that $g'(y) = f'(z)$ entails $y \sim z$ hence $y \in f(\mathrm{E}_a)$, for all $z \in \mathrm{E}_c$.

For all $e \in \mathrm{E}_d \backslash g'(\mathrm{E}_b)$, since (f', g', D) is jointly surjective then there exists a $z \in \mathrm{E}_c$ such that $f'(z) = e$, hence $d(e) = d \circ f'(z) = f'^{<\alpha} \circ c(z)$. Then, for all $y \in \mathrm{E}_b$, if $g'(y) \mid d(e)$ then there exists a $z' \in \mathrm{E}_c$ such that $z' \mid c(z)$ and $g'(y) = f'(z')$, and therefore $y \in f(\mathrm{E}_a)$, which proves property (2).

If part. We first build a monograph C: let $\mathrm{E}_c = \mathrm{E}_d \backslash g'(\mathrm{E}_b \backslash f(\mathrm{E}_a))$, and $c = d|_{\mathrm{E}_c}$. Suppose there exists an edge $e \in D$ such that $d(e)$ is not an E_c-sequence, then there exists $y \in \mathrm{E}_b \backslash f(\mathrm{E}_a)$ such that $g'(y) \mid d(e)$, hence by (2) we have $e \in g'(\mathrm{E}_b)$ and there exists $y' \in \mathrm{E}_b$ such that $e = g'(y')$, so that $d(e) = d \circ g'(y') = g'^{<\alpha} \circ b(y')$, hence there is a $y'' \mid b(y')$ such that $g'(y) = g'(y'')$ and by (1) we get $y = y''$, hence $y \mid b(y')$. If there were an $x \in \mathrm{E}_a$ such that $y' = f(x)$ then $b(y') = f^{<\alpha} \circ a(x)$ and y would belong to $f(\mathrm{E}_a)$. Hence $y' \notin f(\mathrm{E}_a)$ which proves that $e \in g'(\mathrm{E}_b \backslash f(\mathrm{E}_a))$, i.e., that $e \notin \mathrm{E}_c$. Thus $C = (\mathrm{E}_c, c)$ is a monograph.

Let f' be the canonical injection from E_c to E_d, it is obvious that f' is a morphism from C to D.

For all $x \in \mathrm{E}_a$, if $g' \circ f(x) \notin \mathrm{E}_c$ then there exists a $y \in \mathrm{E}_b \backslash f(\mathrm{E}_a)$ such that $g'(y) = g'(f(x))$, but by (1) we have either $y = f(x)$ or $y \in f(\mathrm{E}_a)$, and both are impossible. Hence $g' \circ f(\mathrm{E}_a) \subseteq \mathrm{E}_c$ and we let g be the function $g' \circ f$ with codomain E_c; it is obvious that g is a morphism as are g' and f, and that $f' \circ g = g' \circ f$.

There remains to prove that (g', f', D) is a pushout of (A, f, g). Let $g'' : B \to D'$ and $f'' : C \to D'$ be morphisms such that $f'' \circ g = g'' \circ f$. If there is a morphism $h : D \to D'$ such that $f'' = h \circ f$ and $g'' = h \circ g'$, then

- $h(e) = f''(e)$ for all $e \in \mathrm{E}_c$, and
- $h(g'(y)) = g''(y)$ for all $y \in \mathrm{E}_b$.

But if $g'(y) \notin \mathrm{E}_c$, i.e., if $y \in \mathrm{E}_b \backslash f(\mathrm{E}_a)$ then by (1) the value of y is determined by $g'(y)$, so that h is unique. We now see that such a function exists since, for all $e \in \mathrm{E}_c \cap g'(\mathrm{E}_b) = g' \circ f(\mathrm{E}_a)$, and all $x \in \mathrm{E}_a$ such that $e = g'(y)$ where $y = f(x)$, we have

$$f''(e) = f'' \circ f' \circ g(x) = f'' \circ g(x) = g'' \circ f(x) = g''(y).$$

We finally see that this function is a morphism. For all $e \in \mathrm{E}_d$, if $e \in \mathrm{E}_c$ then

$$d' \circ h(e) = d' \circ f''(e) = f''^{<\alpha} \circ c(e) = h^{<\alpha} \circ d(e),$$

otherwise there exists $y \in \mathrm{E}_b \backslash f(\mathrm{E}_a)$ such that $e = g'(y)$ and then

$$d' \circ h(e) = d' \circ g''(y) = g''^{<\alpha} \circ b(y) = h^{<\alpha} \circ g'^{<\alpha} \circ b(y) = h^{<\alpha} \circ d \circ g'(y) = h^{<\alpha} \circ d(e),$$

hence $d' \circ h = h^{<\alpha} \circ d$, so that $h : D \to D'$. □

Note that C is finite whenever D is finite. This proves that this gluing condition is also valid in **FMonoGr**, and it is obviously also the case in **StdMonoGr**, O-**MonoGr** and O-**StdMonoGr** for every set O of ordinals.

6.5 Terminal objects and products

The construction of products of monographs and the related question of the existence of terminal objects (since products can be formed as pullbacks on terminal objects) are major differences between **Graphs** and **MonoGr**. We now see that some categories of monographs do not have terminal objects.

Definition 6.12 (monographs M_α)**.** *For every ordinal $\alpha > 0$ let a_α be the functional relation that to every $\lambda < \alpha$ associates the unique $\{0\}$-sequence of length λ. Let $\mathrm{M}_\alpha \stackrel{\text{def}}{=} (\alpha, \mathrm{a}_\alpha)$.*

It is clear that M_α is a standard α-monograph, since a_α is a functional relation from α to $\alpha^{<\alpha}$, and $\mathrm{a}_\alpha(0) = \varnothing$, i.e., 0 is a node of M_α.

Lemma 6.13. *For all ordinals $\alpha > 0$, β and every β-monograph B, if there is a morphism $f : \mathrm{M}_\alpha \to B$ then $\alpha \leqslant \beta$.*

Proof. α is the grade of M_α, since for any $\lambda < \alpha$ there is an edge of length λ, that is $\ell_{\mathrm{a}_\alpha}(\lambda) = \lambda$, hence $\mathrm{a}_\alpha(\lambda) \notin \alpha^{<\lambda}$, and therefore M_α is not a λ-monograph. By the existence of f the grade α of M_α is less than the grade of B, hence $\alpha \leqslant \beta$. □

Theorem 6.14. MonoGr, **StdMonoGr** *and* **FMonoGr** *have no terminal object.*

Proof. Suppose that B is a terminal monograph, then there is an ordinal β such that B is a β-monograph, and there is a morphism from $\mathrm{M}_{\beta+1}$ to B. By Lemma 6.13 this implies that $\beta + 1 \leqslant \beta$, a contradiction. This still holds if B is standard since $\mathrm{M}_{\beta+1}$ is standard. And it also holds if B is a finite ω-monograph, since then β can be chosen finite, and then $\mathrm{M}_{\beta+1}$ is also a finite ω-monograph. □

Products of monographs are difficult to define for the simple reason that we are not generally able to combine edges of different lengths in a reversible way. It is however possible to generalize the method for building products of graphs to some pairs of monographs.

Definition 6.15. *Any two α-monographs A, B are said to be* $\times$-compatible *if $\ell_a \circ (a(x)) = \ell_b \circ (b(y))$ for all $(x, y) \in \mathrm{E}_a \times \mathrm{E}_b$ such that $\ell_a(x) = \ell_b(y)$. In this case let*

$$\mathrm{E}_{a \times b} \stackrel{\text{def}}{=} \bigcup_{\lambda < \alpha} \ell_a^{-1}[\lambda] \times \ell_b^{-1}[\lambda]$$

and $a \times b$ be the functional relation that to all $(x, y) \in \mathrm{E}_{a\times b}$ maps the $(\mathrm{E}_a \times \mathrm{E}_b)$*-sequence* $(a(x), b(y))$*. The* product *of A and B is*

$$A \times B \stackrel{\text{def}}{=} (\mathrm{E}_{a\times b},\ a \times b).$$

Note that, if $(x, y) \in \mathrm{E}_{a\times b}$ then $\ell_a(x) = \ell_b(y)$, hence $a(x)$ and $b(y)$ are functions with the same domain $\ell_a(x)$, hence $(a(x), b(y))$ is a function from this domain to the product of their codomains (see Section 2), in this case $\mathrm{E}_a \times \mathrm{E}_b$. Of course, the product $A \times B$ is an α-monograph if and only if $a \times b \subseteq \mathrm{E}_{a\times b} \times \mathrm{E}_{a\times b}^{<\alpha}$, hence iff the functions $(a(x), b(y))$ are $\mathrm{E}_{a\times b}$-sequences.

Lemma 6.16. *For any $\times$-compatible α-monographs A and B, $(A \times B, \pi_1, \pi_2)$ is a product in* **MonoGr**.

Proof. $A \times B$ is a monograph since, for all $(x, y) \in \mathrm{E}_{a\times b}$ and all $\iota < \ell_{a\times b}(x, y) = \ell_a(x) = \ell_b(y)$, we have $(a \times b)(x, y)_\iota = (a(x)_\iota, b(y)_\iota)$ and

$$\ell_a(a(x)_\iota) = \ell_a \circ (a(x))_\iota = \ell_b \circ (b(y))_\iota = \ell_b(b(y)_\iota),$$

hence $(a(x)_\iota, b(y)_\iota) \in \mathrm{E}_{a\times b}$ and $(a \times b)(x, y)$ is therefore an $\mathrm{E}_{a\times b}$-sequence.

We also see that $\pi_1 : A \times B \to A$ is a morphism since $\pi_1^{<\alpha} \circ (a \times b)(x, y) = a(x) = a \circ \pi_1(x, y)$, and similarly $\pi_2 : A \times B \to B$.

For any monograph C and morphisms $f : C \to A$ and $g : C \to B$, we have $\ell_a(f(z)) = \ell_c(z) = \ell_b(g(z))$ for all $z \in \mathrm{E}_c$, hence $h = (f, g)$ is a function from E_c to $\mathrm{E}_{a\times b}$. We also have

$$(a \times b) \circ h(z) = (a \circ f(z), b \circ g(z)) = (f^{<\alpha} \circ c(z), g^{<\alpha} \circ c(z)) = h^{<\alpha} \circ c(z),$$

hence $h : C \to A \times B$ is a morphism. It is obvious that h is the unique morphism such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. $\square$

Theorem 6.17. *The categories* **StdMonoGr**, O-**StdMonoGr** *and* $\{\alpha\}$-**MonoGr** *have products for every set of ordinals O and every ordinal α.*

Proof. By Lemma 6.16 since every pair A, B of standard monographs or $\{\alpha\}$-monographs is $\times$-compatible. Also, if $A \times B$ exists then obviously $\mathrm{tr}(A \times B) = \mathrm{tr}(A) \cap \mathrm{tr}(B)$, hence the product of O-monographs is an O-monograph. $\square$

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