HAL
open science

A free boundary problem arising in PDE optimization

Giuseppe Buttazzo, Edouard Oudet, Bozhidar Velichkov

To cite this version:

Giuseppe Buttazzo, Edouard Oudet, Bozhidar Velichkov. A free boundary problem arising in PDE optimization. Calculus of Variations and Partial Differential Equations, 2015, 54 (4), pp.3829-3856. 10.1007/s00526-015-0923-1 . hal-02009504

HAL Id: hal-02009504

https://hal.univ-grenoble-alpes.fr/hal-02009504

Submitted on 6 Feb 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A free boundary problem arising in PDE optimization

Giuseppe Buttazzo, Edouard Oudet, Bozhidar Velichkov

April 25, 2015

Abstract

A free boundary problem arising from the optimal reinforcement of a membrane or from the reduction of traffic congestion is considered; it is of the form

$$\sup_{\int_D \theta\, dx = m} \ \inf_{u \in H^1_0(D)} \int_D \Big(\frac{1+\theta}{2} |\nabla u|^2 - fu \Big)\, dx.$$

We prove the existence of an optimal reinforcement θ and that it has some higher integrability properties. We also provide some numerical computations for θ and u.

Keywords:shape optimization, membrane reinforcement, free boundary, obstacle problems.
2010 Mathematics Subject Classification: 49J45, 35R35, 49M05, 35J25

1 Introduction

In the present paper we consider a variational problem related to various models arising in several fields among which shape optimization, optimal transport and elasto-plasticity. Here below we discuss the statement of the problem in some of them, the original motivation for the paper being the first one, for which we refer to [7], [5] and references therein.

Optimal reinforcement of a membrane. Let $D \subset \mathbb{R}^d$ be a bounded open set with a smooth boundary and let $f \in L^2(D)$; D represents (when $d = 2$) a membrane, that we assume fixed at its boundary, and f represents a given load. The deformation u of the membrane is then obtained as the solution of the variational problem

$$\min \Big\{ \int_D \Big(\frac{1}{2} |\nabla u|^2 - fu \Big)\, dx \ : \ u \in H^1_0(D) \Big\},$$

where the elasticity coefficient in D is taken equal to 1, or equivalently of the PDE

$$-\Delta u = f \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad \partial D.$$

We now want to rigidify the membrane by adding a reinforcement: this is modeled by a density function $\theta \geq 0$ which corresponds to changing the elasticity coefficient of D from 1 to $1+\theta$. The total amount m of reinforcement is given and the ultimate goal is to increase as much as possible the rigidity of the membrane. This translates into the minimization of the elastic compliance, or equivalently into the maximization of the elastic energy, and we end up with the following optimization problem:

$$\sup_{\int_D \theta\, dx = m} \ \inf_{u \in H^1_0(D)} \int_D \Big(\frac{1+\theta}{2} |\nabla u|^2 - fu \Big)\, dx. \tag{1.1}$$

Reduction to a traffic congestion problem. Consider D as an urban region with given traffic sources f; if σ denotes the exiting traffic flux in D (or equivalently towards the boundary ∂D)

and if the cost of the traffic congestion in D is measured by the quantity $\int_D \frac{1}{2}|\sigma|^2\,dx$ (see for instance [10], [6]), we may obtain σ by solving the minimization problem

$$\min\Big\{\int_D \frac{1}{2}|\sigma|^2\,dx \ : \ -\operatorname{div}\sigma = f \text{ in } D\Big\}.$$

Suppose that there is some given total amount m of resources that can be spent in order to decrease the traffic congestion: we assume that investing the amount $\theta\,dx$ near a point x_0 produces a lower congestion, measured by the quantity $\frac{1}{2(1+\theta)}|\sigma|^2$. Therefore, minimizing the total congestion cost leads to the problem

$$\min\Big\{\int_D \frac{1}{2}\frac{|\sigma|^2}{1+\theta}\,dx \ : \ -\operatorname{div}\sigma = f \text{ in } D\Big\},$$

whose dual form is well-known to coincide with the optimization problem (1.1).

The variational problem we deal with has been considered in the literature under various forms, below we mention two of them.

The elastic-plastic torsion problem. Let $D \subset \mathbb{R}^2$ be the section of a thin rod which is being twisted and let f be a constant corresponding to the twist of the rod. The function $u \in H^1_0(D)$ is a potential related to the elastic energy of the twisted rod. The response of some materials in this situation is of elastic-plastic type which means that the gradient of the potential u cannot grow beyond a certain threshold κ, i.e. u solves the variational problem

$$\min\Big\{\frac{1}{2}\int_D |\nabla u|^2\,dx - f\int_D u\,dx \ : \ u \in H^1_0(D),\ |\nabla u| \le \kappa\Big\},$$

the *elastic region* being given by $\{|\nabla u| < \kappa\}$ while the *plastic* one by $\{|\nabla u| = \kappa\}$.

In [2] Brezis proved that the potential u of the elastic-plastic torsion problem is in fact the solution of the problem

$$-\operatorname{div}\big((1+\theta)\nabla u\big) = f \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

where θ is precisely the solution of the optimization problem (1.1) for some $m > 0$. In fact θ in this case can be interpreted as the response of the material to the deformation of the rod.

Several authors studied the regularity of the torsion function u and the free boundary $\partial\{|\nabla u| < \kappa\}$. The optimal $C^{1,1}$ regularity of u was proved by Wiegner [24] and Evans [14], while Caffarelli and Friedman provided a detailed study of the free boundary in [8].

Optimal transport problem In [1] Bouchitté and Buttazzo considered a problem similar to (1.1) without the underlying membrane, that is with $1+\theta$ replaced by θ, and made the connection of the optimization problem (1.1) with the optimal transport theory. Several authors (see for instance [13], [12], [19]) studied the summability properties of the function θ (called *transport density*) in relation to the summability of the function f (the positive and negative parts f^+ and f^- are usually called *marginals*). The usual setting considered in the literature is with D convex, Neumann conditions at ∂D, together with the zero average condition $\int_D f\,dx = 0$. In our case the Dirichlet boundary condition requires additional regularity of ∂D to get summability properties of θ. Moreover, the presence of the underlying membrane gives raise to a free boundary problem that we analyze in detail.

The paper is organized as follows. The precise mathematical setting of the optimization problem and our main results are described in Section 2, while the proof is carried out in Section 3 . Section 5 is devoted to the *elastic-plastic torsion problem*, in which $f = 1$. Section 6 deals with some numerical computations providing the solutions $\bar{u}$ and $\bar{\theta}$ in some interesting situations.

2 Setting of the problem and main result

Let $D \subset \mathbb{R}^d$ be a given bounded open set and $f \in L^2(D)$. For a given function $u \in H^1_0(D)$ and $\theta \in L^1_+(D) := \Big\{\theta \in L^1(D) \ : \ \theta \geq 0\Big\}$, the functional $J_f : H^1_0(D) \times L^1_+(D) \to \mathbb{R}$ is defined as

$$J_f(u,\theta) = \frac{1}{2}\int_D (1+\theta)|\nabla u|^2\,dx - \int_D uf\,dx,$$

and the energy $E_f(\theta)$ is given by

$$E_f(\theta) = \min\Big\{J_f(u,\theta) \ : \ u \in H^1_0(D)\Big\}. \tag{2.1}$$

The minimum in (2.1) is always achieved due to the fact that every minimizing sequence is bounded in $H^1_0(D)$ and so weakly compact in $H^1_0(D)$. The minimizer is also unique due to the strict convexity of the functional J_f with respect to the first variable. If we denote it by u_θ we have that $u_\theta \in H^1_0(D)$, $|\nabla u_\theta| \in L^2(\theta\,dx)$ and u_θ is the weak solution of

$$-\operatorname{div}\big((1+\theta)\nabla u_\theta\big) = f \quad \text{in} \quad D, \qquad u_\theta \in H^1_0(D), \tag{2.2}$$

i.e. for every $\varphi \in H^1_0(D)$ such that $|\nabla\varphi| \in L^2(\theta dx)$ we have

$$\int_D (1+\theta)\nabla u_\theta \cdot \nabla\varphi\,dx = \int_D f\varphi\,dx.$$

Testing the equation (2.2) with $\varphi = u_\theta$ and integrating by parts leads to

$$E_f(\theta) = J_f(u_\theta,\theta) = -\frac{1}{2}\int_D u_\theta f\,dx.$$

We consider the optimization problem

$$\max_{\theta\in\mathcal{A}_m} E_f(\theta), \tag{2.3}$$

where, for a given $m > 0$, the admissible set $\mathcal{A}_m$ is given by

$$\mathcal{A}_m = \Big\{\theta \in L^1_+(D) \ : \ \int_D \theta(x)\,dx \leq m\Big\}.$$

Due to the monotonicity of $E_f(\theta)$ with respect to θ the formulation of the problem with the alternative admissible set

$$\widetilde{\mathcal{A}}_m = \Big\{\theta \in L^1_+(D) \ : \ \int_D \theta(x)\,dx = m\Big\},$$

is equivalent to the one with $\mathcal{A}_m$.

Remark 2.1. It is interesting to notice that the optimization problem (2.3) is meaningful even if the load f is assumed to be a measure; in this case we have to allow θ to be a measure too, and for every nonnegative measure θ the functional $J_f(u,\theta)$ and the energy $E_f(\theta)$ are defined by

$$\begin{aligned} J_f(u,\theta) &= \frac{1}{2}\int_D (1+\theta)|\nabla u|^2\,dx - \int_D u\,df, \qquad \forall u \in C^\infty_c(D) \\ E_f(\theta) &= \inf\Big\{J_f(u,\theta) \ : \ u \in C^\infty_c(D)\Big\}. \end{aligned}$$

If f is a measure of course we may have $E_f(\theta) = -\infty$ for some measures θ; this happens for instance when f concentrates on sets of dimension smaller than $d-1$ and θ is the Lebesgue measure. However, these "singular" measures θ are ruled out from our analysis because of the maximization problem (2.3) we are dealing with.

Remark 2.2. Testing the energy $E_f(\theta)$ by the function $u = 0$ gives immediately the inequality

$$E_f(\theta) \le 0 \qquad \forall \theta \in \mathcal{A}_m.$$

A more careful analysis (see for instance [1]) shows the better inequality

$$E_f(\theta) \le -\frac{I^2(f)}{2(|D| + m)} \qquad \forall \theta \in \mathcal{A}_m$$

where

$$I(f) = \sup\Big\{ \int_D u\, df \ : \ u \in C_c^\infty(D),\ |\nabla u| \le 1 \Big\}.$$

On the other hand, if $f \in H^{-1}(D)$ the energy E_f remains always bounded from below.:

$$\begin{aligned} J_f(u,\theta) &\ge \frac{1}{2}\int_D (1+\theta)|\nabla u|^2\, dx - \|f\|_{H^{-1}(D)}\|u\|_{H_0^1(D)} \\ &\ge \frac{1}{2}\int_D |\nabla u|^2\, dx - \frac{1}{2}\big(\|f\|^2_{H^{-1}(D)} + \|u\|^2_{H_0^1(D)}\big) \ge -\frac{1}{2}\|f\|^2_{H^{-1}(D)}. \end{aligned}$$

Taking the minimum over $u \in H_0^1(D)$ we obtain

$$E_f(\theta) \ge -\frac{1}{2}\|f\|^2_{H^{-1}(D)} \qquad \forall \theta \in \mathcal{A}_m.$$

When $f \in L^\infty(D)$, setting $M = \|f\|_\infty$ and using the inequalities $E_f(\theta) \ge E_f(0) \ge E_M(0)$, which come from the maximum principle, we also have the inequality

$$E_f(\theta) \ge -\frac{\|f\|^2_{L^\infty}}{2}\int_D w_D\, dx,$$

where w_D is the solution of

$$-\Delta w_D = 1 \quad \text{in} \quad D, \qquad w_D \in H_0^1(D).$$

Our main result is the following.

Theorem 2.3. *Let $D \subset \mathbb{R}^d$ be a bounded open set satisfying the external ball condition and let $f \in L^\infty(D)$. Then there is a solution to the problem* (2.3)*. Moreover, any solution θ of* (2.3) *has the following properties:*

(i) (higher integrability) *$\theta \in L^p(D)$, for every $p \ge 1$;*

(ii) (min-max exchange) *the energy $E_f(\theta)$ satisfies*

$$E_f(\theta) = \min_{u \in H_0^1(D)} \ \frac{1}{2}\int_D |\nabla u|^2\, dx + \frac{m}{2}\|\nabla u\|^2_{L^\infty} - \int_D uf\, dx; \tag{2.4}$$

(iii) (regularity of the state function) *the solution u_θ of* (2.2) *is also the minimizer of the right-hand side of* (2.4)*. In particular, if $f \in C^2(D)$ then $u_\theta \in C^{1,1}(D)$;*

(iv) (optimality condition) *$\theta = 0$ a.e. on the set $\{|\nabla u_\theta| < \|\nabla u_\theta\|_{L^\infty}\}$.*

In the case when the set D is convex we have a slightly stronger result.

Theorem 2.4. *Let $D \subset \mathbb{R}^d$ be a bounded convex set, $r \in (d, +\infty]$ and $f \in L^r(D)$. Then all the conclusions of Theorem* 2.3 *hold. Moreover for the maximizer θ we have that $\theta \in L^r(D)$.*

Before we pass to the proof of Theorem 2.3 and Theorem 2.4 we give some preliminary remarks in order to clarify the assumptions on D and f as well as to outline the main steps of the proof.

The *main difficulty* in the existence result Theorem 2.3 is the lack of compactness in the space of admissible functions $\mathcal{A}_m$. Our approach consists in considering the optimization problem (2.3) on the admissible class

$$\mathcal{A}_{m,p} = \Big\{\theta \in L^p(D),\ \theta \geq 0,\ \Big(\int_D \theta^p\, dx\Big)^{1/p} \leq m\Big\}, \tag{2.5}$$

and then to pass to the limit as $p \to 1$. In fact, due to the weak compactness of the unit ball in L^p (for $p > 1$) we can easily obtain the existence of an optimal reinforcement θ_p for

$$\max\big\{E_f(\theta)\ :\ \theta \in \mathcal{A}_{m,p}\big\}.$$

We discuss this problem in the beginning of Section 3. The rest of the proof is dedicated to the passage to the limit as $p \to 1$. We notice that at this point we work with not just a maximising sequence of L^1 functions but with a family go maxima that obey some regularity properties we may use in order to get the desired convergence result.

Our goal is to find a *uniform estimate* for θ_p in some Lebesgue space $L^r(D)$ with $r > 1$. In order to do that we use an estimate from [12] in a slightly more general form. Roughly speaking, it is of the form

$$\int_D \theta_p^r\, dx \leq C_r \int_D |f(x)|^r\, dx - \int_{\partial D} F_r(|\nabla u_{\theta_p}|) H_{\partial D}\, d\mathcal{H}^{d-1},$$

where C_r is a constant depending on r, F_r is a polynomial with positive coefficients depending only on r and $H_{\partial D}$ is the mean curvature of ∂D. In the case when D is convex the boundary integral is positive, which immediately provides the uniform estimate which finally leads to the result of Theorem 2.4. On the other hand, if D is not convex then we should estimate the boundary term in a uniform way. At this point the bound from below of the mean curvature is natural to be imposed, but we have to estimate also the gradient of the solution on the boundary. We do this by constructing a barrier for u_{θ_p} at each point of the boundary. In order to do this we use the external ball condition, which is stronger than just the bound on the mean curvature, and the assumption $f \in L^\infty$ which allows us to describe the barrier as an explicit solution in a suitably chosen domain. The construction of the barrier and the uniform estimate is proved in Section 3.

In the case when f *is a constant* the proof of Theorem 2.3 can be obtained in a more direct way. In fact once the bound on the gradient is established on ∂D one can extend it in the interior of D by using the Payne-Philippin maximum principle [18]. In this case also the boundedness of the optimal θ is achieved. We notice that this result was already proved by Brezis in [2] where the optimal θ is given through an explicit formula.

Once we prove that the family θ_p is uniformly bounded in L^r, we take, as a natural candidate for a solution, the weak limit of θ_p as $p \to 1$. In order to prove that the limit is a minimizer of (2.3), we use the characterisation of the torsion function u_p as minima of functionals not depending only on f and p.

There is *an alternative approach* to problem (2.3). In fact, the existence of an optimal reinforcement can be obtained for very general D and f if we relax the problem to the space of finite Borel measures on the compact set $\overline{D}$. Moreover, for an optimal measure θ the min-max exchange equality (2.4) still holds. Now under the assumption that D is convex one can use the regularity result for u_θ proved in [4] and then writing the equation for θ as an optimal transport problem in which θ corresponds to the density of transport rays one can obtain the absolute continuity and the summability of θ using the results from [12] and [19]. We discuss this approach in the separate Section 4.

Remark 2.5. It is interesting to analyze the behavior of the solution u_m of (2.4) as $m \to +\infty$. Suppose that $f \geq 0$ and set $v_m = mu_m$. We have that v_m solves the minimization problem

$$\min_{v \in H_0^1(D)} \frac{1}{2m} \int_D |\nabla v|^2 \, dx + \frac{1}{2} \|\nabla v\|^2_{L^\infty(D)} - \int_D vf \, dx \, .$$

The Γ-limit, as $m \to +\infty$ of the cost functional in the last line is

$$\frac{1}{2} \|\nabla v\|^2_{L^\infty(D)} - \int_D vf \, dx \, ,$$

hence $v_m = mu_m$ converges in $L^2(D)$ to the solution $\bar{v}$ of the minimization problem

$$\min_{v \in H_0^1(D)} \frac{1}{2} \|\nabla v\|^2_{L^\infty(D)} - \int_D vf \, dx \, ,$$

which is given by $\bar{v}(x) = C_D d_{\partial D}(x)$, where $d_{\partial D}(x)$ is the Euclidean distance from $x \in D$ to the boundary ∂D and $C_D = \int_D d_{\partial D}(x) f(x) \, dx$.

3 Existence of optimal reinforcement

In this section we carry out the proofs of Theorem 2.3 and Theorem 2.4. In order to make the presentation as clear as possible we divide the main steps of the proof into different subsections.

3.1 Approximating problems: existence and regularity for $p > 1$

In this section we consider the problem of finding an optimal reinforcement in the admissible class of reinforcements $\mathcal{A}_{m,p}$ given by (2.5). Precisely, for every $p > 1$ and $f \in L^2(D)$, we consider the optimization problem

$$\max \big\{ E_f(\theta) \ : \ \theta \in \mathcal{A}_{m,p} \big\} = \max_{\theta \in \mathcal{A}_{m,p}} \ \min_{u \in H_0^1(D)} J_f(u, \theta). \tag{3.1}$$

Interchanging the sup and the inf in (3.1) we obtain the inequality

$$\sup_{\theta \in \mathcal{A}_{m,p}} \ \min_{u \in H_0^1(D)} J_f(u, \theta) \leq \inf_{u \in H_0^1(D)} \ \sup_{\theta \in \mathcal{A}_{m,p}} J_f(u, \theta). \tag{3.2}$$

We now notice that in the Hölder inequality $\int \theta \varphi \, dx \leq \|\theta\|_{L^p} \|\varphi\|_{L^q}$ the equality is achieved if and only if $\theta = \|\theta\|_{L^p} \|\varphi\|_{L^q}^{1-q} \varphi^{q-1}$. Thus, for a fixed $u \in H_0^1(\Omega)$ the supremum on the right-hand side of (3.2) with respect $\theta \in \mathcal{A}_{m,p}$ can be computed explicitly:

$$\sup_{\theta \in \mathcal{A}_{m,p}} J_f(u, \theta) = \frac{1}{2} \int_D |\nabla u|^2 \, dx + \frac{m}{2} \Big(\int_D |\nabla u|^{2q} \, dx \Big)^{1/q} - \int_D fu \, dx,$$

where $q = p/(p-1)$ is the dual exponent of $p > 1$. If moreover $u \in W_0^{1,2q}(D)$, then the supremum is in fact a maximum and is achieved for

$$\theta = m |\nabla u|^{2(q-1)} \Big(\int_D |\nabla u|^{2q} \, dx \Big)^{-(q-1)/q}.$$

Thus, for a given u, we can recover in a unique way the candidate for optimal θ in (3.1) and thus it is now sufficient to optimize in u. We make this argument precise in the following proposition.

Proposition 3.1. *For every $p > 1$ and $f \in L^2(D)$ the optimization problem* (3.1) *admits a unique solution θ_p, given by*

$$\theta_p = m|\nabla u_p|^{2(q-1)}\Big(\int_D |\nabla u_p|^{2q}\,dx\Big)^{-(q-1)/q}, \tag{3.3}$$

where $q = p/(p-1)$ is the dual exponent of p and u_p is the solution of the auxiliary problem

$$\min\Big\{\frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{m}{2}\Big(\int_D |\nabla u|^{2q}\,dx\Big)^{1/q} - \int_D fu\,dx \;:\; u \in H^1_0(D)\Big\}. \tag{3.4}$$

Moreover, if $D \subset \mathbb{R}^d$ is a bounded domain with $C^{1,\alpha}$ boundary and $f \in L^\infty(D)$, then $u_p \in C^{1,\beta}(\bar{D})$ for a constant β depending on α, p, d, Ω and $\|f\|_{L^\infty}$. In particular, θ_p is Hölder continuous up to the boundary.

Proof. As we already noticed, by exchanging the max and the min in the original problem (3.1) we get

$$\sup_{\theta\in\mathcal{A}_{m,p}} \min_{u\in H^1_0(D)} J_f(u,\theta) \le \inf_{u\in H^1_0(D)} \Big\{\frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{m}{2}\Big(\int_D |\nabla u|^{2q}\,dx\Big)^{1/q} - \int_D fu\,dx\Big\}. \tag{3.5}$$

By the direct methods of the calculus of variations we have that the infimum in the right-hand side of (3.5) is achieved, the minimizer u_p is unique and satisfies (weakly in $W^{1,2q}_0(D)$) the equation

$$-\Delta u_p - mC_p\Delta_{2q}u_p = f \quad \text{in } D, \qquad u_p \in W^{1,2q}_0(D),$$

where the constant C_p is given by $C_p = \|\nabla u_p\|^{-2(q-1)}_{L^{2q}}$ and, for $r > 1$, Δ_r is the r-Laplacian operator

$$\Delta_r u = \operatorname{div}\big(|\nabla u|^{r-2}\nabla u\big).$$

Defining θ_p as in (3.3), we get that u_p satisfies

$$-\operatorname{div}\big((1+\theta_p)\nabla u_p\big) = f \quad \text{in } D, \qquad u_p \in H^1_0(D),$$

in sense of distributions in D and so u_p is the unique minimizer of $J_f(\cdot,\theta_p)$ in $H^1_0(\Omega)$ and

$$E_f(\theta_p) = J_f(u_p,\theta_p) = \frac{1}{2}\int_D |\nabla u_p|^2\,dx + \frac{m}{2}\Big(\int_D |\nabla u_p|^{2q}\,dx\Big)^{1/q} - \int_D fu_p\,dx,$$

which together with (3.5) and the minimality of u_p gives

$$\sup_{\theta\in\mathcal{A}_{m,p}} E_f(\theta) \le E_f(\theta_p).$$

By the choice of θ_p we have $\int_D \theta_p^p\,dx = m^p$ and by the fact that u_p is the unique minimizer of (3.4), we get that θ_p is the unique solution of (3.1).

The internal $C^{1,\beta}$-regularity of u_p (and, as a consequence, the Hölder-regularity of θ_p) is classical and can be found in [16], while the up to the boundary version was proved in Theorem 1 of [17]. □

3.2 The limit, as $p \to 1$, of the state functions u_p

Let $D \subset \mathbb{R}^d$ be an open set of finite measure, $f \in L^2(D)$ be a given function, $m > 0$ and $p > 1$ two fixed numbers; we denote by $q = p/(p-1)$ the dual exponent of $p > 1$.

We define the functional $F_p : L^2(D) \to \mathbb{R}$ as

$$F_p(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_D |\nabla u|^2\,dx + \dfrac{m}{2}\Big(\int_D |\nabla u|^{2q}\,dx\Big)^{1/q} - \int_D fu\,dx, & \text{if } u \in W^{1,2q}_0(D),\\ +\infty, & \text{otherwise,}\end{cases} \tag{3.6}$$

while $F_1 : L^2(D) \to \mathbb{R}$ is given by

$$F_1(u) = \begin{cases} \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{m}{2}\|\nabla u\|^2_{L^\infty(D)} - \int_D fu\,dx, & \text{if } u \in W^{1,\infty}_0(D), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.7}$$

For $p \geq 1$ we denote by u_p the unique minimizer of F_p.

Our main objective in this subsection is to show that u_p converges to u_1 in some suitable functional space (a natural candidate being $L^2(D)$) and that the sequence of norms $\|\nabla u_p\|_{L^{2q}}$ does not degenerate as $p \to 1$, a fact that we use later in the uniform estimate of $\|\theta_p\|_{L^\infty}$. Before we continue with the result from this section we recall the notion of Γ-convergence introduced by De Giorgi:

Definition 3.2. *Let X be a given metric space. We say that a sequence of functionals $F_n : X \to \mathbb{R} \cup \{+\infty\}$ Γ-converges to the functional $F : X \to \mathbb{R} \cup \{+\infty\}$, if*

- ($\Gamma - \limsup$) *For every sequence $x_n \in X$ converging to $x \in X$ we have*
$$F(x) \leq \liminf F_n(x_n);$$
- ($\Gamma - \liminf$) *For every $x \in X$ there is a sequence $x_n \in X$ converging to x and such that*
$$F(x) \geq \limsup F_n(x_n).$$

Proposition 3.3. *Let $D \subset \mathbb{R}^d$ be an open set of finite measure and let $f \in L^2(D)$. With the notation introduced in* (3.6) *and* (3.7) *we have:*

(i) The sequence of functionals $(F_p)_{p>1}$ Γ-converges in $L^2(D)$ to the functional F_1.

(ii) The sequence of minima $(u_p)_{p>1}$ converges strongly in $H^1(D)$ to u_1.

(iii) $\lim_{p\to 1} \|\nabla u_p\|_{L^{2q}} = \|\nabla u_1\|_{L^\infty}$.

Proof. The proof is well-known, let us give it for the sake of completeness. Suppose that

$$v_p \to v \quad \text{in} \quad L^2(D) \qquad \text{and} \qquad \lim_{p\to 1} F_p(v_p) < +\infty.$$

Then there is a constant $C > 0$ such that for $p > 1$ small enough we have

$$\frac{1}{2}\|\nabla v_p\|^2_{L^2} + \frac{m}{2}\|\nabla v_p\|^2_{L^{2q}} - \int_D v_p f\,dx \leq C. \tag{3.8}$$

Thus, by the Gagliardo-Nirenberg-Sobolev inequality we have that v_p is uniformly bounded in $H^1_0(D)$ and so we can suppose that v_p converges weakly in $H^1_0(D)$ to $v \in H^1_0(D)$ (which gives the semicontinuity of the first term of (3.8)). We now notice that

$$\|\nabla v\|_{L^\infty(D)} = \sup\Big\{ \int_D \nabla v \cdot \nu\,dx \ : \ \nu \in C^\infty_c(D;\mathbb{R}^d),\ \int_D |\nu|\,dx \leq 1 \Big\}.$$

On the other hand for a fixed $\nu \in C^\infty_c(D;\mathbb{R}^d)$ with $\int_D |\nu|\,dx \leq 1$, we have

$$\begin{aligned} \int_D \nabla v \cdot \nu\,dx &= \lim_{p\to\infty} \int_D \nabla v_p \cdot \nu\,dx \\ &\leq \Big(\int_D |\nabla v_p|^{2q}\,dx\Big)^{1/(2q)} \Big(\int_D |\nu|^{2q/(2q-1)}\,dx\Big)^{(2q-1)/(2q)} \\ &\leq \Big(\liminf_{p\to 1} \|\nabla v_p\|_{L^{2q}}\Big)\Big(\lim_{p\to 1} \|\nu\|_{L^{2p/(p+1)}}\Big) \\ &\leq \|\nu\|_{L^1} \liminf_{p\to 1} \|\nabla v_p\|_{L^{2q}}, \end{aligned}$$

which proves the semicontinuity of the second term in (3.8).

In order or prove the $\Gamma-\limsup$ inequality consider a function $v \in W_0^{1,\infty}(D)$ and take v_p to be the constant family $v_p = v$, for every $p > 1$. Since in this case we have $\|\nabla v\|_{L^{2q}} \to \|\nabla v\|_{L^\infty}$ as $q \to +\infty$, we obtain

$$F(v) = \lim_{p\to 1} F_p(v),$$

which concludes the proof of *(i)*.

We now prove *(ii)*. Since u_p is the minimizer of F_p and $F_p(0) = 0$, we have that $F_p(u_p) \le 0$, which gives

$$\frac{1}{2}\|\nabla u_p\|^2_{L^2(D)} \le \int_D u_p f\,dx \le \|u_p\|_{L^2}\|f\|_{L^2} \le \lambda_1(D)^{-1/2}\|\nabla u_p\|_{L^2}\|f\|_{L^2},$$

which gives the uniform (in p) bound $\|\nabla u_p\|_{L^2} \le 2\|f\|_{L^2}\lambda_1(D)^{-1/2}$. Thus, there is a subsequence of u_p converging in L^2 to some $u \in L^2(D)$. Since F_p Γ-converges to F_1, u is necessarily the minimizer u_1 of F_1 (and since for every subsequence the limit is the same u_p converges to u_1 in L^2). Moreover, by the optimality of u_p and the $\Gamma-\liminf$ inequality we have

$$\limsup_{p\to 1} F_p(u_p) \le \liminf_{p\to 1} F_p(u_1) = F_1(u_1) \le \liminf_{p\to 1} F_p(u_p),$$

which together with the fact that $\int_D u_p f\,dx \to \int_D uf\,dx$ gives that

$$\|\nabla u_1\|^2_{L^2} + m\|\nabla u_1\|^2_{L^\infty} = \lim_{p\to 1}\Big\{\|\nabla u_p\|^2_{L^2} + m\|\nabla u_p\|^2_{L^{2q}}\Big\}.$$

Now since both the terms are semicontinuous with respect to the L^2 convergence of u_p to u_1 we get that

$$\|\nabla u_1\|^2_{L^2} = \lim_{p\to 1}\|\nabla u_p\|^2_{L^2} \qquad \text{and} \qquad \|\nabla u_1\|^2_{L^\infty} = \lim_{p\to 1}\|\nabla u_p\|^2_{L^{2q}}.$$

The first equality proves that $u_p \to u_1$ strongly in $H^1(D)$, while the second equality is precisely *(iii)*. □

3.3 Construction of a barrier for u_p

In order to obtain uniform estimates for u_p and θ_p we impose some mild geometric assumptions on the domain D. Precisely we assume that D *satisfies a uniform external ball condition that is there is some* $\rho > 0$ *such that for every point of the boundary* $x_0 \in \partial D$ *there is a ball* $B_\rho(y_0) \subset \mathbb{R}^d \setminus D$ *such that* $x_0 \in \partial B_\rho(y_0)$. This assumption allows us to construct a barrier for u_p and thus to give an estimate on the gradient $|\nabla u_p|$ on the boundary ∂D in terms of the diameter of D and the largest possible ρ. In the case when D is convex we have an estimate only in terms of the diameter $\mathrm{diam}(D)$. In this estimate we use a weak maximum (or comparison) principle which we state below in a general form.

Lemma 3.4 (Weak maximum principle)**.** *Let* $D \subset \mathbb{R}^d$ *be a bounded open set and let* $G : [0,+\infty) \to [0,+\infty)$ *be a convex function such that* $G'(0) > 0$. *For a function* $f \in L^2(D)$ *we denote by* $u_{\Omega,f}$ *the unique minimizer of the problem*

$$\min\Big\{\frac{1}{2}\int_D G\big(|\nabla u|^2\big)\,dx - \int_D uf\,dx \ :\ u \in H^1_0(D)\Big\}.$$

(a) If $f, g \in L^2(D)$ *are such that* $f \ge g$, *then* $u_{\Omega,f} \ge u_{\Omega,g}$.

(b) If $\omega \subset \Omega$ *is a bounded open set and* $f \ge 0$ *a.e. on* Ω, *then* $u_{\Omega,f} \ge u_{\omega,f}$.

We are now in a position to construct our barrier.

Lemma 3.5 (Pointwise boundary estimate)**.** *Let $D \subset \mathbb{R}^d$ be a bounded open set whose boundary ∂D is locally a graph of a $C^{1,\alpha}$ function for some $\alpha \in (0,1)$. Let $x_0 \in \partial D$ and let $B_\rho(y_0)$ be a ball in $\mathbb{R}^d \setminus D$ tangent to ∂D in x_0. Then, the minimizer u_p of F_p satisfies the gradient estimate*

$$\Big(1 + mC_p|\nabla u_p(x_0)|^{2(q-1)}\Big)|\nabla u_p(x_0)| \le \|f\|_{L^\infty(D)} \left(1 + \frac{\operatorname{diam}(D)}{\rho}\right)^{d-1} \operatorname{diam}(D). \tag{3.9}$$

Proof. Without loss of generality we assume $y_0 = 0$. Let R be the diameter of D. Then $D \subset B_{R+\rho} \setminus \bar{B}_\rho$. We set for simplicity

$$M = \|f\|_{L^\infty}\ , \qquad C = mC_p = m\Big(\int_D |\nabla u_p|^{2q}\,dx\Big)^{-1/p}$$

and we stress that M and C are fixed constants in the proof below. Let u be the solution of the equation

$$-\operatorname{div}\Big(\big(1 + C|\nabla u|^{2(q-1)}\big)\nabla u\Big) = M \quad \text{in } D, \qquad u \in W_0^{1,2q}(D),$$

and let U be the solution of

$$-\operatorname{div}\Big(\big(1 + C|\nabla U|^{2(q-1)}\big)\nabla U\Big) = M \quad \text{in } B_{R+\rho} \setminus \bar{B}_\rho, \qquad U \in W_0^{1,2q}(B_{R+\rho} \setminus \bar{B}_\rho). \tag{3.10}$$

By the maximum principle, applied to the operator associated to the function $G(t) = t + Ct^{q-1}$, we have $|u_p| \le u \le U$. On the other hand, the solution U of (3.10) is radially symmetric, i.e. a function of r that we still denote by U and that solves

$$-r^{1-d}\partial_r\Big(r^{d-1}\big(1 + C|\partial_r U|^{2(q-1)}\big)\partial_r U\Big) = M \quad \text{in }]\rho, \rho + R[, \qquad U(\rho) = U(\rho + R) = 0. \tag{3.11}$$

Let $\rho_1 \in]\rho, \rho + R[$ be a point where U achieves its maximum. Integrating (3.11) we have

$$\rho^{d-1}\big(1 + C|\partial_r U(\rho)|^{2(q-1)}\big)\partial_r U(\rho) = \int_\rho^{\rho_1} Mr^{d-1}\,dr \le MR(R+\rho)^{d-1},$$

which gives (3.9) since $\partial_r U(\rho) > 0$. □

Taking the largest possible ρ in the pointwise estimate (3.9) we obtain the following result.

Lemma 3.6 (Boundary estimate for a regular domain)**.** *Let $D \subset \mathbb{R}^d$ be a bounded open set whose boundary ∂D is locally a graph of a $C^{1,\alpha}$ function for some $\alpha \in (0,1)$.*

(i) If D is convex, then we have

$$\Big(1 + mC_p\|\nabla u_p\|^{2(q-1)}_{L^\infty(\partial D)}\Big)\|\nabla u_p\|_{L^\infty(\partial D)} \le \operatorname{diam}(D)\|f\|_{L^\infty(D)}.$$

(ii) If D satisfies the external ball condition with radius ρ, then

$$\Big(1 + mC_p\|\nabla u_p\|^{2(q-1)}_{L^\infty(\partial D)}\Big)\|\nabla u_p\|_{L^\infty(\partial D)} \le \|f\|_{L^\infty(D)} \operatorname{diam}(D) \left(1 + \frac{\operatorname{diam}(D)}{\rho}\right)^{d-1}.$$

3.4 Uniform estimate for θ_p

In this section we extend the boundary estimate for θ_p to the entire domain D. We need a uniform estimate on the norm $\|u_p\|_{L^\infty}$, which is classical and so we give only the precise statement here.

Lemma 3.7. *Let $f \in L^r(D)$ for some $r > d/2$. Then we have the estimate*

$$\|u_p\|_{L^\infty} \leq \frac{C_d}{d/2 - 1/r} \|f\|_{L^r} |D|^{d/2-1/r},$$

where C_d is a dimensional constant.

We now give our main a priori estimate for θ_p. The statement might appear as a generalization of an estimate by De Pascale-Evans-Pratelli, but the proof is precisely the one they gave in [12]. We reproduce the proof below for the sake of completeness and in order to show that the presence of a general functional G and the non-convexity of the set D do not influence the final result. In fact the only difference is that since the domain is not convex there is a boundary term that appears in the final estimate.

Lemma 3.8 (De Pascale-Evans-Pratelli)**.** *Let D be a bounded open set with smooth boundary and let $f \in L^r(D)$ for $r \geq 2$ and $r > d/2$. Let $G : [0, +\infty) \to [0, +\infty)$ be a convex function such that $G'(0) > 0$ and let $u \in C^1(\bar{D}) \cap H^2_{loc}(D)$ be the solution of*

$$-\operatorname{div}\big(G'(|\nabla u|^2)\nabla u\big) = f \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D. \tag{3.12}$$

Then for every $\varepsilon \in (0, 1)$ we have the estimate

$$\begin{aligned}\int_D \big|G'(|\nabla u|^2)\big|^r |\nabla u|^2 \, dx \leq 3\varepsilon \int_D \big|G'(|\nabla u|^2)\big|^r \, dx + \Big(\frac{(r-1)^r}{\varepsilon^{r-1}} + \frac{1}{\varepsilon^{2r-1}}\Big)\|u\|^r_{L^\infty}\|f\|^r_{L^r} \\ - \frac{(r-1)^2\|u\|^2_{L^\infty}}{\varepsilon} \int_{\partial D} H\big|G'(|\nabla u|^2)\big|^r |\nabla u|^2 \, d\mathcal{H}^{d-1},\end{aligned}$$

where H is the mean curvature of ∂D with respect to the outer normal.

Proof. We first notice that under the assumptions that D and f are smooth, we have that u is also smooth up to the boundary ∂D. Thus we can use the multiplication technique from [12] in order to obtain the L^p estimate of $G'(|\nabla u|^2)$. For the sake of simplicity we set $\sigma = G'(|\nabla u|^2)$. We first test the equation (3.12) with the function $\sigma^{r-1}u \in H^1_0(D)$ obtaining

$$\begin{aligned}\int_D \sigma^r |\nabla u|^2 \, dx + (r-1)\int_D u\sigma^{r-1}\nabla u \cdot \nabla\sigma \, dx &= \int_D f u \sigma^{r-1} \, dx \\ &\leq \|u\|_{L^\infty}\|f\|_{L^r}\left(\int_D \sigma^r \, dx\right)^{1-1/r}.\end{aligned} \tag{3.13}$$

Our main objective is to estimate the second term of the left-hand side. Multiplying (3.12) by $\operatorname{div}(\sigma^{r-1}\nabla u) = (\sigma^{r-1}u_j)_j$ we get

$$\begin{aligned}\int_D (\sigma u_i)_i(\sigma^{r-1}u_j)_j \, dx &= -\int_D f(\sigma^{r-1}u_j)_j \, dx = -\int_D f(\sigma^{r-2}\sigma u_j)_j \, dx \\ &= -\int_D f\sigma^{r-2}(\sigma u_i)_i \, dx - (r-2)\int_D f\sigma^{r-2}\sigma_j u_j \, dx \\ &\leq \int_D f^2\sigma^{r-2} \, dx + (r-2)\int_D |f|\,\sigma^{r-2}|\nabla\sigma \cdot \nabla u| \, dx.\end{aligned} \tag{3.14}$$

On the other hand, we can integrate by parts the left-hand side getting

$$\begin{aligned}\int_D (\sigma u_i)_i(\sigma^{r-1}u_j)_j \, dx &= -\int_D \sigma u_i(\sigma^{r-1}u_j)_{ji} \, dx + \int_{\partial D} (\sigma^{r-1}u_j)_j \sigma u_i n_i \, d\mathcal{H}^{d-1} \\ &= \int_D (\sigma u_i)_j(\sigma^{r-1}u_j)_i \, dx + \int_{\partial D}\Big[(\sigma^{r-1}u_j)_j\sigma u_i n_i - \sigma u_i(\sigma^{r-1}u_j)_i n_i\Big] \, d\mathcal{H}^{d-1} \\ &= \int_D (\sigma u_i)_j(\sigma^{r-1}u_j)_i \, dx + \int_{\partial D} \sigma^r\big(u_{jj}u_i n_i - u_{ij}u_i n_j\big) \, d\mathcal{H}^{d-1} \\ &= \int_D (\sigma u_i)_j(\sigma^{r-1}u_j)_i \, dx + \int_{\partial D} \sigma^r u_n(\Delta u - u_{nn}) \, d\mathcal{H}^{d-1},\end{aligned}$$

where u_n and u_{nn} indicate the first and the second derivatives in the direction of the exterior normal n to ∂D. Developing the term in the volume integral we have

$$\int_D (\sigma u_i)_i (\sigma^{r-1} u_j)_j \, dx = \int_D \Big(\sigma^r \|Hess(u)\|_2^2 + (r-1)\sigma^{r-2} |\nabla\sigma \cdot \nabla u|^2 + r\sigma^{r-1}\sigma_j u_i u_{ij} \Big) \, dx \\ + \int_{\partial D} \sigma^r u_n (\Delta u - u_{nn}) \, d\mathcal{H}^{d-1},$$

where $\|Hess(u)\|_2^2 = \sum_{i,j} u_{ij}^2 \le 0$. We first notice that $\sigma_j = 2u_k u_{kj} G''(|\nabla u|^2)$ and thus, by the convexity of G we have $\sigma_j u_i u_{ij} \ge 0$. On the other hand, on the boundary ∂D we have

$$\Delta u = u_{nn} + H u_n,$$

where H is the mean curvature. Thus we have the inequality

$$\int_D (\sigma u_i)_i (\sigma^{r-1} u_j)_j \, dx \ge \int_D (r-1)\sigma^{r-2} |\nabla\sigma \cdot \nabla u|^2 \, dx + \int_{\partial D} H\sigma^r |\nabla u|^2 \, d\mathcal{H}^{d-1},$$

which together with (3.14) gives

$$\begin{aligned} \int_D \sigma^{r-2} |\nabla\sigma \cdot \nabla u|^2 \, dx &\le \int_D |f|^2 \sigma^{r-2} \, dx - \int_{\partial D} H\sigma^r |\nabla u|^2 \, d\mathcal{H}^{d-1} \\ &\le \|f\|_{L^r}^2 \|\sigma\|_r^{r-2} - \int_{\partial D} H\sigma^r |\nabla u|^2 \, d\mathcal{H}^{d-1}. \end{aligned}$$

Using (3.13) we can now repeatedly use the Young inequality $A^\alpha B^\beta \le \varepsilon\alpha A + \varepsilon^{\alpha/\beta}\beta B$ for $0 \le \alpha, \beta \le 1$ and $\alpha + \beta = 1$ obtaining

$$\begin{aligned} \int_D \sigma^r |\nabla u|^2 \, dx &\le (r-1) \int_D \sigma^{r-1} \|u\|_{L^\infty} |\nabla u \cdot \nabla\sigma| \, dx + \|u\|_{L^\infty} \|f\|_{L^r} \|\sigma\|_{L^r}^{r-1} \\ &\le \varepsilon \int_D \sigma^r \, dx + \frac{(r-1)^2 \|u\|_{L^\infty}^2}{\varepsilon} \int_D \sigma^{r-2} |\nabla u \cdot \nabla\sigma|^2 \, dx + \|u\|_{L^\infty} \|f\|_{L^r} \|\sigma\|_{L^r}^{r-1} \\ &\le \varepsilon \int_D \sigma^r \, dx + \frac{(r-1)^2 \|u\|_{L^\infty}^2}{\varepsilon} \|f\|_{L^r}^2 \|\sigma\|_r^{r-2} + \|u\|_{L^\infty} \|f\|_{L^r} \|\sigma\|_{L^r}^{r-1} \\ &\qquad - \frac{(r-1)^2 \|u\|_{L^\infty}^2}{\varepsilon} \int_{\partial D} H\sigma^r |\nabla u|^2 \, d\mathcal{H}^{d-1} \\ &\le 3\varepsilon \int_D \sigma^r \, dx + \Big(\frac{(r-1)^r}{\varepsilon^{r-1}} + \frac{1}{\varepsilon^{2r-1}} \Big) \|u\|_{L^\infty}^r \|f\|_{L^r}^r \\ &\qquad - \frac{(r-1)^2 \|u\|_{L^\infty}^2}{\varepsilon} \int_{\partial D} H\sigma^r |\nabla u|^2 \, d\mathcal{H}^{d-1}, \end{aligned}$$

which is precisely the claim. □

In order to give a uniform estimate for θ_p on every domain satisfying the external ball condition and not only on the regular ones, we use an approximation argument. The following lemma that we use is well known in the γ-convergence theory.

Lemma 3.9. *Suppose that $D \subset \mathbb{R}^d$ is a bounded open satisfying the external ball condition and that D_n is a sequence of open sets of finite measure containing D and such that $|D_n \setminus D| \to 0$. Let $f \in L^2(D_1)$, $p \in [1, +\infty)$ be fixed and let u_p be the minimizer of F_p on D and u_p^n on D_n. Then the sequence u_p^n converges to u_p strongly in $H^1(\mathbb{R}^d)$ and $W^{1,2q}(\mathbb{R}^d)$.*

Proof. By the Sobolev inequality it is immediate to check that the sequence u_n^p is uniformly bounded in $H^1(\mathbb{R}^d)$ and $W^{1,2q}(\mathbb{R}^d)$ and so we may suppose that up to a subsequence it converges in L^2 to a function u. Moreover, all the functions u_p^n are uniformly bounded in L^∞ and so, the

function u is zero almost everywhere on $\mathbb{R}^d \setminus D$. Now, by [**?**] we have that $u \in H^1_0(D)$. Moreover, for any function $v \in H^1_0(D) \subset H^1_0(D_n)$ we have

$$F_p(u) \leq \liminf_{n\to\infty} F_p(u_p^n) \leq F_p(v),$$

which proves that $u = u_p$ is the minimizer of F_p on D. Moreover, since $F_p(u) \geq F_p(u_p^n)$ for every n, we get that $F_p(u_p^n) \to F_p(u)$. By the strong L^2 convergence of u_p^n to u_p we obtain also that $\|\nabla u_p^n\|_{L^2} \to \|\nabla u_p\|_{L^2}$ and $\|\nabla u_p^n\|_{L^{2q}} \to \|\nabla u_p\|_{L^{2q}}$, which concludes the proof. □

Proposition 3.10. *Let $D \subset \mathbb{R}^d$ be a bounded open set of finite perimeter satisfying the external ball condition and let $f \in L^\infty(\mathbb{R}^d)$. Then, for every $r \geq d$, there are constants $\delta > 0$, depending on D and f, and C, depending on r, on the dimension d, the perimeter $P(D)$, the diameter $diam(D)$, the radius R of the external ball, the norms $\|f\|_{L^\infty(D)}$ and $\|u_1\|_{L^\infty}$, such that*

$$\|\theta_p\|_{L^r(D)} \leq C \quad \textit{for every} \quad p \in (1, 1+\delta).$$

Proof. Suppose first that D has smooth boundary. We recall the notations u_p for the minimizer of the functional F_p in $H^1_0(D)$, θ_p for the optimal reinforcement, and $C_p = \|\nabla u_p\|_{2q}^{-2(q-1)}$, where $q = p/(p-1)$. Setting

$$G_p(t) = t + \frac{mC_p}{q} t^q$$

we have that u_p is the minimizer of the functional

$$H^1_0(D) \ni u \mapsto \frac{1}{2}\int_D G_p(|\nabla u|^2)\,dx - \int_D uf\,dx.$$

Moreover, we have

$$\theta_p = m\left(\frac{|\nabla u_p|}{\|\nabla u_p\|_{L^{2q}}}\right)^{2(q-1)} = mC_p|\nabla u_p|^{2(q-1)} = G_p'(|\nabla u_p|^2) - 1.$$

By Lemma 3.8 and the mean curvature estimate $H_{\partial D} \geq -1/R$, we have that

$$\begin{aligned}\int_D \sigma_p^r |\nabla u_p|^2\,dx \leq 3\varepsilon \int_D \sigma_p^r\,dx + \Big(\frac{(r-1)^r}{\varepsilon^{r-1}} + \frac{1}{\varepsilon^{2r-1}}\Big)\|u_p\|_{L^\infty}^r \|f\|_{L^r}^r \\ + \frac{(r-1)^2\|u_p\|_{L^\infty}^2}{\varepsilon R}\int_{\partial D} \sigma_p^r |\nabla u_p|^2\,d\mathcal{H}^{d-1},\end{aligned} \tag{3.15}$$

We denote by S and B the sets

$$S = \{x \in \overline{D} \ : \ |\nabla u_p(x)| \leq \|\nabla u_p\|_{L^{2q}}\} \qquad \text{and} \qquad B = \{x \in \overline{D} \ : \ |\nabla u_p(x)| > \|\nabla u_p\|_{L^{2q}}\},$$

and we notice that we have the inequality $\sigma_p \leq 1 + m$ on S, where we set for simplicity $\sigma_p = G_p(|\nabla u_p|^2)$. We now estimate separately the three terms on the right-hand side. For the first one, after integrating separately on S and B, we get

$$\int_D \sigma_p^r\,dx = \int_S \sigma_p^r\,dx + \int_B \sigma_p^r\,dx \leq (1+m)^r|D| + \|\nabla u_p\|_{L^{2q}}^{-2}\int_B \sigma_p^r|\nabla u_p|^2\,dx. \tag{3.16}$$

We can estimate the second term simply by using Lemma 3.7, while for the third one we have

$$\begin{aligned}\int_{\partial D}\sigma_p^r|\nabla u_p|^2\,d\mathcal{H}^{d-1} &= \int_{S\cap\partial D}\sigma_p^r|\nabla u_p|^2\,d\mathcal{H}^{d-1}+\int_{B\cap\partial D}\sigma_p^r|\nabla u_p|^2\,d\mathcal{H}^{d-1}\\ &\le (1+m)^{r-2}\int_{S\cap\partial D}\sigma_p^2|\nabla u_p|^2\,d\mathcal{H}^{d-1}\\ &\quad+\|\nabla u_p\|_{L^{2q}}^{2-r}\int_{B\cap\partial D}\sigma_p^r|\nabla u_p|^r\,d\mathcal{H}^{d-1}\\ &\le (1+m)^{r-2}P(D)\|f\|_{L^\infty}^2\operatorname{diam}(D)^2\left(1+\frac{\operatorname{diam}(D)}{\rho}\right)^{2(d-1)}\\ &\quad+\|\nabla u_p\|_{L^{2q}}^{2-r}P(D)\|f\|_{L^\infty}^r\operatorname{diam}(D)^r\left(1+\frac{\operatorname{diam}(D)}{\rho}\right)^{r(d-1)}.\end{aligned}$$

Now taking in (3.15) $\varepsilon=\|\nabla u_p\|_{L^{2q}}^2/6$ and absorbing part of the first term in the right-hand side of (3.15) into the left-hand side we get

$$\begin{aligned}\frac{1}{2}\int_D\sigma_p^r|\nabla u_p|^2\,dx &\le \frac{1}{2}\|\nabla u_p\|_{L^{2q}}^2(1+m)^r|D|\\ &\quad+6^{2r-1}\Big(\frac{(r-1)^r}{\|\nabla u_p\|_{L^{2q}}^{2r-2}}+\frac{1}{\|\nabla u_p\|_{L^{2q}}^{4r-2}}\Big)\|u_p\|_{L^\infty}^r\|f\|_{L^r}^r\\ &\quad+\frac{6(r-1)^2\|u_p\|_{L^\infty}^2}{\|\nabla u_p\|_{L^{2q}}^2R}\Big[(1+m)^{r-2}P(D)\|f\|_{L^\infty}^2\operatorname{diam}(D)^2\left(1+\frac{\operatorname{diam}(D)}{\rho}\right)^{2(d-1)}\\ &\quad+\|\nabla u_p\|_{L^{2q}}^{2-r}P(D)\|f\|_{L^\infty}^r\operatorname{diam}(D)^r\left(1+\frac{\operatorname{diam}(D)}{\rho}\right)^{r(d-1)}\Big].\end{aligned}$$

We now notice that the same inequality holds for irregular D. In fact we can approximate D by a sequence of smooth sets D_n containing D such that $|D_n\setminus D|\to 0$, $P(D_n)\le 2P(D)$, $\operatorname{diam}(D_n)\le 2\operatorname{diam}(D)$. By Lemma 3.9 the right-hand side of the above estimate passes to the limit, while the left-hand side is semicontinuous which finally gives the estimate passes to the limit. We now consider $\delta>0$ such that

$$\frac{1}{2}\|u_1\|_{L^\infty(D)}\le\|u_1\|_{L^\infty(D)}\le\frac{3}{2}\|u_1\|_{L^\infty(D)},\quad\text{for every}\quad p\in(1,1+\delta).$$

Then there is a constant $C=C(d,r,m,\|f\|_{L^\infty},P(D),\operatorname{diam}(D),\|u_1\|_{L^\infty})$ such that

$$\int_D\sigma_p^r|\nabla u_p|^2\,dx\le C,\quad\text{for every}\quad p\in(1,1+\delta).$$

Now the conclusion follows by the inequality $\int_D\theta_p^r\,dx\le\int_D\sigma_p^r\,dx$ and by applying one more time the estimate (3.16). □

Uniform L^∞ estimate for θ_p in the case of constant force. In the case $f=const$ on D a uniform estimate on θ_p can be obtained in a more direct way by using the Payne-Philippin maximum principle [18] that we recall below. The idea is similar to the one used by Kawohl in [15] for the infinity Laplacian.

Theorem 3.11 (Payne-Philippin maximum principle [18])**.** *Let $D\subset\mathbb{R}^d$ be a bounded open set with $C^{2+\varepsilon}$ boundary and let $g:[0,+\infty)\to[0,+\infty)$ be a C^2 function such that $2sg'(s)+g(s)>0$ on $[0,+\infty)$. Then, the function*

$$\widetilde{G}(x)=\int_0^{|\nabla u(x)|^2}\big(2sg'(s)+g(s)\big)\,ds,$$

achieves its maximum on the boundary ∂D, where the function u is the solution to the equation

$$-\operatorname{div}\big(g(|\nabla u|^2)\nabla u\big)=1\quad in\quad D,\qquad u=0\quad on\quad\partial D.$$

We are now in a position to give a uniform estimate on $\|\theta_p\|_{L^\infty(D)}$.

Proposition 3.12. *Let $D \subset \mathbb{R}^d$ be a bounded open set satisfying an external ball condition with radius $R \in (0, +\infty]$, where by $R = +\infty$ is intended that D is convex. Then, there is $\varepsilon > 0$ such that for $p \in (1, 1+\varepsilon]$ we have the estimate*

$$\theta_p \leq 2\|\nabla u_1\|^{-2}_{L^\infty(D)} M^2 \operatorname{diam}(D)^2 \left(1 + \frac{\operatorname{diam}(D)}{R}\right)^{2(d-1)}.$$

For convex D we have

$$\theta_p \leq 2\|\nabla u_1\|^{-2}_{L^\infty} M^2 \operatorname{diam}(D)^2. \tag{3.17}$$

Proof. We consider only the case of D convex since the two cases are analogous. We first prove (3.17) for regular domains and we then argue by approximation. Indeed, suppose first that D has regular $C^{2,\varepsilon}$ boundary. Then u_p is a solution of

$$-\operatorname{div}\big(g_p(|\nabla u_p|^2)\nabla u_p\big) = M \quad \text{in } D, \qquad u_p = 0 \quad \text{on } \partial D,$$

where g_p is given by

$$g_p(s) = s + mC_p s^{q-1},$$

and is sufficiently smooth when $q > 1$ is large (i.e. p close to 1). By the Payne-Philippin theorem we have that the function

$$\widetilde{G}_p(x) = \frac{3}{2}|\nabla u_p|^2 + \frac{2q-1}{q} mC_p |\nabla u_p|^{2q},$$

assumes its maximum on the boundary of D. On the other hand, by (3.17) on ∂D we have

$$\widetilde{G}_p \leq \frac{2q-1}{q}|\nabla u|^2\Big(1 + mC_p|\nabla u|^{2(q-1)}\Big) \leq \frac{2q-1}{q} M^2 \operatorname{diam}(D)^2,$$

and so on the entire D we have the estimate

$$mC_p|\nabla u|^{2q} \leq \frac{q}{2q-1} G_p \leq M^2 \operatorname{diam}(D)^2,$$

The optimal potential θ_p satisfies

$$\theta_p^p = \theta_p^{q/(q-1)} = \Big(mC_p|\nabla u_p|^{2(q-1)}\Big)^{q(q-1)} = \Big(m^{p/q} C_p^{1/(q-1)}\Big) mC_p |\nabla u_p|^{2q}.$$

We now recall that by the definition of C_p we get

$$C_p^{1/(q-1)} = \Big(\|\nabla u_p\|^{-2q/p}_{L^{2q}}\Big)^{1/(q-1)} = \|\nabla u_p\|^{-2}_{L^{2q}}.$$

Thus we obtain

$$\theta_p \leq \Big(m^{p/q}\|\nabla u_p\|^{-2}_{L^{2q}} M^2 \operatorname{diam}(D)^2\Big)^{1/p}.$$

The case of irregular D follows by approximation with smooth convex sets containing D and then applying Lemma 3.9. □

3.5 Conclusion of the proof of Theorem 2.3 and Theorem 2.4

Proof of Theorem 2.3. Let for $p > 1$, $u_p \in W_0^{2q}(D)$ be the minimizer of F_p and let θ_p be the optimal potential relative to u_p. By the fact that $\theta_p \in \mathcal{A}_{m,p}$ and by Proposition 3.10 we have that there is a constant $\bar{C}$ such that for $p > 1$ small enough we have

$$\int_D \theta_p^p \, dx = m^p \qquad \text{and} \qquad \int_D \theta_p^r \, dx \le C,$$

for some $r \ge d$. In particular, θ_p is uniformly bounded in $L^2(D)$ and so, up to a subsequence, θ_p converges weakly in $L^2(D)$ to a nonnegative function $\bar{\theta} \in L^2(D)$. Moreover, we have

$$\int_D \bar{\theta} \, dx = \lim_{p\to 1} \int_D \theta_p \, dx \le \liminf_{p\to 1} \|\theta_p\|_{L^p} |D|^{1/q} = m.$$

On the other hand, by Proposition 3.3 we have that as $p \to 1$ the solutions u_p converge strongly in $H_0^1(D)$ to the minimum u_1 of the functional F_1. Thus, for every fixed smooth function with compact support $\varphi \in C_c^\infty(D)$ we have

$$\int_D f\varphi \, dx = \int_D (1+\theta_p)\nabla u_p \cdot \nabla\varphi \, dx \xrightarrow[p\to 1]{} \int_D (1+\bar{\theta})\nabla u_1 \cdot \nabla\varphi \, dx.$$

Thus, we have that u_1 is in fact the solution of the equation

$$-\operatorname{div}\big((1+\bar{\theta})\nabla u_1\big) = f \quad \text{in } D, \qquad u_1 = 0 \quad \text{on } \partial D,$$

and after integration by parts

$$E_f(\bar{\theta}) = \frac{1}{2}\int_D (1+\bar{\theta})|\nabla u_1|^2 \, dx - \int_D f u_1 \, dx = -\frac{1}{2}\int_D f u_1 \, dx.$$

By the convergence of u_p to u_1 in $L^2(D)$ and by Proposition 3.3 we have

$$E_f(\bar{\theta}) = -\frac{1}{2}\int_D f u_1 \, dx = \lim_{p\to 1} -\frac{1}{2}\int_D f u_p \, dx = \lim_{p\to 1} E_f(\theta_p) = \lim_{p\to 1} F_p(u_p) = F_1(u_1),$$

which proves that

$$E_f(\bar{\theta}) = \min_{u\in H_0^1(D)} F_1(u). \tag{3.18}$$

On the other hand we have the general min-max inequality

$$\sup_{\theta\in\mathcal{A}_m} E_f(\bar{\theta}) = \sup_{\theta\in\mathcal{A}_m} \min_{u\in H_0^1(D)} J_f(\theta,u) \le \min_{u\in H_0^1(D)} \sup_{\theta\in\mathcal{A}_m} J_f(\theta,u) = \min_{u\in H_0^1(D)} F_1(u),$$

which concludes the proof that $\bar{\theta}$ is a solution to the problem (4.1).

Claim *(i)* follows by the fact that $\bar{\theta} \in L^r(D)$ and that r can be chosen arbitrary large.

Claim *(ii)* is in fact (3.18) and the claim *(iv)* follows directly by this equality.

The regularity of the minimizer u_1 of F_1 *(iii)* was proved by Evans in [14]. □

Proof of Theorem 2.4. We first consider the case $r < +\infty$. We notice that in the case of a convex set there is no boundary term in the De Pascale-Evans-Pratelli [12] estimate. Thus, for $f \in L^r(D)$ there is a constant C_r such that for every p large enough

$$\|\theta_p\|_{L^r} \le C_r\big(1 + \|f\|_{L^r}\big).$$

The rest of the proof follows as in the proof of Theorem 2.3. □

Remark 3.13. By using methods of optimal transportation theory slightly finer results can be obtained ([20], private communication):

- if D has the exterior ball condition and $f \in L^p(D)$ with $p \in [1,+\infty]$, then $\bar{\theta} \in L^p(D)$;
- if D is any bounded open set and $f \in L^p_{loc}(D)$ with $p \in [1,+\infty]$, then $\bar{\theta} \in L^p_{loc}(D)$.

4 Alternative approach to the optimal reinforcement problem

Let $D \subset \mathbb{R}^d$ be a bounded open set with a smooth boundary. The right hand-side f represents a given force on the membrane D. We allow f to be a general signed measure $f \in \mathcal{M}(D)$. A reinforcement of the membrane is given by a measure μ in the class $\mathcal{M}^+(D)$ of nonnegative measures on D. For a vertical displacement $u \in C^1_c(D)$, the energy of the membrane subjected to the force f and reinforced by μ is given by the functional

$$J_f(u,\mu) := \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\int_D |\nabla u|^2\,d\mu - \int_D u\,df\ ,$$

and we define the Dirichlet energy functional

$$E_f(\mu) := \inf\Big\{J_f(u,\mu)\ :\ u \in C^1_c(D)\Big\}.$$

As we noticed in Remark 2.1 for some measures f and μ we may have $E_f(\mu) = -\infty$. This occurs for instance if the force $f \notin H^{-1}(D)$ and μ is the Lebesgue measure on D. However these cases are ruled out from our discussion, because we are interested in the maximal reinforcement of the membrane. In fact, for any measure f there exists a measure μ, corresponding to the transport density of a suitable Monge optimal transport problem, such that $E_f(\mu) > -\infty$ (see for instance [1]). The optimization problem we are interested is the following:

$$\max\Big\{E_f(\mu) :\ \mu \in \mathcal{A}_m\Big\}, \tag{4.1}$$

where the parameter $m > 0$ represents the available quantity of reinforcement material and the admissible set $\mathcal{A}_m$ is given by

$$\mathcal{A}_m = \Big\{\mu \in \mathcal{M}^+(D)\ :\ \mu(D) \le m\Big\}.$$

A first existence result in the class of nonnegative measures μ is the following.

Proposition 4.1. *Let $D \subset \mathbb{R}^d$ be a bounded smooth open set and let $m > 0$. Then for every signed measure $f \in \mathcal{M}(D)$ the optimization problem* (4.1) *admits a solution $\bar\mu \in \mathcal{M}^+(D)$ with $\bar\mu(D) = m$ and $\bar\mu(\partial D) = 0$. Moreover, we have that*

$$E_f(\bar\mu) = \inf_{u \in W^{1,\infty}_0(\bar D)} \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{m}{2}\|\nabla u\|^2_{L^\infty} - \int_D u\,df.$$

Proof. We first notice that for every fixed $u \in C^1_c(D)$ the map $\mu \mapsto J_f(u,\mu)$ is continuous for the weak* convergence. Hence the map $\mu \mapsto E_f(\mu)$, being an infimum of continuous maps, is weak* upper semicontinuous. The conclusion follows by the weak* compactness of the $\mathcal{A}_m$ and the fact seen above that there is at least μ such that $E_f(\mu)$ is finite. The last claim follows by the monotonicity of the functional E_f and the fact that for every measure $\mu \in \mathcal{A}_m$ it holds $E_f(\mu) = E_f(\mu\lfloor D)$.

To prove the last claim we notice that the functional $J_f : \mathcal{A}_m \times C^1_c(D) \to \mathbb{R}$ satisfies the assumptions:

- J_f is concave and upper semicontinuous in the first variable with respect to the weak-$*$ convergence;
- J_f is convex in the second variable.

According to a classical result (see for instance [11]) we may interchange the inf and the sup for J_f obtaining

$$\sup_{\mu \in \mathcal{A}_m}\ \inf_{u \in C^1_c(D)} J_f(\mu,u) = \inf_{u \in C^1_c(D)}\ \sup_{\mu \in \mathcal{A}_m} J_f(\mu,u). \tag{4.2}$$

The supremum with respect to μ is easy to compute and we have

$$\sup_{\mu\in\mathcal{A}_m} J_f(\mu,u) = \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{m}{2}\|\nabla u\|^2_{L^\infty} - \int_D u\,df$$

as required. □

Therefore, we end up with a variational problem for the functional

$$F_1(u) = \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{m}{2}\|\nabla u\|^2_{\infty} - \int_D u\,df. \tag{4.3}$$

The existence and the regularity of the minimizers of F_1 has been widely studied in the literature (see for instance [4] and [14]). We summarize the known results below.

Theorem 4.2. *For every measure $f \in \mathcal{M}(D)$ the optimization problem*

$$\min\Big\{F_1(u)\ :\ u\in H^1_0(D)\Big\}$$

admits a unique solution $\bar u$. Moreover,

(i) if $f\in L^p(D)$ with $1<p<+\infty$, then $u\in W^{2,p}(D)$; in particular if $p>d$, then the solution $\bar u$ is $C^{1,\alpha}(\bar D)$ for some $\alpha>0$;

(ii) If D is convex and $f\in L^\infty(D)$ then $\Delta\bar u\in L^\infty(D)$;

(iii) if $f\in C^2(D)$, then $\bar u\in C^{1,1}(D)$.

Proof. The existence follows by the direct methods of the calculus of variations while the uniqueness is a consequence of the strict convexity of the functional F_1. The solution $\bar u$ is clearly Lipschitz and setting $M=\|\nabla\bar u\|_{L^\infty}$ we have that $\bar u$ is also the solution of the problem

$$\min\Big\{\frac{1}{2}\int_D |\nabla u|^2\,dx - \int_D u\,df\ :\ u\in W^{1,\infty}_0(D),\ \|\nabla u\|_{L^\infty}\le M\Big\}.$$

The claims *(i)* and *(ii)* were proved in [4], while *(iii)* was proved in [14]. □

Let now $f\in L^p(D)$ with $p>d$ and let $\bar\mu$ be the solution to the problem (4.1). We notice that $J_f(\bar\mu,\cdot)$ can be extended from a functional on $C^1_c(D)$ to a functional on $C^1(D)\cap C_0(\bar D)$. Thus, we can use the minimizer $\bar u$ as a test function in $J_f(\bar\mu,\cdot)$ obtaining

$$F_1(\bar u)\ge J_f(\bar\mu,\bar u)\ge \inf_{u\in C^1(D)\cap C_0(\bar D)} J_f(\bar\mu,u) = E_f(\bar\mu) = F_1(\bar u),$$

where the last equality follows by (4.2). Thus the second inequality is an equality and implies that $\bar u$ is the minimizer of the functional $J_f(\bar\mu,\cdot)$ in $C^1(D)\cap C_0(\bar D)$ and so it satisfies the Euler-Lagrange equation

$$-\operatorname{div}\big((1+\bar\mu)\nabla\bar u\big) = f \quad \text{in } D, \qquad \bar u = 0 \quad \text{on } \partial D. \tag{4.4}$$

Using $\bar u$ as a test function in (4.4) we get

$$\int_D |\nabla\bar u|^2\,dx + \int_D |\nabla\bar u|^2\,d\bar\mu = \int_D \bar u f\,dx.$$

On the other hand, since $\bar u$ is a minimizer of F_1, we have that the function $t\mapsto F_1(t\bar u)$ has a minimum in $t=1$, which gives

$$\int_D |\nabla\bar u|^2\,dx + m\|\nabla\bar u\|^2_{L^\infty} = \int_D \bar u f\,dx.$$

Putting together the two identities we have that

$$\int_D |\nabla \bar{u}|^2 \, d\bar{\mu} = m\|\nabla \bar{u}\|^2_{L^\infty} = mM^2,$$

which together with (4.4) gives the Monge-Kantorovich transport equation

$$\begin{cases} -\operatorname{div}(\bar{\mu}\nabla \bar{u}) = f + \Delta \bar{u} \quad \text{in } D, \qquad \bar{u} = 0 \quad \text{on } \partial D, \\ |\nabla \bar{u}| \le M, \\ |\nabla \bar{u}| = M \quad \text{on } \operatorname{spt}(\bar{\mu}). \end{cases} \tag{4.5}$$

This equation was widely studied in the framework of optimal transport theory. In particular, it was proved in [12], [19] and [13] that the integrability properties of the right-hand side give the integrability of $\bar{\mu}$. Thus we can prove directly Theorem 2.4 using these optimal transport results.

Alternative Proof of Theorem 2.4. Let $f \in L^r(D)$ for some $r > d$. By Proposition 4.1 there is a solution $\bar{\mu}$ to the problem (4.1). By the regularity result of Brezis and Stampacchia [4] the minimizer $\bar{u}$ of F_1 is $C^{1,\alpha}$ for some $\alpha > 0$ and $\Delta \bar{u} \in L^r(D)$. Thus $\bar{\mu}$ and $\bar{u}$ solve (4.5), whose right-hand side is in $L^r(D)$. Now by the summability results from [12] and [19] we have that $\mu \in L^r(D)$, which concludes the proof in the case $r < \infty$. If $f \in L^\infty$, then again by [4] we have that $\Delta \bar{u} + f \in L^\infty$. Now by [19] (and also [20]) we have that also $\bar{\mu} \in L^\infty(D)$. □

5 The elastic-plastic torsion problem

The solution of the optimization problem (4.1) is related to the solution of the variational minimization problem (4.3). In fact, the solution of (4.3) is precisely the torsion function $\bar{u}$ associated to the optimal elasticity term θ. On the other hand, in the case $f \equiv 1$, the problem (4.3) is equivalent to the well-known of *elastic-plastic torsion problem.* In fact if $\bar{u} \in W^{1,\infty}(\bar{D})$ is the solution of the variational problem (4.3), then setting $\kappa = \|\nabla \bar{u}\|_{L^\infty}$, it is immediate to check that $\bar{u}$ is also the solution of

$$\min\Big\{\frac{1}{2}\int_D |\nabla u|^2 \, dx - \int_D u \, dx \ : \ u \in H^1_0(D), \ \|\nabla u\|_{L^\infty} \le \kappa\Big\}. \tag{5.1}$$

This problem was a subject of an intense study in the past. The first results are due to Ting who studied the regularity of $\bar{u}$ and of the free boundary $\partial\{|\nabla \bar{u}| < \kappa\}$ in the case of a square [21], a regular [22] and an irregular polygon [23]. The key observation is that the problem (5.1) is equivalent to the following *obstacle problem*:

$$\min\Big\{\frac{1}{2}\int_D |\nabla u|^2 \, dx - \int_D u \, dx \ : \ u \in H^1_0(D), \ u \le \kappa \, d_{\partial D}\Big\}, \tag{5.2}$$

where by $d_{\partial D} : D \to [0, +\infty)$ we denote the distance function

$$d_{\partial D}(x) = \min\big\{|x - y| \ : \ y \in \partial D\big\}.$$

For a general domain $D \subset \mathbb{R}^d$ the equivalence of (5.1) and (5.2) was proved by Brezis and Sibony [3] and is based on a bound on the Lipschitz constant of the solution of (5.2). In fact since we have that

$$\Big\{u \in H^1_0(D) \ : \ |\nabla u| \le k\Big\} \subset \Big\{u \in H^1_0(D) \ : \ u \le \kappa \, d_{\partial D}\Big\},$$

it is sufficient to prove that the solution of the obstacle problem (5.2) is κ-Lipschitz. This may not be true in the case of a general force term f when large oscillations can be produced directly by f and so the equivalence of the two problems does not hold in general (see [8]).

The regularity of $\bar{u}$ is due to several authors. In [4] Brezis and Stampacchia showed that $\bar{u} \in C^{1,\alpha}(D)$, for every $\alpha \in (0,1)$. The optimal regularity $\bar{u} \in C^{1,1}(D)$ was obtained by Caffarelli-Riviere [9], Evans [14] and Wiegner [24]. In the case of planar domains with boundary which is a union of C^3 curves the free boundary was characterized by Caffarelli and Friedman [8].

In this case the optimal reinforcement θ is supported on the set $\{\bar{u} = \kappa d_{\partial D}\}$ and satisfies the Hamilton-Jacobi equation

$$\nabla\theta \cdot \nabla d_{\partial D} + \theta \Delta d_{\partial D} + \frac{1}{\kappa} + \Delta d_{\partial D} = 0 \quad \text{on } \{\bar{u} = \kappa d_{\partial D}\}.$$

This fact was used by Brezis in [2] to obtain θ explicitly and also to show that θ is continuous. We notice that the regularity of θ for a general nonconstant f is an open question. We conclude this section with an example in which the optimal reinforcement θ can be easily computed; other cases are treated numerically in Section 6.

Example 5.1 (The radial case). Let D be the unit ball of $\mathbb{R}^d$ and $f = 1$. We are then dealing with the optimal torsion problem. Passing to polar coordinates gives the optimization problem

$$\min\Big\{\frac{1}{2}\int_0^1 r^{d-1}|u'|^2\,dr + \frac{m}{2d\omega_d}\|u'\|_\infty^2 - \int_0^1 r^{d-1}u\,dr \ : \ u(1) = 0\Big\},$$

where ω_d is the measure of D. After some elementary calculations we find that the solution $\bar{u}$ is given by

$$\bar{u}(r) = \begin{cases} \dfrac{a^2 - r^2}{2d} + \dfrac{a(1-a)}{d} & \text{if } r \in [0,a] \\ \dfrac{a}{d}(1-r) & \text{if } r \in [a,1] \end{cases}$$

where $a = a_m$ is a suitable number in $]0,1[$. Optimizing with respect to a we obtain that a_m is the unique solution of

$$a^{d+1} - (d+1)a\Big(1 + \frac{m}{\omega_d}\Big) + d = 0.$$

It remains to compute the optimal function $\bar{\theta}$. By the Euler-Lagrange equation of

$$\int_{a_m}^1 \frac{1+\theta}{2} r^{d-1}|u'|^2\,dr - \int_{a_m}^1 r^{d-1}u\,dr$$

we obtain the expression of $\bar{\theta}$, given by

$$\bar{\theta}(r) = \frac{r}{a_m} - 1.$$

6 Approximation of the free boundary problem

6.1 A discrete convex constrained formulation

We introduce in this section, a convergent discretization of the problem

$$\min\Big\{\int_D \Big(\frac{1}{2}|\nabla u|^2 - fu\Big)\,dx + \frac{m}{2}\|\nabla u\|_\infty^2 \ : \ u \in H_0^1(D)\Big\} \tag{6.1}$$

for a domain $D \subset \mathbb{R}^d$. We focus our description on the case of the dimension 2. It is straightforward to generalize this approach to the three dimensional case. Our approach relies both on classical techniques in approximation of linear partial differential equations and on a suitable reformulation of this non-smooth convex energy. Assume τ_h is a given mesh made of simplexes which approximates the computational domain D. In the context of P_1 finite elements, we consider $H^1(D_h)$ the space of functions which are globally continuous and piecewise linear on every simplex of the mesh τ_h. Analogously to the continuous framework, we denote by $H_0^1(D_h)$

the functions of $H^1(D_h)$ which vanish at every boundary point of τ_h. We define K_h and M_h respectively the stiffness and the mass matrices associated to this discretization, more precisely we have

$$K_h = \left(\int_{D_h} \nabla u_i \cdot \nabla u_j \, dx \right)_{i,j} \qquad M_h = \left(\int_{D_h} u_i u_j \, dx \right)_{i,j}$$

where the family (u_i) is a basis of the the vector space $H_0^1(D_h)$. Thus if we denote by λ the coordinates of a function of $H_0^1(D_h)$ in this basis, we can introduce the discrete $E_m(\lambda)$ energy associated to problem (6.1) defined by

$$\frac{1}{2}\lambda^T K_h \lambda_- v_f^T M_h \lambda + \frac{m}{2} \sup_l \{(K_x\lambda)_l^2 + (K_y\lambda)_l^2\}$$

where the exponent T denotes the transpose operator, the coordinates of the vector v_f are given by the evaluation of the fixed function f on every vertex of the mesh and the matrices K_x and K_y are the linear operator associated to the evaluation of partial derivatives ∂_x and ∂_y in $H^1(D_h)$. Notice that K_x and K_y are not square matrices since their first dimension is indexed by the number of simplexes of τ_h. In order to get rid of the sup term which makes the original energy non smooth, we classically introduce a new variable $t \in \mathbb{R}$ and the new quadratic constraints

$$(K_x\lambda)_l^2 + (K_y\lambda)_l^2 \leq t \tag{6.2}$$

where l is an index associated to the set of the simplexes of τ_h. In this context, we define a new equivalent smooth convex constrained optimization problem which is the minimization of the energy $F_m(\lambda, t)$ defined by

$$F_m(\lambda, t) = \frac{1}{2}\lambda^T K_h \lambda_- v_f^T M_h \lambda + \frac{m}{2} t \tag{6.3}$$

for admissible (λ, t) which satisfy the constrained (6.2).

6.2 Numerical results in 2D and 3D

In order to study the numerically solution of problem 6.3 under constraints 6.2, we used a classical interior point solver provided by the so called IPOPT library available from COIN-OR (`http://www.coin-or.org`). Since the cost function and all the constraints are of quadratic type, we provided an exact evaluation of the Hessian matrix with respect to our variable (λ, t). Moreover, noticed that due to this quadratic formulation we can easily define a fixed pattern for the Hessian matrix which allows, through factorization techniques, to speed up dramatically the computation of the descent direction at every step. We present below several experiments based on this numerical approximation.

We first focused our study to the example of the unite disk. We discretize this set by a mesh made of 2×10^4 triangles. This complexity leads to an optimization problem of an approximate number of unknowns of the same order and under 2×10^4 constraints. We represent on the first column of Figure 1 the optimal graphs obtained for $m = 0$, 0.1, 0.5. We added on the graphs the analytically computed radial solution and observe an almost perfect match between our solution and the theoretical one. On the second column of Figure 1, we plot the graph of the gradient of the solution and for $m = 0.1$, 0.5. We underlined the free boundary set by adding a bolted line along its boundary. We reproduce the same experiments for three other geometries on Figures 2, 3 and 4. As expected, we observe in the three cases that for m large enough the free boundary of the problem converges to the singular set of the distance function of the domain.

Finally, to illustrate the efficiency of this approach even in a large scale context, we approximated the optimal profiles on three dimensional problems. In Figures 5 and 6 we discretized a cube and a dodecahedron by meshes made of approximately 10^6 simplexes for $m = 5$. The IPOPT library was able to solve such problems in several minutes on a standard computer. We

plot in Figures 5 and 6 different cuts of the graph of the modulus of the gradient of the solution. As in the two dimensional case we observe a convergence of the free boundary to a complex geometrical set which is very close from the singular set of the distance function.

Acknowledgements. A part of this paper was written during a visit of the authors at the Johann Radon Institute for Computational and Applied Mathematics (RICAM) of Linz. The authors gratefully acknowledge the Institute for the excellent working atmosphere provided. The work of the first author is part of the project 2010A2TFX2 *"Calcolo delle Variazioni"* funded by the Italian Ministry of Research and University. The first author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] G. Bouchitté, G. Buttazzo: *Characterization of optimal shapes and masses through Monge-Kantorovich equation.* J. Eur. Math. Soc., **3** (2001), 139–168.

[2] H. Brezis: *Multiplicateur de Lagrange en torsion "elasto-plastique".* Arch. Rat. Mech. Anal., **41** (1971), 32–40.

[3] H. Brezis, M. Sibony: *Equivalence de deux inequations variationnelles et applications.* Arch. Rat. Mech. Anal., **41** (1971), 254–265.

[4] H. Brezis, G. Stampacchia: *Sur la régularité de la solution d'inéquations elliptiques.* Bull. Soc. Math. France, **96** (1968), 153–180.

[5] D. Bucur, G. Buttazzo: *Variational Methods in Shape Optimization Problems.* Progress in Nonlinear Differential Equations **65**, Birkhäuser Verlag, Basel (2005).

[6] G. Buttazzo, G. Carlier, S. Guarino: *Optimal regions for congested transport.* Math. Model. Numer. Anal. (M2AN), (to appear), available at `http://cvgmt.sns.it`.

[7] G. Buttazzo, G. Dal Maso: *An existence result for a class of shape optimization problems..* Arch. Rational Mech. Anal., **122** (1993), 183–195.

[8] L.A. Caffarelli, A. Friedman: *The free boundary for elastic-plastic torsion problem.* Trans. Amer. Math. Soc., **252** (1979), 65–97.

[9] L.A. Caffarelli, N. Riviere: *On the Lipschitz character of the stress tensor when twisting an elastic-plastic bar.* Arch. Rat. Mech. Anal., **69** (1) (1979), 31–36.

[10] G. Carlier, C. Jimenez, F. Santambrogio: *Optimal transportation with traffic congestion and Wardrop equilibria.* SIAM J. Control Optim., **47** (3) (2008), 1330–1350.

[11] F.H. Clarke: *Multiple integrals of Lipschitz functions in the calculus of variations.* Proc. Amer. Math. Soc., **64** (2) (1977), 260–264.

[12] L. De Pascale, L.C. Evans, A. Pratelli: *Integral Estimates for Transport Densities.* Bulletin of the London Mathematical Society, **36** (3) (2004), 383–395.

[13] L. De Pascale, A. Pratelli: *Regularity properties for Monge Transport Density and for Solutions of some Shape Optimization Problem.* Calc. Var. PDE, **14** (3) (2002), 249–274.

[14] L.C. Evans: *A second order elliptic equation with gradient constraint.* Comm. PDE, **4** (5) (1979), 555–572.

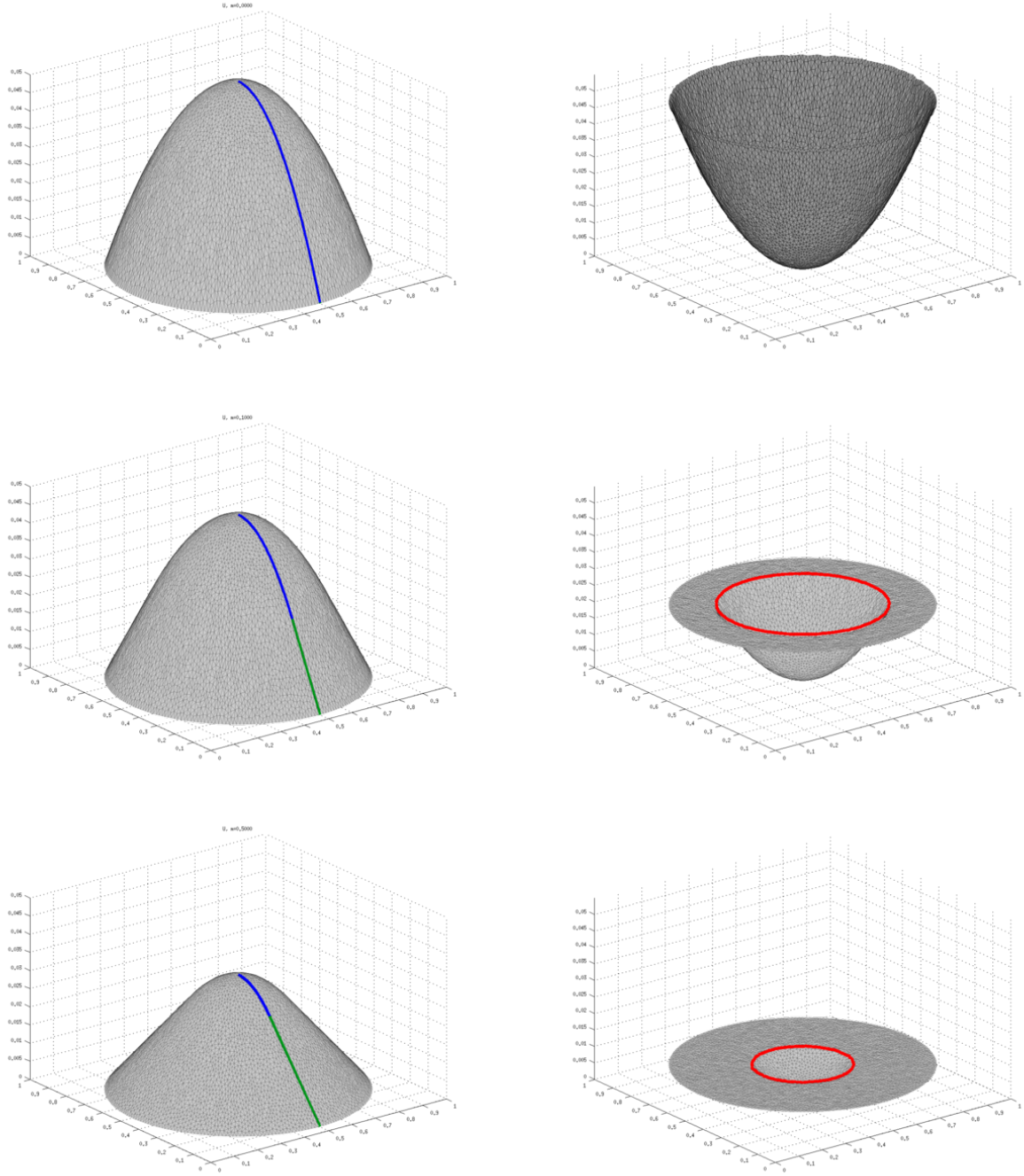

Figure 1: Optimization on a disk for $m = 0, 0.1, 0.5$

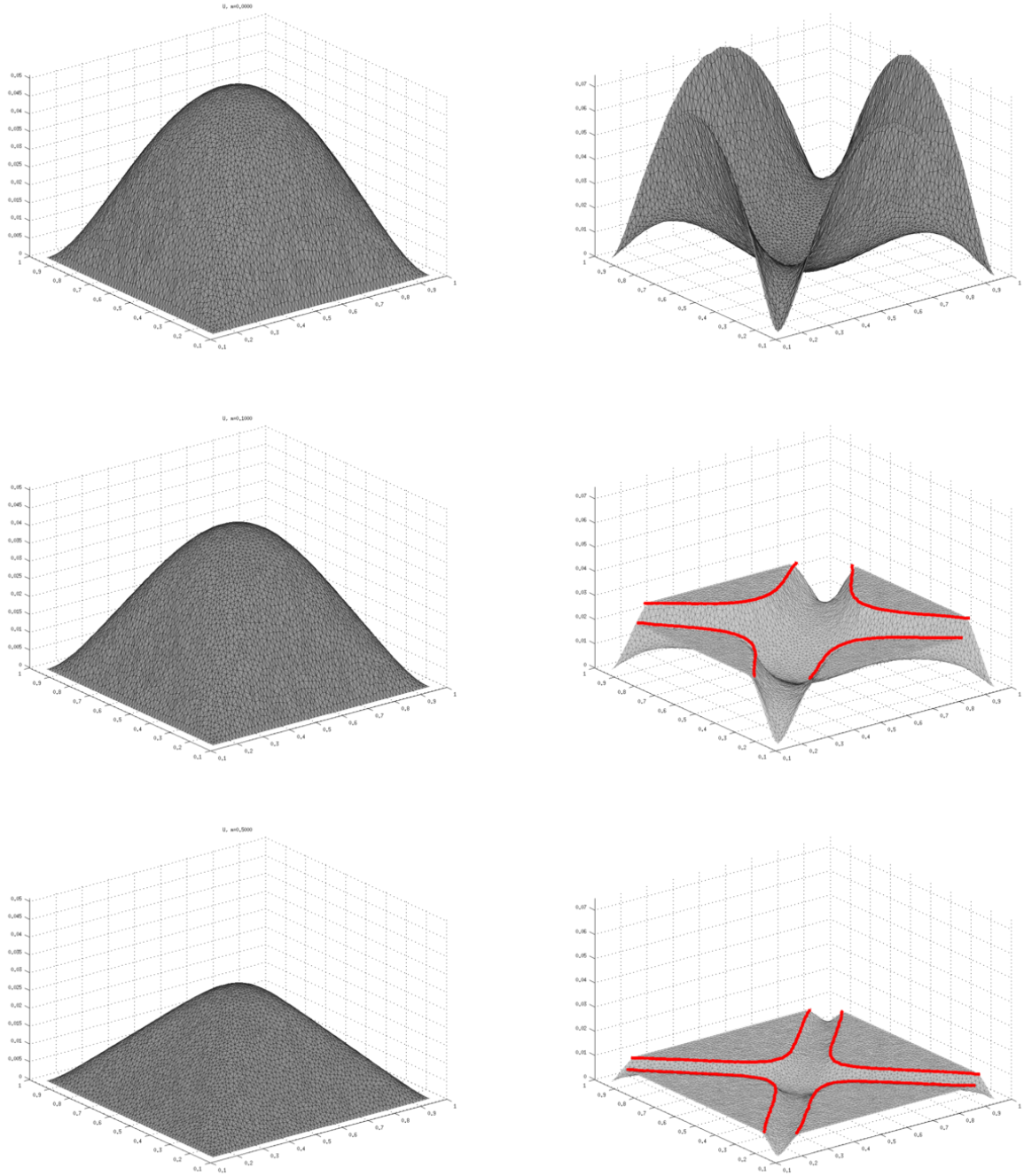

Figure 2: Optimization on a square for $m = 0, 0.1, 0.5$

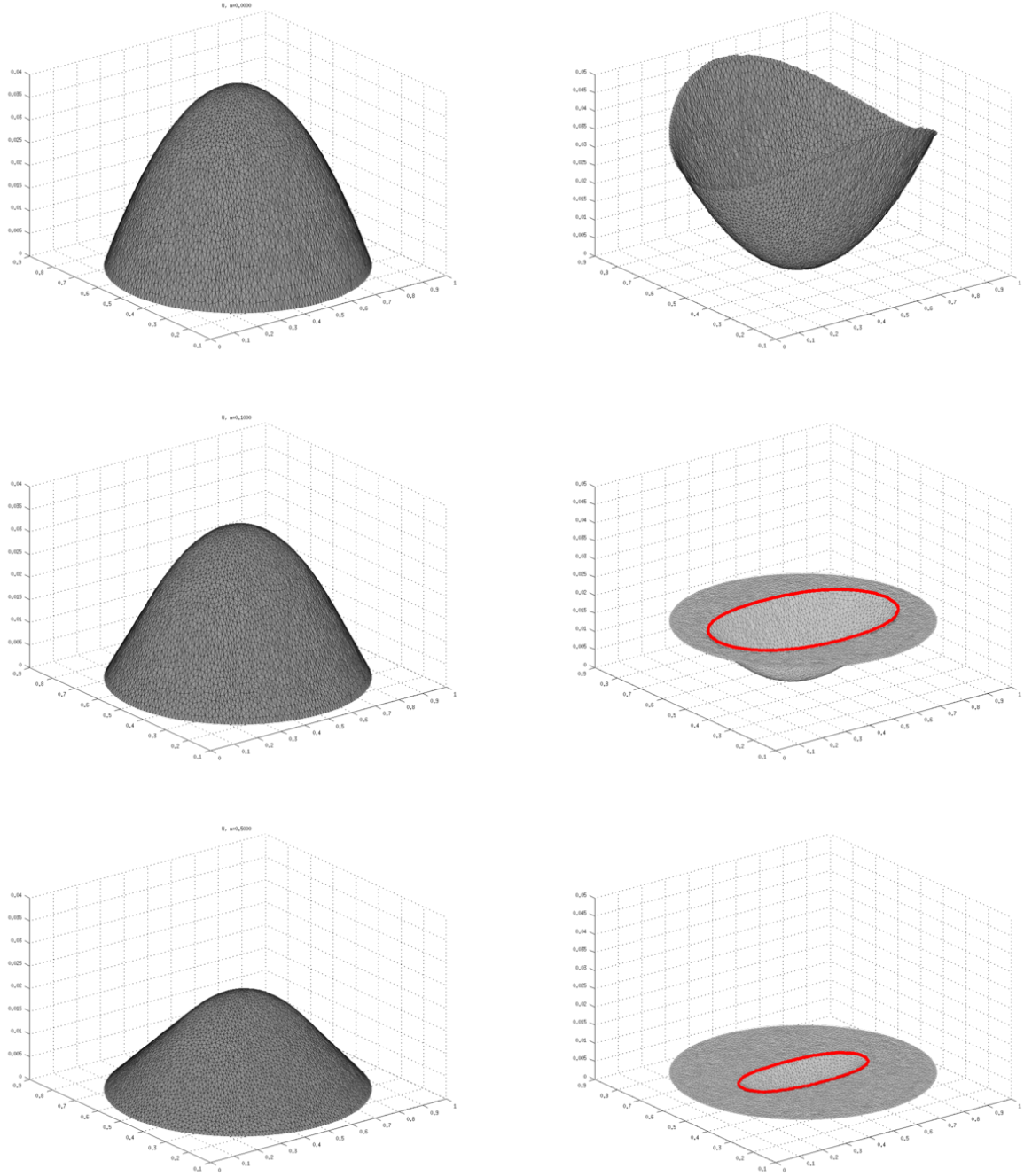

Figure 3: Optimization on an ellipse for $m = 0, 0.1, 0.5$

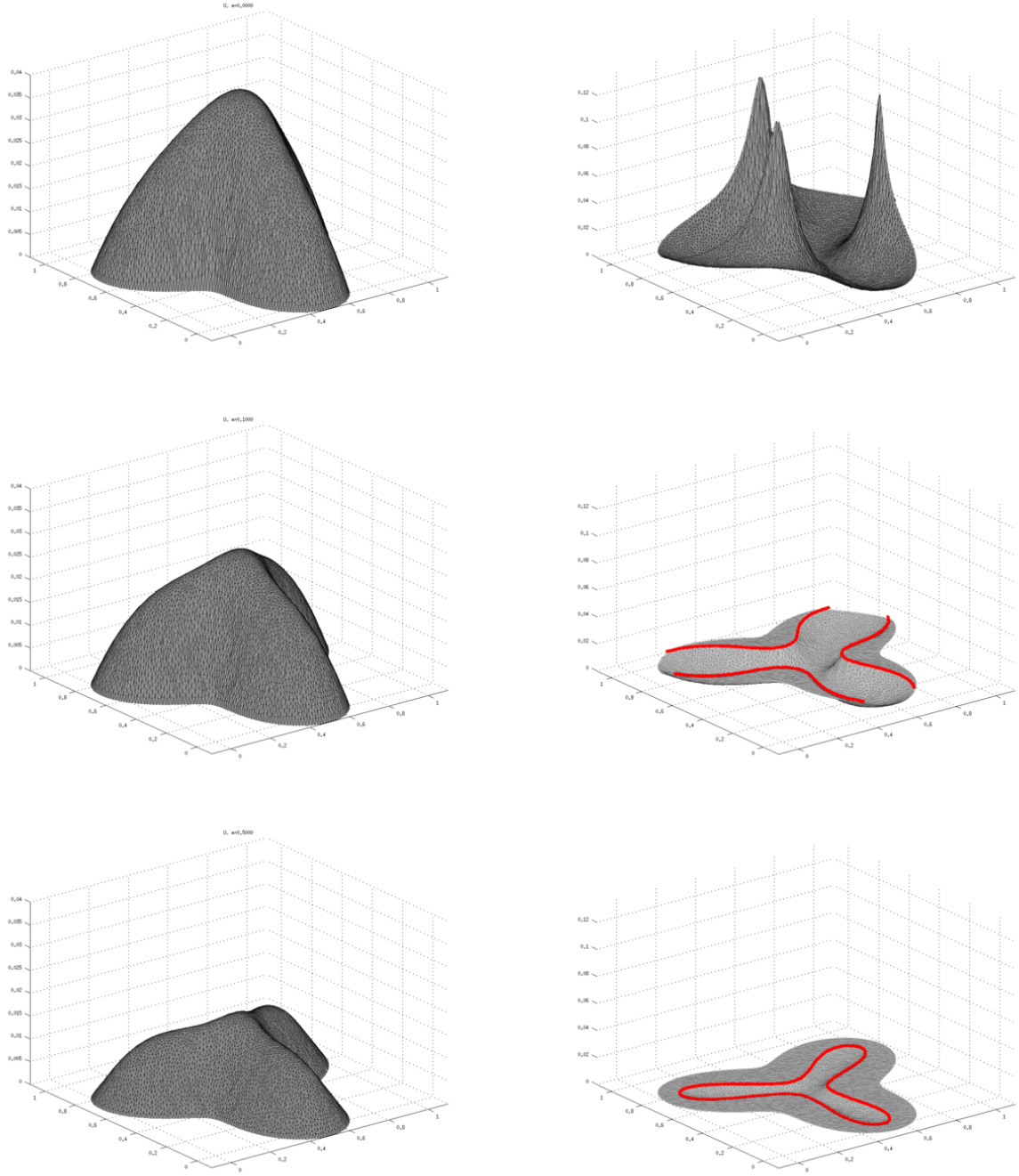

Figure 4: Optimization on a treffle for $m = 0, 0.1, 0.5$

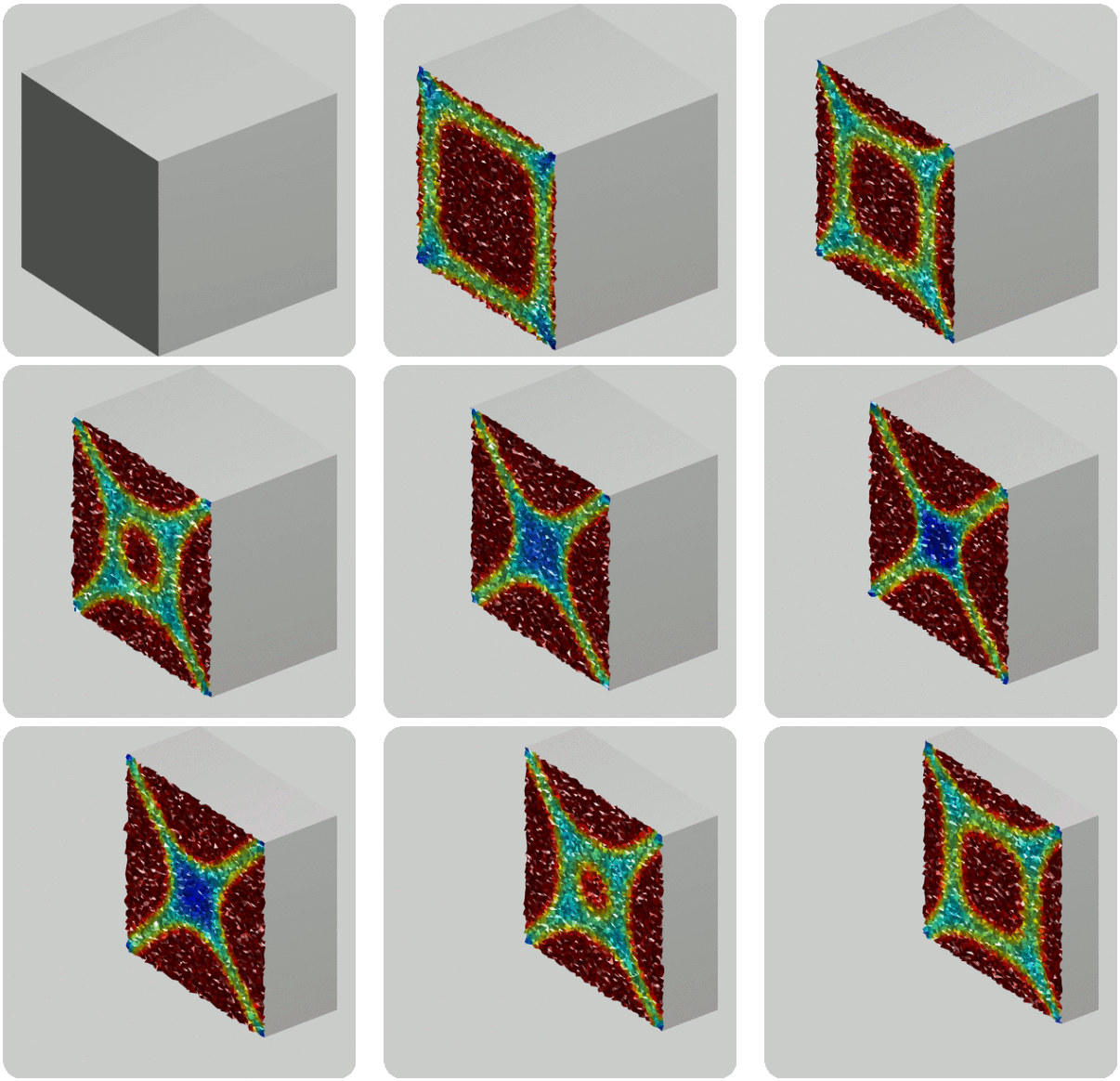

Figure 5: Tomographic view of the optimal solution defined in a cube for $m = 5$. The free boundary is visible in light color and is very close to the cut locus of the cube

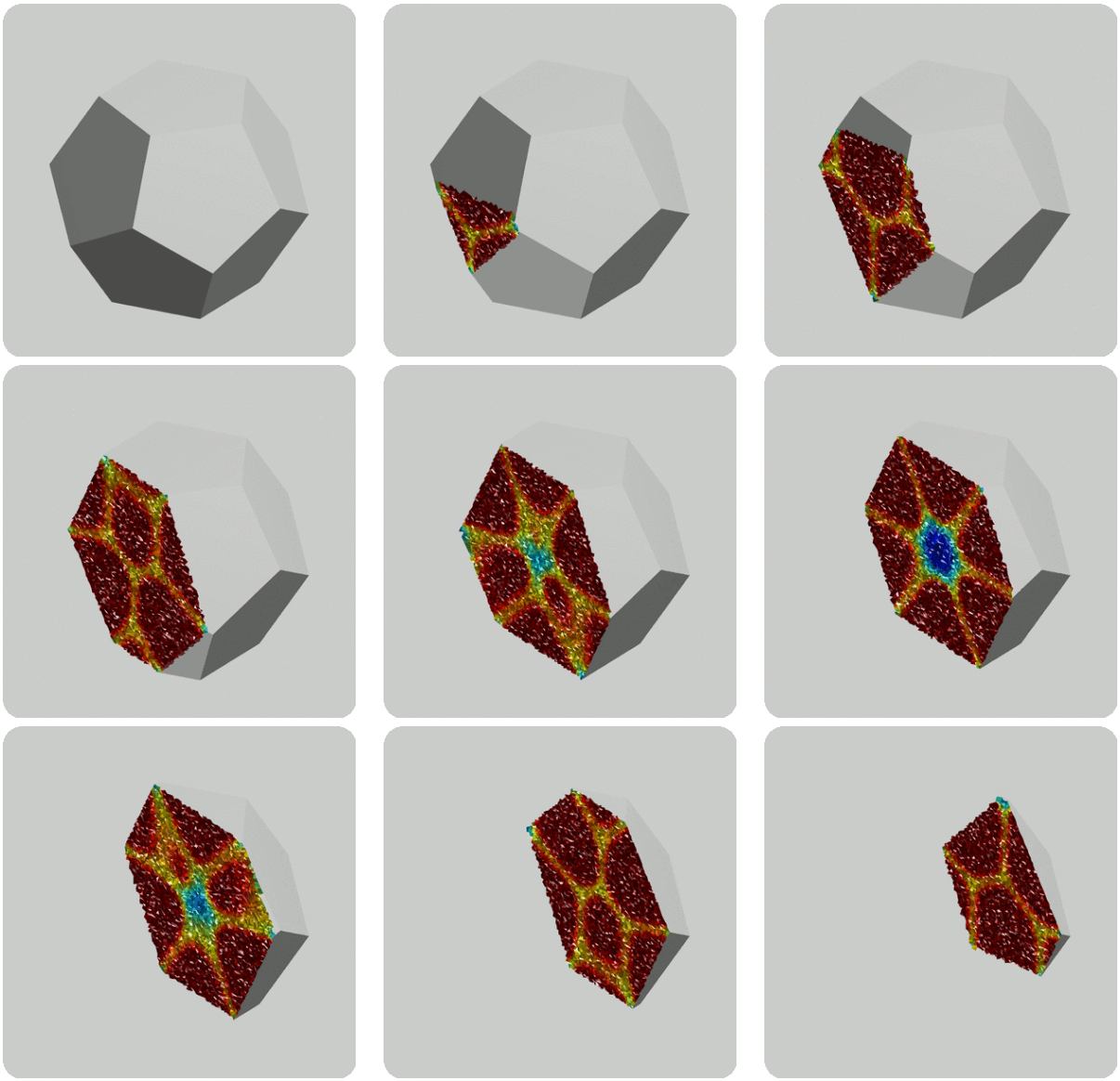

Figure 6: Tomographic view of the optimal solution defined in a dodecahedron for $m = 5$. The free boundary is visible in light color and is very close to the cut locus of the dodecahedron

[15] B. Kawohl: *On a family of torsional creep problems.* J. Reine Angew. Math., **410** (1990), 1–22.

[16] O. Ladyzhenskaya, N. Uraltseva: *Linear and quasilinear elliptic equations.* Academic Press (1968).

[17] G.M. Lieberman: *Boundary regularity for solutions of degenerate elliptic equations.* Nonlinear Analysis: Theory, Methods & Applications, **12** (11) (1988), 1203–1219.

[18] L.E. Payne, G.A. Philippin: *Some applications of the maximum principle in the problem of the torsional creep.* SIAM J. Appl. Math., **33** (3) (1977), 446–455.

[19] F. Santambrogio: *Absolute continuity and summability of transport densities: simpler proofs and new estimates.* Calc. Var., **36** (2009), 343–354.

[20] F. Santambrogio: *Private communication.* .

[21] T.W. Ting: *Elastic-plastic torsion of square bar.* Trans. Amer. Math. Soc., **123** (1966), 369–401.

[22] T.W. Ting: *Elastic-plastic torsion problem II.* Arch. Rat. Mech. Anal., **25** (1967), 342–365.

[23] T.W. Ting: *Elastic-plastic torsion problem III.* Arch. Rat. Mech. Anal., **34** (1969), 228–244.

[24] M. Wiegner: *The $C^{1,1}$-character of solutions of second order elliptic equations with gradient constrained.* Comm. Partial Differential Equations **6** (3) (1981), 361–371.

Giuseppe Buttazzo: Dipartimento di Matematica, Università di Pisa
Largo B. Pontecorvo 5, 56127 Pisa - ITALY
`buttazzo@dm.unipi.it`
`http://www.dm.unipi.it/pages/buttazzo/`

Edouard Oudet: Laboratoire Jean Kuntzmann (LJK), Université Joseph Fourier
Tour IRMA, BP 53, 51 rue des Mathématiques, 38041 Grenoble Cedex 9 - FRANCE
`edouard.oudet@imag.fr`
`http://www-ljk.imag.fr/membres/Edouard.Oudet/`

Bozhidar Velichkov: Laboratoire Jean Kuntzmann (LJK), Université Joseph Fourier
Tour IRMA, BP 53, 51 rue des Mathématiques, 38041 Grenoble Cedex 9 - FRANCE
`bozhidar.velichkov@imag.fr`
`http://www.velichkov.it`