# Variational Approximation of Functionals Defined on 1-dimensional Connected Sets: The Planar Case

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# Variational approximation of functionals defined on 1-dimensional connected sets: the planar case

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**Abstract**

In this paper we consider variational problems involving 1-dimensional connected sets in the Euclidean plane, such as the classical Steiner tree problem and the irrigation (Gilbert-Steiner) problem. We relate them to optimal partition problems and provide a variational approximation through Modica-Mortola type energies proving a full $\Gamma$-convergence result. We also introduce a suitable convex relaxation and develop the corresponding numerical implementations. The proposed methods are quite general and the results we obtain can be extended to $n$-dimensional Euclidean space or to more general manifold ambients, as shown in the companion paper [11].

## 1 Introduction

Connected one dimensional structures play a crucial role in very different areas like discrete geometry (graphs, networks, spanning and Steiner trees), structural mechanics (crack formation and propagation), inverse problems (defects identification, contour segmentation), etc. The modeling of these structures is a key problem both from the theoretical and the numerical point of view. Most of the difficulties encountered in studying such one dimensional objects are related to the fact that they are not canonically associated to standard mathematical quantities. In this article we plan to bridge the gap between the well-established methods of multi-phase modeling and the world of one dimensional connected sets or networks. Whereas we strongly believe that our approach may lead to new points of view in quite different contexts, we restrict here our exposition to the study of two standard problems in the Calculus of Variations which are respectively the classical Steiner tree problem and the Gilbert-Steiner problem (also called the irrigation problem).

The Steiner Tree Problem (STP) [22] can be described as follows: given $N$ points $P_1, \dots, P_N$ in a metric space $X$, (e.g. $X$ a graph, with $P_i$ assigned vertices), find a connected (sub-)graph $F \subset X$ containing the points $P_i$ and having minimal length. Such

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an optimal graph $F$ turns out to be a tree and is thus called a *Steiner Minimal Tree* (SMT). In case $X = \mathbb{R}^d$, $d \geq 2$ endowed with the Euclidean $\ell^2$ metric, one refers often to the *Euclidean* or *geometric* STP, while for $X = \mathbb{R}^d$ endowed with the $\ell^1$ (Manhattan) distance or for $X$ contained in a fixed grid $\mathcal{G} \subset \mathbb{R}^d$ one refers to the *rectilinear* STP.

Here we will adopt the general metric space formulation of [31]: given a metric space $X$, and given a compact (possibly infinite) set of terminal points $A \subset X$ , find

$$\text{(STP)} \qquad \inf\{\mathcal{H}^1(S),\, S \text{ connected},\, S \supset A\},$$

where $\mathcal{H}^1$ indicates the 1-dimensional Hausdorff measure on $X$. Existence of solutions for (STP) relies on Golab's compactness theorem for compact connected sets, and it holds true also in generalized cases (e.g. $\inf \mathcal{H}^1(S)$, $S \cup A$ connected).

Problems like (STP) are relevant for the design of optimal transport channels or networks connecting given endpoints, for example the optimal design of net routing in VLSI circuits in the case $d = 2, 3$. The Steiner Tree Problem has been widely studied from the theoretical and numerical point of view in order to efficiently devise constructive solutions, mainly through combinatoric optimization techniques. Finding a Steiner Minimal Tree is known to be a NP hard problem (and even NP complete in certain cases), see for instance [6, 7] for a comprehensive survey on PTAS algorithms for (STP).

The situation in the Euclidean case is theoretically well understood: given $N$ points $P_i \in \mathbb{R}^d$ a SMT connecting them always exists, the solution being in general not unique (think for instance to symmetric configurations of the endpoints $P_i$). The SMT is a union of segments connecting the endpoints, possibly meeting at 120° in at most $N-2$ further branch points, called *Steiner points*.

Nonetheless, the quest of computationally tractable approximating schemes for (STP) has recently attracted a lot of attention in the Calculus of Variations community, due to different variational interpretations of (STP) as respectively a size minimization problem for 1-dimensional connected sets [27, 20], an optimal branched transport problem [10, 16], or even a Plateau problem in a suitable class of vector distributions endowed with some algebraic structure [27, 24], to be solved by finding suitable *calibrations* [25]. Several authors have proposed different approximations of the problem, whose validity is essentially limited to the planar case, mainly using a phase field based approach together with some coercive regularization, see e.g. [12, 19, 29, 13].

Our aim is to propose a variational approximation for (STP) and for the Gilbert-Steiner irrigation problem (in the equivalent formulations of [34, 23]) in the Euclidean case $X = \mathbb{R}^d$, $d \geq 2$. In this paper we focus on the planar case $d = 2$ and prove a genuine $\Gamma$-convergence result (see Theorem 3.9 and Proposition 3.3) by considering integral functionals of Modica-Mortola type [26]. In the companion paper [11] we rigorously prove that certain integral functionals of Ginzburg-Landau type (see [1]) yield a variational approximation for (STP) and of the irrigation problem valid in any dimension $d \geq 3$. This approach is related to the interpretation of (STP) as a Plateau problem in a cobordism class of integral currents with multiplicities in a suitable normed group as studied by Marchese and Massaccesi in [24] (see also [27] for the planar case). Our method is quite general and may be easily adapted to a variety of situations (e.g. in manifolds or more general metric space ambients, with densities or anisotropic norms, etc.).

The plan of the paper is as follows: in Section 2 we reformulate (STP) and the irrigation problem as a suitable modification of the optimal partition problem in the planar case. In section 3, we state and prove our main $\Gamma$-convergence results, respectively Proposition 3.3 and Theorem 3.9. Inspired by [18], we introduce in section 4 a convex relaxation of the corresponding energies. In Section 5 we present our approximating scheme for (STP) and for the Gilbert-Steiner problem and illustrate its flexibility in different situations, showing how our convex formulation is able to recover multiple solutions whereas $\Gamma$-relaxation detects any locally minimizing configuration. Finally, in Section 6 we propose some examples and generalizations that are extensively studied in the companion paper [11].

# 2 Irrigation-type problems for Euclidean graphs and optimal partitions

In this section we describe some optimization problems on Euclidean graphs with fixed endpoints set $A$, like (STP) or irrigation-type problems, following the approach of [24, 23], and we rephrase them as optimal partition-type problems in the planar case $\mathbb{R}^2$.

## 2.1 Acyclic graphs and rank one tensor valued measures

Let $A = \{P_1, \dots, P_N\} \subset \mathbb{R}^d$, $d \geq 2$, be a given set of $N$ distinct points, with $N > 2$. Define $\mathcal{G}(A)$ to be the set of *acyclic graphs* $L$ connecting the endpoints set $A$ such that $L$ can be described as the union $L = \cup_{i=1}^{N-1} \lambda_i$, where $\lambda_i$ are simple rectifiable curves with finite length having $P_i$ as initial point and $P_N$ as final point, oriented by $\mathcal{H}^1$-measurable unit vector fields $\tau_i$ satisfying $\tau_i(x) = \tau_j(x)$ for $\mathcal{H}^1$-a.e. $x \in \lambda_i \cap \lambda_j$ (i.e. the orientation of $\lambda_i$ is coherent with that of $\lambda_j$ on their intersection).

For $L \in \mathcal{G}(A)$, if we identify the curves $\lambda_i$ with the vector measures $\Lambda_i = \tau_i \otimes \mathcal{H}^1 \llcorner \lambda_i$, all the information concerning this acyclic graph $L$ is encoded in the rank one tensor valued measure $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, where the $\mathcal{H}^1$-measurable vector field $\tau \in \mathbb{R}^d$ carrying the orientation of the graph $L$ satisfies $\operatorname{spt} \tau = L$, $|\tau| = 1$, $\tau = \tau_i$ $\mathcal{H}^1$-a.e. on $\lambda_i$, and the $\mathcal{H}^1$-measurable vector field $g \in \mathbb{R}^{N-1}$ has components $g_i$ satisfying $g_i \cdot \mathcal{H}^1 \llcorner L = \mathcal{H}^1 \llcorner \lambda_i = |\Lambda_i|$, with $|\Lambda_i|$ the total variation measure of the vector measure $\Lambda_i = \tau \otimes \mathcal{H}^1 \llcorner \lambda_i$. Observe that $g_i \in \{0, 1\}$ a.e. for any $1 \leq i \leq N-1$.

**Definition 2.1** *Given any graph $L \in \mathcal{G}(A)$, we call the above constructed $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ the canonical $\mathbb{R}^d \otimes \mathbb{R}^{N-1}$-valued measure representation of the acyclic graph $L$.*

**Remark 2.2** Observe that for any $1 \leq i \leq N-1$ the measures $\Lambda_i$ verify the property

$$\operatorname{div} \Lambda_i = \delta_{P_i} - \delta_{P_N}. \tag{2.1}$$

To any compact connected set $K \supset A$ with $\mathcal{H}^1(K) < +\infty$, i.e. to any candidate minimizer for (STP), we associate in a canonical way an acyclic graph $L \in \mathcal{G}(A)$ connecting $\{P_1, \dots, P_N\}$ such that $\mathcal{H}^1(L) \leq \mathcal{H}^1(K)$ (see e.g. Lemma 2.1 in [24]). Given such a graph $L \in \mathcal{G}(A)$ canonically represented by the tensor valued measure $\Lambda$, the measure $\mathcal{H}^1 \llcorner L$ corresponds to the smallest positive measure dominating $|\Lambda_i| = \mathcal{H}^1 \llcorner \lambda_i$ for $1 \le i \le N-1$, where $|\Lambda_i|$ is the total variation measure of the vector measure $\Lambda_i = \Lambda \cdot e_i = \tau \otimes (g \cdot e_i)\mathcal{H}^1 \llcorner L$. It is thus given by $\mathcal{H}^1 \llcorner L = \sup_i \mathcal{H}^1 \llcorner \lambda_i = \sup_i |\Lambda_i|$, the supremum of the total variation measures $|\Lambda_i|$.

**Remark 2.3** An equivalent definition of the measure $\mu = \sup_{1 \le i \le N-1} \mu_i$, for $\mu_i$ positive Radon measures on $\mathbb{R}^d$, can be given by duality: we have, for any positive $\psi \in C_c^0(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \psi \, d\mu = \sup \left\{ \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \varphi_i \, d\mu_i \, , \ \varphi_i \in C_c^0(\mathbb{R}^d), \ \sum_{i=1}^{N-1} \varphi_i(x) \le \psi(x) \right\} .$$

**Remark 2.4 (graphs as $G$-currents)** In [24], the rank one tensor measure $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ identifying a graph in $\mathbb{R}^d$ is defined as a *current* with coefficients in the group $\mathbb{Z}^{N-1} \subset \mathbb{R}^{N-1}$. For $\omega \in \mathcal{D}^1(\mathbb{R}^d)$ a smooth compactly supported differential 1-form and $\vec{\varphi} = (\varphi_1, ..., \varphi_{N-1}) \in [\mathcal{D}(\mathbb{R}^d)]^{N-1}$ a smooth test (vector) function, one sets

$$\langle \Lambda, \omega \otimes \vec{\varphi} \rangle := \int_{\mathbb{R}^d} \langle \omega \otimes \vec{\varphi}, \tau \otimes g \rangle \, d|\Lambda| = \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \langle \omega, \tau \rangle \varphi_i g_i d|\Lambda| = \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \langle \omega, \tau \rangle \varphi_i \, d|\Lambda_i| \, .$$

Moreover, fixing a norm $\Psi$ on $\mathbb{R}^{N-1}$, one may define the $\Psi$-total variation of the current $\Lambda$ as

$$||\Lambda||_\Psi = |\Lambda|_\Psi(\mathbb{R}^d) = \sup \left\{ \langle \Lambda, \omega \otimes \vec{\varphi} \rangle \, , \ \ |\omega(x)| \le 1 \, , \ \Psi^*(\vec{\varphi}(x)) \le 1 \right\} \, , \tag{2.2}$$

where $\Psi^*$ is the dual norm to $\Psi$ w.r.t. the scalar product on $\mathbb{R}^{N-1}$. Remark that (2.2) defines the $\Psi$-total variation for a generic $d \times (N-1)$ matrix valued measure $\Lambda$.

## 2.2 Irrigation-type functionals

In this section we consider functionals defined on acyclic graphs connecting a fixed set $A = \{P_1, \ldots, P_N\} \subset \mathbb{R}^d$, $d \ge 2$, by using their canonical representation as rank one tensor valued measures, in order to identify the graph with an *irrigation plan* from the point sources $\{P_1, \ldots, P_{N-1}\}$ to the target point $P_N$. We focus here on suitable energies in order to describe the irrigation problem and the Steiner tree problem in a common framework as in [24, 23]. We observe moreover that the irrigation problem with one point source $(I_\alpha)$ introduced by Xia [34], in the equivalent formulation of [23], approximates the Steiner tree problem as $\alpha \to 0$ in the sense of $\Gamma$-convergence (see Proposition 2.6).

We first introduce some additional notation: let be given positive measures $\mu_i$ on $\mathbb{R}^d$, for $i = 1, ..., M$ to form a $\mathbb{R}^M$-valued vector measure $\vec{\mu}$. Let $|\vec{\mu}|_1 = \sum_i \mu_i$, so that $\vec{\mu} = g|\vec{\mu}|_1$ with $g \in \mathbb{R}^M$, $0 \le g_i \le 1$ for $1 \le i \le M$, $\sum_i g_i = 1$. Accordingly, we denote $|\vec{\mu}|_\infty$ the supremum measure $|\vec{\mu}|_\infty = \sup_i \mu_i = (\sup_i g_i) \, |\vec{\mu}|_1$. For $p \ge 1$ define the measure $|\vec{\mu}|_p := |g|_p |\vec{\mu}|_1$, with $|g|_p = (\sum_i g_i^p)^{1/p}$ the $\ell^p$ norm of $g \in \mathbb{R}^M$. We have the coerciveness property

$$\frac{1}{M} |\vec{\mu}|_1 \le |\vec{\mu}|_\infty \le |\vec{\mu}|_q \le |\vec{\mu}|_p \le |\vec{\mu}|_1, \quad \forall \, 1 < p < q \, , \forall \, \vec{\mu} \, . \tag{2.3}$$

More generally, for $\Psi$ a norm on $\mathbb{R}^M$, we define the measure $\Psi(\vec{\mu}) := \Psi(g)|\vec{\mu}|_1$. In particular, we have the characterization

$$\Psi(\vec{\mu})(\mathbb{R}^d) = \sup\left\{\sum_{i=1}^{M}\int_{\mathbb{R}^d}\varphi_i\,d\mu_i,\ 0\le\varphi_i\in C_c^0(\mathbb{R}^d)\ \forall\, 1\le i\le M,\quad \Psi^*(\vec{\varphi})\le 1\right\} \tag{2.4}$$

with $\Psi^*$ the dual norm to $\Psi$ w.r.t. the Euclidean structure on $\mathbb{R}^M$. The total variation of the measure $\Psi(\vec{\mu})$ coincides with the $\Psi$-total variation $||\Lambda||_\Psi$ as defined in (2.2), where $\Lambda = \tau\otimes\vec{\mu}$ for any $|\vec{\mu}|_1$-measurable unit vector field $\tau\in\mathbb{R}^d$.

Let $\Lambda = \tau\otimes\vec{\mu} = \tau\otimes g|\vec{\mu}|_1$ be a rank one $\mathbb{R}^d\otimes\mathbb{R}^M$-valued measure with $|\tau| = 1$. For $0<\alpha\le 1$ define

$$\mathcal{F}^\alpha(\Lambda) = \int_{\mathbb{R}^d}|g|_{1/\alpha}d|\vec{\mu}|_1 = |\vec{\mu}|_{1/\alpha}(\mathbb{R}^d) \tag{2.5}$$

and

$$\mathcal{F}^0(\Lambda) = \int_{\mathbb{R}^d}|g|_\infty d|\vec{\mu}|_1 = \int_{\mathbb{R}^d}\left(\sup_{1\le i\le M}\mu_i\right) = |\vec{\mu}|_\infty(\mathbb{R}^d). \tag{2.6}$$

In other words, $\mathcal{F}^\alpha(\Lambda) = ||\Lambda||_{\Psi_\alpha}$, $\mathcal{F}^0(\Lambda) = ||\Lambda||_{\Psi_0}$ are total variation-type functionals, with respect to the norms $\Psi_\alpha = |\cdot|_{\ell^{1/\alpha}}$ and $\Psi_0 = |\cdot|_{\ell^\infty}$.

When $\Lambda = \tau\otimes g\mathcal{H}^1\llcorner L$ is the canonical representation of an acyclic graph $L\in\mathcal{G}(A)$, so that in particular we have $|\tau| = 1$ and $g_i\in\{0,1\}$ for $1\le i\le M$, we deduce

$$\mathcal{F}^0(\Lambda) = \int_{\mathbb{R}^d}|g|_\infty\,d\mathcal{H}^1\llcorner L = \mathcal{H}^1(L)\,,\qquad \mathcal{F}^\alpha(\Lambda) = \int_{\mathbb{R}^d}|g|_{1/\alpha}\,d\mathcal{H}^1\llcorner L = \int_L|\theta|^\alpha d\mathcal{H}^1\,,$$

where $\theta(x) = \sum_i g_i(x)^{1/\alpha} = \sum_i g_i(x)\in\mathbb{Z}$, and $0\le\theta(x)\le M$. We thus recognize that minimizing the functional $\mathcal{F}^\alpha$ among graphs $L$ connecting $P_1,\dots,P_{N-1}$ to $P_N$ solves the irrigation problem with sources $P_1,\dots,P_{N-1}$ and target $P_N$ (see [23]), while minimizing $\mathcal{F}^0$ among graphs $L$ with endpoints set $\{P_1,\dots,P_N\}$ solves (STP) in $\mathbb{R}^d$.

Since both $\mathcal{F}^\alpha$ and $\mathcal{F}^0$ are total variation-type functionals (thanks to the key coerciveness property 2.3), minimizers do exist in the class of rank one tensor valued measures. The fact that the minimization problem within the class of canonical tensor valued measures representing acyclic graphs has a solution in that class is a consequence of compactness properties of Lipschitz maps (in $\mathbb{R}^2$, it follows alternatively by the compactness theorem in the $SBV$ class [5]). Actually, existence of minimizers in the canonically oriented graph class in $\mathbb{R}^2$ can be deduced as a byproduct of our $\Gamma$-convergence result (see Corollary 3.7 and Corollary 3.8) and in $\mathbb{R}^d$, for $d>2$, by the parallel $\Gamma$-convergence analysis contained in the companion paper [11].

**Remark 2.5** A minimizer of $\mathcal{F}^0$ (resp. $\mathcal{F}^\alpha$) among tensor valued measures $\Lambda$ representing admissible graphs corresponds necessarily to the canonical representation of a minimal graph, i.e. $g_i\ge 0\ \forall\,1\le i\le N-1$. Indeed if $g_i = -g_j$ on a connected arc in $\lambda_i\cap\lambda_j$, with $\lambda_i$ going from $P_i$ to $P_N$ and $\lambda_j$ going from $P_j$ to $P_N$, this implies that $\lambda_i\cup\lambda_j$ contains a cycle, hence $\Lambda$ cannot be a minimizer.

We conclude this section by observing in the following proposition that the Steiner tree problem can be seen as the limit of irrigation problems (cf. [29], [23]).

**Proposition 2.6** *The functional $\mathcal{F}^0$ is the $\Gamma$-limit, as $\alpha \to 0$, of the functionals $\mathcal{F}^\alpha$ with respect to the convergence of measures.*

**Proof:** Observe that $|g|_p \leq |g|_q$ for any $1 \leq q < p \leq +\infty$, $g \in \mathbb{R}^{N-1}$, and moreover $|g|_q \to |g|_\infty$ as $q \to +\infty$. Hence, we have that, for the $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, $\mathcal{F}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |g|_{1/\alpha} d\mathcal{H}^1 \llcorner L$ is a monotonic decreasing sequence as $\alpha \to 0$, so that $\mathcal{F}^\alpha \xrightarrow{\Gamma} \mathcal{F}^0$ by elementary properties of $\Gamma$-convergence, see for instance Remark 1.40 of [15]. $\square$

### 2.3 Acyclic graphs and partitions of $\mathbb{R}^2$

This section is dedicated to the two-dimensional case. The following result, which is an instance of the constancy theorem for currents or the Poincaré's lemma for distributions (see [21]), states that two acyclic graphs having the same endpoints set give rise to a partition of $\mathbb{R}^2$, or equivalently (see [5]), that their oriented difference corresponds to the orthogonal distributional gradient of a piecewise integer valued function having bounded total variation.

**Lemma 2.7** *Let $\{P, R\} \subset \mathbb{R}^2$ and let $\lambda$, $\gamma$ be simple rectifiable curves from $P$ to $R$ oriented by $\mathcal{H}^1$-measurable unit vector fields $\tau'$, $\tau''$. Define as above $\Lambda = \tau' \otimes \mathcal{H}^1 \llcorner \lambda$ and $\Gamma = \tau'' \otimes \mathcal{H}^1 \llcorner \gamma$.*

*Then there exists a function $u \in SBV(\mathbb{R}^2; \mathbb{Z})$ such that, denoting $Du$ and $Du^\perp$ respectively the measures representing the gradient and the orthogonal gradient of $u$, we have $Du^\perp = \Gamma - \Lambda$.*

**Proof:** Consider simple oriented polygonal curves $\lambda_k$ and $\gamma_k$ connecting $P$ to $R$ such that the Hausdorff distance to respectively $\lambda$ and $\gamma$ is less than $\frac{1}{k}$ and the length of $\lambda_k$ (resp. $\gamma_k$) converges to the length of $\lambda$ (resp. $\gamma$). We can also assume without loss of generality that $\lambda_k$ and $\gamma_k$ intersect only transversally in a finite number of points $m_k \geq 2$. Let $\tau'_k$, $\tau''_k$ be the $\mathcal{H}^1$-measurable unit vector fields orienting $\lambda_k$, $\gamma_k$ and define the measures $\Lambda_k = \tau'_k \otimes \mathcal{H}^1 \llcorner \lambda_k$ and $\Gamma_k = \tau''_k \otimes \mathcal{H}^1 \llcorner \gamma_k$.

For a given $k \in \mathbb{N}$ consider the closed curve $\sigma_k = \lambda_k \cup \gamma_k$ oriented by $\tau_k = \tau''_k - \tau'_k$ (i.e. we reverse the orientation of $\lambda_k$). Fix a direction $e \in \mathbb{R}^2$ and a vector $v \in \mathbb{R}^2$ so that the line $r(t) = v + te$, $t \in \mathbb{R}$, intersects $\sigma_k$ transversally at $x_j = r(t_j)$, for $t_1 < t_2 < \cdots < t_M$. Fix $s_0 < t_1$ and set $u_k(r(s_0)) = 0$, and $u_k(x) = 0$ for $x$ in the connected component of $\mathbb{R}^2 \setminus \sigma_k$ containing $r(s_0)$. For $j \geq 1$ fix $s_j \in (t_j, t_{j+1})$ and set $u_k(r(s_j)) = u_k(r(s_{j-1})) - \operatorname{sign}(e \cdot \tau_k(x_j)^\perp)$. Extend $u_k$ to be piecewise constant to the connected component of $\mathbb{R}^2 \setminus \sigma_k$ containing $r(s_j)$. Fix now a new direction $e$ and a new vector $v$ and repeat the procedure until $u_k$ remains defined on the whole of $\mathbb{R}^2 \setminus \sigma_k$.

The map $u_k$ is well defined. Indeed suppose $y_1$ and $y_2$ belong to the same connected component of $\mathbb{R}^2 \setminus \sigma_k$ and consider the arc $\beta$ connecting them in the complement of $\sigma_k$. Let $r(t) = ty_2 + (1-t)y_1$, $t \in \mathbb{R}$, be the line passing through $\{y_1, y_2\}$ and suppose w.l.o.g.

that it intersects $\sigma_k$ transversally at $x_j = r(t_j)$, for $t_1 < t_2 < \cdots < t_M$. By construction we have

$$u_k(y_1) - u_k(y_2) = \sum_{j \text{ s.t. } 0<t_j<1} \operatorname{sign}((y_2 - y_1) \cdot \tau_k(x_j)^\perp).$$

On the other hand the arc $\beta$ together with the segment $[y_1, y_2]$ form an oriented boundary $e = \partial B$, so that by the Green formula we have

$$0 = \int_B \operatorname{div} \Sigma_k^\varepsilon = \int_{\partial B} \Sigma_k^\varepsilon \cdot \nu_e$$

where $\Sigma_k^\varepsilon = \Sigma_k * \eta_\varepsilon$ is a regularization of the measure $\Sigma_k = \Gamma_k - \Lambda_k$ and $\nu_e$ is the exterior normal to $B$. It follows, passing to the limit as $\varepsilon \to 0$, after direct computations,

$$0 = \sum_{j \text{ s.t. } 0<t_j<1} \operatorname{sign}(\nu_e(x_j) \cdot \tau_k(x_j)) = \sum_{j \text{ s.t. } 0<t_j<1} \operatorname{sign}(\nu_e(x_j)^\perp \cdot \tau_k(x_j)^\perp),$$

and since $\nu_e(x_j)^\perp = \pm(y_2 - y_1)$ we have $u_k(y_2) = u_k(y_1)$.

We deduce that $Du_k^\perp = \tau_k \otimes \mathcal{H}^1 \llcorner \sigma_k = \Gamma_k - \Lambda_k$, hence $|Du_k|(\mathbb{R}^2) = \mathcal{H}^1(\sigma_k)$ and $\|u_k\|_{L^1(\mathbb{R}^2)} \leq C|Du_k|(\mathbb{R}^2)$ by Poincaré's inequality in $BV$. Hence $u_k \in SBV(\mathbb{R}^2; \mathbb{Z})$ is an equibounded sequence in norm, and by Rellich compactness theorem there exists a subsequence still denoted $u_k$ converging in $L^1(\mathbb{R}^2)$ to a $u \in SBV(\mathbb{R}^2; \mathbb{Z})$. Taking into account that we have $Du_k^\perp = \Gamma_k - \Lambda_k$, we deduce in particular that $Du^\perp = \Gamma - \Lambda$ as desired. $\square$

**Remark 2.8** Let $A \subset \mathbb{R}^2$ as above. For $i = 1, ..., N-1$ let $\gamma_i$ be the segment joining $P_i$ to $P_N$, denote $\tau_i = \frac{P_N - P_i}{|P_N - P_i|}$ its orientation, and identify $\gamma_i$ with the vector measure $\Gamma_i = \tau_i \otimes \mathcal{H}^1 \llcorner \gamma_i$. Then $G = \cup_{i=1}^{N-1} \gamma_i$ is an acyclic graph connecting the endpoints set $A$ and $\mathcal{H}^1(G) = |\Gamma|(\mathbb{R}^2)$, where $|\Gamma| = \sup_i |\Gamma_i|$.

Taking into account Lemma 2.7 we have

**Corollary 2.9** *Let $A = \{P_1, \ldots, P_N\} \subset \mathbb{R}^2$ be a set of terminal points and $G \in \mathcal{G}(A)$ (for instance the acyclic graph considered in Remark 2.8). For any acyclic graph $L \in \mathcal{G}(A)$, denoting $\Gamma$ (resp. $\Lambda$) the canonical tensor valued representation of $G$ (resp. $L$), we have*

$$\mathcal{H}^1(L) = \int_{\mathbb{R}^2} \sup_i |\Lambda_i| = \int_{\mathbb{R}^2} \sup_i |Du_i^\perp - \Gamma_i|$$

*for suitable $u_i \in SBV(\mathbb{R}^2; \mathbb{Z})$, $1 \leq i \leq N-1$.*

Thus, fixing the family of measures $\Gamma = (\Gamma_1, \ldots, \Gamma_{N-1})$ as in Remark 2.8, we are led to consider the minimization for $U = (u_1, \ldots, u_{N-1}) \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ of the functional

$$\mathcal{F}^0(U) \equiv \mathcal{F}^0(DU^\perp - \Gamma) = \int_{\mathbb{R}^2} \sup_i |Du_i^\perp - \Gamma_i|.$$

We have already seen that to each acyclic graph $L \in \mathcal{G}(A)$ we can associate a $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ such that $\mathcal{H}^1(L) = \mathcal{F}^0(U)$. Moreover, for minimizers, we have the following

**Remark 2.10** To each minimizer $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ of $\mathcal{F}^0$ we can find an acyclic graph $L$ connecting the terminal points $P_1,\dots,P_N$ and such that $\mathcal{F}^0(U) = \mathcal{H}^1(L)$.

To prove this fact, let $U = (u_1,\dots,u_{N-1})$ be a minimizer of $\mathcal{F}^0$ in $SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$, and denote $\Lambda_i = \Gamma_i - Du_i^{\perp}$. Then $\Lambda_i = \tau_i \otimes \mathcal{H}^1 \llcorner \lambda_i$ and $\lambda_i$ necessarily contains a simple rectifiable curve $\lambda_i'$ connecting $P_i$ to $P_N$ since we have $\operatorname{div}\Lambda_i = \delta_{P_i} - \delta_{P_N}$ (use for instance the decomposition theorem for rectifiable 1-currents, as in [23]). Consider the canonical rank one tensor measure $\Lambda'$ associated to the acyclic subgraph $L' = \lambda_1' \cup \cdots \cup \lambda_{N-1}'$ connecting $P_1,\dots,P_{N-1}$ to $P_N$. Then by Lemma 2.7, there exists $U' = (u_1',\dots,u_{N-1}') \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ such that $Du_i'^{\perp} = \Gamma_i - \Lambda_i'$ and in particular $\mathcal{H}^1(L') = \mathcal{F}^0(U') \le \mathcal{F}^0(U)$.

We thus have a relationship between (STP) and the minimization of $\mathcal{F}^0$ over functions in $SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$, and a similar connection can be made between the $\alpha$-irrigation problem and minimization over the same space of

$$\mathcal{F}^\alpha(U) \equiv \mathcal{F}^\alpha(DU^\perp - \Gamma).$$

The aim of the next section is then to provide an approximation of the functionals $\mathcal{F}^\alpha$ in the sense of $\Gamma$-convergence through Modica-Mortola type energies.

**Remark 2.11** In the case $P_1,\dots,P_N \in \partial\Omega$ with $\Omega \subset \mathbb{R}^2$ a convex set, we may choose $G = \cup_{i=1}^{N-1}\gamma_i$ with $\gamma_i$ connecting $P_i$ to $P_N$ and $\operatorname{spt}\gamma_i \subset \partial\Omega$. We deduce by Corollary 2.9 that for any acyclic graph $L \in \mathcal{G}(A)$

$$\mathcal{H}^1(L) = \int_\Omega \sup_i |Du_i^\perp|$$

for suitable $u_i \in SBV(\Omega;\mathbb{Z})$ such that (in the trace sense) $u_i = 1$ on $\gamma_i \subset \partial\Omega$ and $u_i = 0$ elsewhere in $\partial\Omega$, $1 \le i \le N-1$. We recover here an alternative formulation of the optimal partition problem in a convex planar set $\Omega$ as studied for instance in [3] and [4].

## 3 Γ-convergence

In this section we state and prove our main Γ-convergence result, namely Proposition 3.3 and Theorem 3.9.

**Lemma 3.1** *Let $\Psi$ be a norm on $\mathbb{R}^M$, and $\varepsilon > 0$. Consider positive Radon measures $\mu_{1,\varepsilon},\dots,\mu_{M,\varepsilon}$ and $\mu_1,\dots,\mu_M$ satisfying*

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^d} \varphi\, d\mu_{i,\varepsilon} \ge \int_{\mathbb{R}^d} \varphi\, d\mu_i \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}^d),\ \varphi \ge 0,\ \forall\, 1 \le i \le M\,. \tag{3.1}$$

*Then, setting $\vec{\mu}_\varepsilon = (\mu_{1,\varepsilon},\dots,\mu_{M,\varepsilon})$ and $\vec{\mu} = (\mu_1,\dots,\mu_M)$, we have*

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^d} \varphi\, d\Psi(\vec{\mu}_\varepsilon) \ge \int_{\mathbb{R}^d} \varphi\, d\Psi(\vec{\mu}) \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d),\ \varphi \ge 0. \tag{3.2}$$

**Proof:** Fix $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi \geq 0$. By (2.4) we deduce that

$$\int_{\mathbb{R}^d} \varphi d\Psi(\vec{\mu}_\varepsilon) = \sup \left\{ \sum_i \int_{\mathbb{R}^d} \varphi \xi_i d\mu_{i,\varepsilon}\,, \quad \xi_i \geq 0, \quad \Psi^*(\vec{\xi}) \leq 1 \right\}\,,$$

so that

$$\begin{aligned} \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^d} \varphi\, d\Psi(\vec{\mu}_\varepsilon) &\geq \liminf_{\varepsilon\to 0} \sum_i \int_{\mathbb{R}^d} \varphi \xi_i d\mu_{i,\varepsilon} \\ &\geq \sum_i \int_{\mathbb{R}^d} \varphi \xi_i\, d\mu_i \end{aligned} \tag{3.3}$$

for any $\xi_i \geq 0$, $\Psi^*(\vec{\xi}) \leq 1$, whence, again by (2.4), conclusion (3.2) follows.

□

**Lemma 3.2** *Let $\Psi$ be a norm on $\mathbb{R}^M$, and $\varepsilon > 0$. Consider positive Radon measures $\mu_{1,\varepsilon}, \ldots, \mu_{M,\varepsilon}$ and $\mu_1, \ldots, \mu_M$ satisfying*

*(i) $\mu_{i,\varepsilon} \rightharpoonup \mu_i$ and $\mu_{i,\varepsilon}(\mathbb{R}^d) \to \mu_i(\mathbb{R}^d)$ for all $i = 1, \ldots, M$,*

*(ii) there exist $M$ measurable functions $\varphi_1, \ldots, \varphi_M$ such that*

$$\int_{\mathbb{R}^d} d\Psi(\vec{\mu}) = \sum_{i=1}^M \int_{\mathbb{R}^d} \varphi_i\, d\mu_i \tag{3.4}$$

*and*

$$\int_{\mathbb{R}^d} d\Psi(\vec{\mu}_\varepsilon) = \sum_{i=1}^M \int_{\mathbb{R}^d} \varphi_i\, d\mu_{i,\varepsilon} + o(1). \tag{3.5}$$

*Then, setting $\vec{\mu}_\varepsilon = (\mu_{1,\varepsilon}, ..., \mu_{M,\varepsilon})$ and $\vec{\mu} = (\mu_1, ..., \mu_M)$, we have*

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^d} \varphi\, d\Psi(\vec{\mu}_\varepsilon) \leq \int_{\mathbb{R}^d} \varphi\, d\Psi(\vec{\mu}) \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d),\ \varphi \geq 0. \tag{3.6}$$

**Proof:** Fix $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi \geq 0$. By direct calculations

$$\begin{aligned} \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^d} \varphi\, d\Psi(\vec{\mu}_\varepsilon) &\overset{(3.5)}{=} \limsup_{\varepsilon\to 0} \sum_i \int_{\mathbb{R}^d} \varphi_i \varphi\, d\mu_{i,\varepsilon} \leq \sum_i \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^d} \varphi_i \varphi\, d\mu_{i,\varepsilon} \\ &\overset{(i)}{=} \sum_i \int_{\mathbb{R}^d} \varphi_i \varphi\, d\mu_i \overset{(3.4)}{=} \int_{\mathbb{R}^d} \varphi\, d\Psi(\vec{\mu}). \end{aligned}$$

□

We consider now Modica–Mortola functionals for functions having a prescribed jump part. Due to summability issues for the absolutely continuous part of the gradient, we work in local spaces.

**Proposition 3.3** *For any fixed $i \in \{1, ..., N-1\}$ consider the measure $\Gamma_i = \tau_i \otimes \mathcal{H}^1 \llcorner \gamma_i$ defined in Remark 2.8, and consider the Modica-Mortola type functionals*

$$F^i_\varepsilon(u, B) = \int_B e^i_\varepsilon(u)\, dx = \int_B |Du^\perp - \Gamma_i|^2 + \frac{1}{\varepsilon^2} W(u)\, dx, \tag{3.7}$$

*defined for $u \in H_i \equiv W^{1,2}_{loc}(\mathbb{R}^2 \setminus \gamma_i) \cap BV(\mathbb{R}^2)$ and $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$ open, where $W(u) \geq 0$ is a smooth 1-periodic potential vanishing on $\mathbb{Z}$ (take for example, $W(u) = \sin^2(\pi u)$). Let $c_0 = 2\int_0^1 \sqrt{W(s)}\, ds$ ($c_0 = \frac{2}{\pi}$ for $W(s) = \sin^2(\pi s)$). We have:*

1. **(Compactness and lower bound inequality)** *For any $u_{i,\varepsilon} \in W^{1,2}_{loc}(\mathbb{R}^2 \setminus \gamma_i) \cap BV(\mathbb{R}^2)$ such that $F^i_\varepsilon(u_{i,\varepsilon}, B) \leq C(B)$ for any $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$ open, there exists $u_i \in SBV(\mathbb{R}^2; \mathbb{Z})$ such that (up to a subsequence) $u_{i,\varepsilon} \to u_i$ in $L^1(\mathbb{R}^2)$. Moreover,*

$$\liminf_{\varepsilon \to 0} \int \varepsilon \varphi_i e^i_\varepsilon(u_{i,\varepsilon})\, dx \geq c_0 \int \varphi_i d|Du_i^\perp - \Gamma_i| \tag{3.8}$$

*for any $\varphi_i \in C^\infty_c(\mathbb{R}^2 \setminus \{P_1, ..., P_N\})$, $\varphi_i \geq 0$.*

2. **(Upper bound (in)-equality)** *For any $\Lambda_i = \tau \otimes \mathcal{H}^1 \llcorner \lambda_i$, with $\lambda_i$ a simple rectifiable curve joining $P_i$ to $P_N$, let $u_i \in SBV(\mathbb{R}^2; \mathbb{Z})$, such that $Du_i^\perp = \Gamma_i - \Lambda_i$. Then there exists a sequence $u_{i,\varepsilon} \in W^{1,2}_{loc}(\mathbb{R}^2 \setminus \gamma_i) \cap BV(\mathbb{R}^2)$ s.t. $u_{i,\varepsilon} \to u_i$ in $L^1(\mathbb{R}^2)$ and*

$$\limsup_{\varepsilon \to 0} \int \varepsilon \varphi_i e^i_\varepsilon(u_{i,\varepsilon})\, dx \leq c_0 \int \varphi_i d|Du_i^\perp - \Gamma_i| \tag{3.9}$$

*for any $\varphi_i \in C^\infty_c(\mathbb{R}^2 \setminus \{P_1, ..., P_N\})$, $\varphi_i \geq 0$.*

**Remark 3.4** Observe that energy-bounded sequences $u_{i,\varepsilon} \in H_i$, that are involved in statements 1) and 2) of Proposition 3.3, verify $\lim_{|x| \to +\infty} u_{i,\varepsilon}(x) = 0$ for any $\varepsilon > 0$, $1 \leq i \leq N-1$.

**Remark 3.5** Proposition 3.3 holds true also in case the measures $\Gamma_i$ are associated to oriented simple polyhedral (or even rectifiable) finite length curves joining $P_i$ to $P_N$.

**Remark 3.6** To avoid statements in local energy spaces in Proposition 3.3 one could consider variants of the functionals $F^i_\varepsilon$ by relying on suitable smoothings $\Gamma_{i,\epsilon} = \Gamma_i * \eta_\varepsilon$ of the measures $\Gamma_i$, with $\eta_\varepsilon$ a symmetric mollifier.

**Proof of Proposition 3.3** Observe first that by the localization property of $\Gamma$-convergence (see [15]) it suffices to prove (3.8) in the case $\varphi_i = \mathbf{1}_B$ the characteristic function of an arbitrary open set (say a ball) $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$. Then the conclusion follows by approximating the test function $\varphi_i$ by simple functions $\sum_k \alpha_k \mathbf{1}_{B_k}$, with $\{B_k\}_k$ a disjoint family of open sets $B_k$, thanks to the regularity of the Radon measures involved in the statement.

We thus fix a ball $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$, and we distinguish two cases, according to whether $B \cap \gamma_i = \emptyset$ or not. In the first case we have

$$F^i_\varepsilon(u, B) \equiv \int_B e^i_\varepsilon(u)\, dx = \int_B |Du^\perp|^2 + \frac{1}{\varepsilon^2} W(u)\, dx$$

and (3.8) follows by the classical result of Modica-Mortola [26] applied to $F^i_\varepsilon(u, B)$.

In the case $B \cap \gamma_i \neq \emptyset$ we follow the arguments of [8], and consider $u_0 = \mathbf{1}_{B^+}$, where $B^+ = \{z \in B \setminus \gamma_i \, : \, (z - z_0) \cdot \nu > 0\}$, for $z_0 \in B \cap \gamma_i$ and $\nu^\perp = \tau_i$, so that $Du_0^\perp = \Gamma_i \llcorner B$. Letting $v = u - u_0$ we have $Dv^\perp = Du^\perp - \Gamma_i$ on $B$ and $W(v) = W(u)$ on $B$ by 1-periodicity of the potential $W$. Hence

$$F^i_\varepsilon(u, B) = \int_B e^i_\varepsilon(u)\, dx = \int_B |Dv^\perp|^2 + \frac{1}{\varepsilon^2} W(v)\, dx \equiv \tilde{F}^i_\varepsilon(v, B)$$

and conclusion (3.8) follow again by applying Modica-Mortola theorem [26] to $\tilde{F}^i_\varepsilon(v, B)$.

To prove (3.9) we may consider w.l.o.g. that $\lambda_i$ is a polyhedral arc made by segments joining the points $P_i = S_0, S_1, \ldots, S_k = P_N$, and that it intersects $\gamma_i$ orthogonally in a finite number of points (it is then possible to conclude in the general case by a density argument). To construct the approximating $u_{i,\varepsilon}$ consider a tubular neighbourhood $U_\delta$ of $\lambda_i$, with $\delta = O(\varepsilon)$ and consider maximal rectangular $\delta$-strips of the segments $[S_j, S_{j+1}]$ contained in $U_\delta \setminus \cup_j B_{\delta'}(S_j)$, for suitable $\delta' = C\delta$, where to perform the classical Modica-Mortola optimal smooth interpolation along the orthogonal direction of each segment $[S_j, S_{j+1}]$ in order to match the values of $u_i$ on the boundary of the strips, then extend smoothly (e.g. a Lipschitz extension) of the resulting map on $\partial B_{2\delta'}(S_j)$ on the whole of $B_{2\delta'}(S_j)$, for any $j = 0, ..., k$. We have that $u_{i,\varepsilon} \in SBV(\mathbb{R}^2)$ and moreover $F^i_\varepsilon(u_{i,\varepsilon}, B) < +\infty$ for any open subset $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$ provided $\varepsilon$ is sufficiently small. One also deduces $u_{i,\varepsilon} \to u_i$ in $L^1(\mathbb{R}^2)$ and $Du_{i,\varepsilon} \rightharpoonup Du_i$ as measures in $\mathbb{R}^2$, with $|Du_{i_\varepsilon}|(\mathbb{R}^2) \to |Du_i|(\mathbb{R}^2)$, by the Modica-Mortola theorem, whence (3.9) follows.

□

**Corollary 3.7 (Γ-convergence)** *Let $\Psi : \mathbb{R}^{N-1} \to [0, +\infty)$ be a norm on $\mathbb{R}^{N-1}$. In the notation of Proposition 3.3, let $H = H_1 \times \cdots \times H_{N-1}$, and consider the functionals*

$$\mathcal{F}^\Psi_\varepsilon(U, B) = \int_B \varepsilon \Psi(\vec{e}_\varepsilon(U))\, dx\,, \qquad \textit{for } U = (u_1, ..., u_{N-1}) \in H, \tag{3.10}$$

$$\mathcal{F}^\Psi(U, B) = \int_B \Psi(g) d|DU^\perp - \Gamma|_1\,, \qquad \textit{for } U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1}), \tag{3.11}$$

*for $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$ open, where we set $\vec{e}_\varepsilon(U) = (e^1_\varepsilon(u_1), ..., e^{N-1}_\varepsilon(u_{N-1}))$, $g = (g_1, ..., g_{N-1})$ and, for $1 \le i \le N-1$,*
*$|Du_i^\perp - \Gamma_i| = g_i |DU^\perp - \Gamma|_1$, with $|DU^\perp - \Gamma|_1 := \sum_{i=1}^{N-1} |Du_i^\perp - \Gamma_i|$. Then we have*

1. **(Compactness and lower bound inequality)** *For all $U_\varepsilon \in H$ such that $\mathcal{F}^\Psi_\varepsilon(U_\varepsilon, B) \le C(B)$, $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$, there exists $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ such that (up to a subsequence) $U_\varepsilon \to U$ in $L^1(\mathbb{R}^2; \mathbb{R}^{N-1})$. Moreover,*

$$\liminf_{\varepsilon \to 0} \mathcal{F}^\Psi_\varepsilon(U_\varepsilon, B) \ge c_0 \mathcal{F}^\Psi(U, B) \tag{3.12}$$

2. **(Upper bound (in)-equality)** *Let $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ be a rank one tensor valued measure canonically representing an acyclic graph $L$ connecting $P_1, ..., P_N$, and let $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ such that $Du_i^\perp = \Gamma_i - \Lambda_i$ for any $i = 1, \ldots, N-1$. Then there exists a sequence $U_\varepsilon \in H$ such that $U_\varepsilon \to U$ in $L^1(\mathbb{R}^2; \mathbb{R}^{N-1})$ and*

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon^\Psi(U_\varepsilon, B) \le c_0 \mathcal{F}^\Psi(U, B) \tag{3.13}$$

*for any open subset $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$.*

**Proof:** To prove (3.12) apply Lemma 3.1, formula (3.2) to the measures $\mu_\varepsilon^i = \varepsilon e_\varepsilon^i(u_{i,\varepsilon})dx$ and $\mu_i = c_0|Du_i^\perp - \Gamma_i| = c_0 g_i |DU^\perp - \Gamma|_1$ that verify (3.1) in view of (3.8).

To prove (3.13), consider w.l.o.g. that each $\Lambda_i = \tau_i \otimes \mathcal{H}^1 \llcorner \lambda_i$ is such that $\lambda_i$ is a polyhedral curve and intersects orthogonally $\gamma_i$ in a finite number of points. Then the support of the measure $\Lambda$ is an acyclic polyhedral graph (oriented by $\tau$) with edges $E_0, \ldots, E_M$ and vertices $\{S_0, \ldots, S_\ell\}$ such that $E_k = [S_{k_1}, S_{k_2}]$ for suitable indexes $k_1, k_2 \in \{0, \ldots, \ell\}$. By finiteness there exists $\eta > 0$ such that given any edge $E_k$ of that graph the sets

$$V_k = \{x \in \mathbb{R}^2, \mathrm{dist}(x, E_k) < \min\{\eta,\, \eta \cdot \mathrm{dist}(x, S_{k_1}),\, \eta \cdot \mathrm{dist}(x, S_{k_2})\}\}$$

are disjoint and their union form an open neighbourhood of $\cup_i \lambda_i \setminus \{S_0, \ldots, S_\ell\}$.

Remark that on $E_k$ we have $\Psi(g) = g \cdot c_k$, for a suitable vector $c_k$ such that $\Psi^*(c_k) = 1$. We are led to define, for $i = 1, \ldots, N-1$, the piecewise constant functions $\varphi_i(x) = c_{k,i}$ whenever $x \in V_k$, and $\varphi_i(x) = 0$ elsewhere in $\mathbb{R}^2$. With respect to those functions, the measures $\mu_i = c_0|\Lambda_i|$ verify (3.4). Denote $\vec{\varphi} = (\varphi_1, ..., \varphi_{N-1})$ and for further convenience consider also the constant extension of $g$ to $V_k$ for any $k$, and to zero outside, and denote again by $g$ this function.

To construct the approximating $u_{i,\varepsilon}$ consider as before a tubular neighbourhood $U_\delta$ of $\cup_k E_k$, with $\delta = O(\varepsilon)$, and consider maximal rectangular $\delta$-strips $R_k$ of the segments $E_k$ contained in $U_\delta \setminus \cup_k B_{\delta'}(S_k)$, for a suitable $\delta' = C\delta$ (for $\varepsilon$ small enough $R_k \subset V_k$). Perform on each $R_k$ the classical Modica-Mortola optimal smooth interpolation along the orthogonal direction of the segment $E_k$ in order to match the values of $u_i$ on the boundary of the strips, then extend it to $\mathbb{R}^2$ by a Lipschitz extension of the resulting map on $\partial B_{2\delta'}(S_k)$ on the whole of $B_{2\delta'}(S_k)$, for any $k = 0, ..., \ell$.

Let $\mu_{i,\varepsilon} = e_\varepsilon^i(u_{i,\varepsilon})dx$, and $\vec{\mu}_\varepsilon = (\mu_{1,\varepsilon}, \ldots, \mu_{N-1,\varepsilon})$. By construction, if $E_k$ is contained in both $\lambda_i \cap \lambda_j$, then the measures $\mu_{i,\varepsilon}$ and $\mu_{j,\varepsilon}$ coincide on $W_{k,\delta'} \equiv V_k \setminus (B_{2\delta'}(S_{k_1}) \cup B_{2\delta'}(S_{k_2}))$, to define a measure $\mu_{\delta',\varepsilon}$ on $W_{\delta'} = \cup_k W_{k,\delta'}$. We have that

$$\vec{\mu}_\varepsilon \llcorner W_{\delta'} = g\mu_{\delta',\varepsilon}$$

and in particular

$$\begin{aligned}
\int_{\mathbb{R}^2} d\Psi(\vec{\mu}_\varepsilon) &= \int_{W_{\delta'}} d\Psi(\vec{\mu}_\varepsilon) + O(\varepsilon) = \int_{W_{\delta'}} \vec{\varphi} \cdot g \, d\mu_{\delta',\varepsilon} + O(\varepsilon) \\
&= \sum_i \int_{W_{\delta'}} \varphi_i d\mu_{i,\varepsilon} + O(\varepsilon) = \sum_i \int_{\mathbb{R}^2} \varphi_i d\mu_{i,\varepsilon} + O(\varepsilon)
\end{aligned}$$

so that the measures $\mu_{i,\varepsilon}$ satisfy 3.5, and also i) and ii) of Lemma 3.2 in view of Proposition 3.3. Hence formula (3.6) applies, and we easily deduce (3.13).

$\square$

**Corollary 3.8 (Convergence of minimizers)** *Let $U_\varepsilon \in H$ be a sequence of minimizers for $\mathcal{F}^\Psi_\varepsilon$ in $H$. Then (up to a subsequence), $U_\varepsilon \to U$ in $L^1(\mathbb{R}^2)$, and $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ is a minimizer of $\mathcal{F}^\Psi$ in $SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$.*

**Proof:** Let $V \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ such that $DV^\perp = \Gamma - \Lambda$, where $\Lambda$ canonically represents an acyclic graph $L \in \mathcal{G}(A)$, and let $V_\varepsilon \in H$ such that $\limsup_{\varepsilon\to 0} \mathcal{F}^\Psi_\varepsilon(V_\varepsilon, B) \le \mathcal{F}^\Psi(V,B)$ for any $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$. Hence we have, since (3.12) holds and $\mathcal{F}^\Psi_\varepsilon(U_\varepsilon, B) \le \mathcal{F}^\Psi_\varepsilon(V_\varepsilon, B)$,

$$\mathcal{F}^\Psi(U,B) \le \liminf_{\varepsilon\to 0} \mathcal{F}^\Psi_\varepsilon(U_\varepsilon, B) \le \limsup_{\varepsilon\to 0} \mathcal{F}^\Psi_\varepsilon(V_\varepsilon, B) \le \mathcal{F}^\Psi(V,B)$$

for any $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$. Given a general $V \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ we can proceed like in Remark 2.10 and find $V'$ such that $DV'^\perp = \Gamma - \Lambda$ and $\mathcal{F}^\Psi(V',B) \le \mathcal{F}^\Psi(V,B)$.

$\square$

Corollaries 3.7 and 3.8 together with Remark 2.10 may be summarized, in case $\mathcal{F}^\Psi$ corresponds respectively to $\mathcal{F}^0$ and $\mathcal{F}^\alpha$ for $0 < \alpha \le 1$, in the following

**Theorem 3.9** *Let $A = \{P_1, .., P_N\} \subset \mathbb{R}^2$ and $\Gamma_i = \tau \otimes \mathcal{H}^1 \llcorner \gamma_i$, for $1 \le i \le N-1$, as in Remark 2.8. For $U = (u_1, ..., u_{N-1})$, $u_i \in W^{1,2}_{loc}(\mathbb{R}^2 \setminus \gamma_i) \cap BV(\mathbb{R}^2)$, define*

$$\mathcal{F}^0_\varepsilon(U,B) = \int_B \varepsilon \sup_{1\le i\le N-1} e^i_\varepsilon(u_i)\, dx\,,$$

*and, for $0 < \alpha \le 1$,*

$$\mathcal{F}^\alpha_\varepsilon(U,B) = \int_B \varepsilon \left( \sum_{i=1}^{N-1} e^i_\varepsilon(u_i)^{1/\alpha} \right)^\alpha dx\,,$$

*where $B \subset \mathbb{R}^2 \setminus \{P_1, ..., P_N\}$ is open and the energy densities $e^i_\varepsilon(u_i)$ are defined in Proposition 3.3, formula (3.7), and let*

$$\mathcal{F}^0(U,B) \equiv \mathcal{F}^0(DU^\perp - \Gamma, B)\,, \qquad \mathcal{F}^\alpha(U,B) \equiv \mathcal{F}^\alpha(DU^\perp - \Gamma, B) \tag{3.14}$$

*be defined as in (2.6) and (2.5). Let moreover $c_0 > 0$ be as defined in Proposition 3.3. Then we have*

$$\mathcal{F}^0_\varepsilon \xrightarrow{\Gamma} c_0\mathcal{F}^0 \qquad \text{and} \qquad \mathcal{F}^\alpha_\varepsilon \xrightarrow{\Gamma} c_0\mathcal{F}^\alpha\,,$$

*where the $\Gamma$-convergence takes place with respect to the strong topology of $L^1(\mathbb{R}^2;\mathbb{R}^{N-1})$. In particular, up to subsequences, minimizers $U_\varepsilon$ of $\mathcal{F}^0_\varepsilon$ converge, as $\varepsilon \to 0$, to $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ with $DU^\perp - \Gamma = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, and $L$ a Steiner Minimal Tree with terminal points in $A$, while minimizers $V_\varepsilon$ of $\mathcal{F}^\alpha_\varepsilon$ converge (up to subsequences), as $\varepsilon \to 0$, to $V \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$, where $DV^\perp - \Gamma = \tau' \otimes g' \cdot \mathcal{H}^1 \llcorner L_\alpha$ represents an optimal $\alpha$-irrigation plan with sources $P_1, \ldots, P_{N-1}$ and target point $P_N$.*

# 4 Convex relaxation

In this section we propose convex positively 1-homogeneous relaxations of the irrigation-type functionals $\mathcal{F}^\alpha$ for $0 \le \alpha < 1$ so as to include the Steiner tree problem corresponding to $\alpha = 0$ (notice that the case $\alpha = 1$ corresponds to the well-known Monge-Kantorovich optimal transportation problem with respect to the Monge cost $c(x, y) = |x - y|$).

More precisely, we consider relaxations of the functional defined by

$$\mathcal{F}^\alpha(\Lambda) = \|\Lambda\|_{\Psi_\alpha} = \int_{\mathbb{R}^d} |g|_{1/\alpha} \, d\mathcal{H}^1 \llcorner L$$

if the measure $\Lambda$ is the canonical representation of an acyclic graph $L$ with terminal points $\{P_1, ..., P_N\} \subset \mathbb{R}^d$, so that in particular, according to Definition 2.1, we can write $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ with $|\tau| = 1$, $g_i \in \{0, 1\}$. For any other $d \times (N-1)$ valued measure $\mu$ on $\mathbb{R}^d$ we set $\mathcal{F}^\alpha(\mu) = +\infty$.

As a preliminary remark observe that, since we are looking for positively 1-homogeneous extensions, the extended functionals $\mathcal{R}^\alpha$ satisfy

$$\mathcal{R}^\alpha(c\Lambda) = |c|\mathcal{F}^\alpha(\Lambda)$$

for any $c \in \mathbb{R}$ and $\Lambda$ as above. As a consequence we have that $\mathcal{R}^\alpha(-\Lambda) = \mathcal{R}^\alpha(\Lambda)$, where $-\Lambda$ represents the same graph $L$ as $\Lambda$ but only with reversed orientation.

## 4.1 Extension to rank one tensor measures

First of all let us discuss the possible positively 1-homogeneous convex relaxations of $\mathcal{F}^\alpha$ on the class of rank one tensor valued Radon measures $\Lambda = \tau \otimes \vec{\mu} = \tau \otimes g \cdot |\vec{\mu}|_1$, where $|\tau| = 1$, $g \in \mathbb{R}^{N-1}$ (cf. Section 2.2).

For a generic $\Lambda = \tau \otimes \vec{\mu} = \tau \otimes g \cdot |\vec{\mu}|_1$ as above, we can consider extensions of the form

$$\mathcal{R}^\alpha(\Lambda) = \int_{\mathbb{R}^d} \Psi^\alpha(g) \, d|\vec{\mu}|_1$$

for a convex positively 1-homogeneous $\Psi^\alpha$ on $\mathbb{R}^{N-1}$ (i.e. a norm) verifying

$$|g|_{1/\alpha} \le \Psi^\alpha(g) \le \Phi^{**}_{\alpha 0}(g) \qquad \text{for all } g \in \mathbb{R}^{N-1}, \tag{4.1}$$

where $\Phi^{**}_{\alpha 0}$ represents the convex positively 1-homogeneous envelope of the function $\Phi_{\alpha 0}(g) := |g|_{1/\alpha}$ for $g_i \in \{0, 1\}$ $\forall i = 1, \dots, N-1$. The expression for $\Phi^{**}_{\alpha 0}$ is given (cf. [23]), for $\alpha > 0$, by

$$\Phi^{**}_{\alpha 0}(g) = \left( \sum_{1 \le i \le N-1} |g_i^+|^{1/\alpha} \right)^\alpha + \left( \sum_{1 \le i \le N-1} |g_i^-|^{1/\alpha} \right)^\alpha, \tag{4.2}$$

and for $\alpha = 0$ by

$$\Phi^{**}_{00}(g) = \sup_{1 \le i \le N-1} g_i^+ - \inf_{1 \le i \le N-1} g_i^-, \tag{4.3}$$

with $g_i^+ = \max\{g_i, 0\}$ and $g_i^- = \min\{g_i, 0\}$. Notice that in [24] the norm $\Psi$ on $\mathbb{R}^{N-1}$ (see Remark 2.4) employed in order to find solutions of (STP) using calibrations corresponds precisely to $\Psi = \Phi_{00}^{**}$.

According to the convex extensions $\Psi^\alpha$ and $\Psi^0$ considered, when it comes to finding minimizers of respectively $\mathcal{R}^\alpha$ and $\mathcal{R}^0$ in suitable classes of weighted graphs with prescribed fluxes at their terminal points, or more generally in the class of rank one tensor valued measures having divergence prescribed by (2.1), the minimizer is not necessarily the canonical representation of an acyclic graph. Let us consider the following example, where the minimizer contains a cycle.

**Example 4.1** Consider the Steiner tree problem for $\{P_1, P_2, P_3\} \subset \mathbb{R}^2$. We claim that the minimizer of $\mathcal{R}^0(\Lambda) = \int_{\mathbb{R}^2} |g|_\infty \, d|\vec{\mu}|_1$ within the class of rank one tensor valued Radon measures $\Lambda = \tau \otimes g \cdot |\vec{\mu}|_1$ satisfying (2.1) is supported on the triangle $L = [P_1, P_2] \cup [P_2, P_3] \cup [P_1, P_3]$, hence it is not acyclic. Denoting $\tau$ the global orientation of $L$ (i.e. from $P_1$ to $P_2$, $P_1$ to $P_3$ and $P_2$ to $P_3$) we actually have as minimizer

$$\Lambda = \tau \otimes \left( \left[\frac{1}{2}, -\frac{1}{2}\right] \cdot \mathcal{H}^1 \llcorner [P_1, P_2] + \left[\frac{1}{2}, \frac{1}{2}\right] \cdot \mathcal{H}^1 \llcorner [P_3, P_2] + \left[\frac{1}{2}, \frac{1}{2}\right] \cdot \mathcal{H}^1 \llcorner [P_3, P_1] \right). \quad (4.4)$$

The proof of the claim follows from Remark 4.2 and Lemma 4.3.

**Remark 4.2 (Calibrations)** A way to prove the minimality of $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ within the class of rank one tensor valued Radon measures satisfying (2.1) is to exhibit a *calibration* for $\Lambda$, i.e. a matrix valued differential form $\omega = (\omega_1, \ldots, \omega_{N-1})$, with $\omega_j = \sum_{i=1}^d \omega_{ij} dx_i$ for measurable coefficients $\omega_{ij}$, such that

- $d\omega_j = 0$ for all $j = 1, \ldots, N-1$;
- $\|\omega\|_* \leq 1$, where $\|\cdot\|_*$ is the dual norm to $\|\tau \otimes g\| = |\tau| \cdot |g|_\infty$, defined as

$$\|\omega\|_* = \sup\{\tau^t \, \omega \, g \;:\; |\tau| = 1, \; |g|_\infty \leq 1\};$$

- $\langle \omega, \Lambda \rangle = \sum_{i,j} \tau_i \omega_{ij} g_j = |g|_\infty$ pointwise, so that

$$\int_{\mathbb{R}^2} \langle \omega, \Lambda \rangle = \mathcal{R}^0(\Lambda).$$

In this way for any competitor $\Sigma = \tau' \otimes g' \cdot |\vec{\mu}'|_1$ we have $\langle \omega, \Sigma \rangle \leq |g'|_\infty$, and moreover $\Sigma - \Lambda = DU^\perp$, for $U \in BV(\mathbb{R}^2; \mathbb{R}^{N-1})$, hence

$$\int_{\mathbb{R}^2} \langle \omega, \Lambda - \Sigma \rangle = \int_{\mathbb{R}^2} \langle \omega, DU^\perp \rangle = \int_{\mathbb{R}^2} \langle d\omega, U \rangle = 0\,.$$

It follows

$$\mathcal{R}^0(\Sigma) \geq \int_{\mathbb{R}^2} \langle \omega, \Sigma \rangle = \int_{\mathbb{R}^2} \langle \omega, \Lambda \rangle = \mathcal{R}^0(\Lambda)\,,$$

i.e. $\Lambda$ is a minimizer within the given class of competitors.

Let us construct a calibration $\omega = (\omega_1, \omega_2)$ for $\Lambda$ in the general case $P_1 \equiv (x_1, 0)$, $P_2 \equiv (x_2, 0)$ and $P_3 \equiv (0, x_3)$, with $x_1 < 0$, $x_1 < x_2$ and $x_3 > 0$.

**Lemma 4.3** *Let $P_1$, $P_2$, $P_3$ defined as above and $\Lambda$ as in (4.4). Consider $\omega = (\omega_1, \omega_2)$ defined as*

$$\omega_1 = \frac{1}{2a}[(x_1 + a)dx + x_3 dy], \quad \omega_2 = \frac{1}{2a}[(x_1 - a)dx + x_3 dy], \quad \textit{for } (x, y) \in B_L$$
$$\omega_1 = \frac{1}{2b}[(x_2 + b)dx + x_3 dy], \quad \omega_2 = \frac{1}{2b}[(x_2 - b)dx + x_3 dy], \quad \textit{for } (x, y) \in B_R$$

*with $B_L$ the left half-plane w.r.t. the line containing the bisector of vertex $P_3$, $B_R$ the corresponding right half-plane and $a = \sqrt{x_1^2 + x_3^2}$, $b = \sqrt{x_2^2 + x_3^2}$. The matrix valued differential form $\omega$ is a calibration for $\Lambda$.*

**Proof:** For simplicity we consider here the particular case $x_1 = -\frac{1}{2}$, $x_2 = \frac{1}{2}$ and $x_3 = \frac{\sqrt{3}}{2}$ (the general case is similar). For this choice of $x_1$, $x_2$, $x_3$ we have

$$\omega_1 = \frac{1}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \omega_2 = -\frac{3}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \text{for } (x, y) \in \mathbb{R}^2,\ x < 0,$$
$$\omega_1 = \frac{3}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \omega_2 = -\frac{1}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \text{for } (x, y) \in \mathbb{R}^2,\ x > 0.$$

The piecewise constant 1-forms $\omega_i$ for $i = 1, 2$ are globally closed in $\mathbb{R}^2$ (on the line $\{x = 0\}$ they have continuous tangential component), $\|\omega\|_* \leq 1$ (cf. Remark 4.2), and taking their scalar product with respectively $(1, 0) \otimes (1/2, -1/2)$, $(-1/2, \sqrt{3}/2) \otimes (1/2, 1/2)$ for $x < 0$ and $(1/2, \sqrt{3}/2) \otimes (1/2, 1/2)$ for $x > 0$ we obtain in all cases $1/2$, i.e. $|g|_\infty$, so that

$$\int_{\mathbb{R}^2} \langle \omega, \Lambda \rangle = \mathcal{R}^0(\Lambda)\,.$$

Hence $\omega$ is a calibration for $\Lambda$.

$\square$

**Remark 4.4** A calibration always exists for minimizers in the class of rank one tensor valued measures as a consequence of Hahn-Banach theorem (see e.g. [24]), while it may be not the case in general for graphs with integer or real weights. The classical minimal configuration for (STP) with 3 endpoints $P_1$, $P_2$ and $P_3$ admits a calibration with respect to the norm $\Phi_{00}^{**}$ in $\mathbb{R}^{N-1}$ (see [24]) and hence it is a minimizer for the relaxed functional $\mathcal{R}^0(\Lambda) = ||\Lambda||_{\Phi_{00}^{**}}$ among all real weighted graphs (and all rank one tensor valued Radon measures satisfying (2.1)). It is an open problem to show whether or not a minimizer of the relaxed functional $\mathcal{R}^0(\Lambda) = ||\Lambda||_{\Phi_{00}^{**}}$ has integer weights.

## 4.2 Extension to general matrix valued measures

Let us turn next to the convex relaxation of $\mathcal{F}^\alpha$ for generic $d \times (N-1)$ matrix valued measures $\mu = (\mu_1, \ldots, \mu_{N-1})$, where $\mu_i$, for $1 \le i \le N-1$, are the vector measures corresponding to the columns of $\mu$. As a first step observe that, due to the positively 1-homogeneous request on $\mathcal{R}^\alpha$, whenever $\Lambda = p \cdot \mathcal{H}^1 \llcorner L = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, with $|\tau| = c \ge 0$ and $g_i \in \{0,1\}$, we must have

$$\mathcal{R}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |\tau||g|_{1/\alpha} \, d\mathcal{H}^1 \llcorner L = \int_{\mathbb{R}^d} \Phi_\alpha(p) \, d\mathcal{H}^1 \llcorner L,$$

with $\Phi_\alpha(p) = |\tau||g|_{1/\alpha}$ is defined only for matrices $p \in K_0$ ($+\infty$ otherwise), where

$$K_0 = \{\tau \otimes g \in \mathbb{R}^{d \times (N-1)}, \ g_i \in \{0,1\}, \ \ |\tau| = c \ge 0\}.$$

Following [18], we look for $\Phi_\alpha^{**}$, the positively 1-homogeneous convex envelope on $\mathbb{R}^{d\times(N-1)}$ of $\Phi_\alpha(\cdot)$. Setting $q = [q_1, \ldots, q_{N-1}]$, with $q_i \in \mathbb{R}^d$ its columns, we have that the convex conjugate function $\Phi_\alpha^*(q) = \sup\{q \cdot p - \Phi_\alpha(p), \ p \in K_0\}$ is given by

$$\begin{aligned}\Phi_\alpha^*(q) &= \sup\left\{ \tau^t \cdot q \cdot g - |\tau| \cdot |g|_{1/\alpha}\,, \quad |\tau| = c \ge 0, \, g = \sum_{i \in J} e_i, \ J \subset \{1, ..., N-1\} \right\} \\ &= \sup\left\{ c \left[ \tau^t \cdot \left( \sum_{j \in J} q_j \right) - |J|^\alpha \right] \,, \quad c \ge 0, \, |\tau| = 1, \, J \subset \{1, ..., N-1\} \right\}.\end{aligned}$$

Hence $\Phi_\alpha^*$ is the indicator function of the convex set

$$K^\alpha = \left\{ q \in \mathbb{R}^{d\times(N-1)}, \ \left| \sum_{i \in J} q_i \right| \le |J|^\alpha \ \ \forall \, J \subset \{1, ..., N-1\} \right\},$$

and in particular, for $\alpha = 0$, it holds (cf. [18])

$$K^0 = \left\{ q \in \mathbb{R}^{d\times(N-1)}, \ \left| \sum_{i \in J} q_i \right| \le 1 \ \ \forall \, J \subset \{1, ..., N-1\} \right\}.$$

It follows that $\Phi_\alpha^{**}$ is the support function of $K^\alpha$, i.e., for $p \in \mathbb{R}^{d\times(N-1)}$,

$$\Phi_\alpha^{**}(p) = \sup_{q \in K^\alpha} p \cdot q = \sup\left\{ p \cdot q \,, \ \left| \sum_{i \in J} q_i \right| \le |J|^\alpha \,, \ J \subset \{1, ..., N-1\} \right\}, \tag{4.5}$$

with $q_1, \ldots, q_{N-1} \in \mathbb{R}^d$ the columns of $q$. We are then led to consider the relaxed functional, for matrix valued test functions $\varphi = (\varphi_1, \ldots, \varphi_{N-1})$,

$$\mathcal{R}^\alpha(\mu) = \int_{\mathbb{R}^d} \Phi_\alpha^{**}(\mu) = \sup\left\{ \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \varphi_i \, d\mu_i, \quad \varphi \in C_c^\infty(\mathbb{R}^d; K^\alpha) \right\}.$$

Observe that for $\mu$ a rank one tensor valued measure the above expression coincides with the one obtained in the previous section choosing $\Psi^\alpha = \Phi^{**}_{0\alpha}$.

In the planar case $d = 2$, consider a $2 \times (N-1)$ matrix valued measure $\mu = (\mu_1, \ldots, \mu_{N-1})$ such that $\operatorname{div} \mu_i = \delta_{P_i} - \delta_{P_N}$. Fix a measure $\Gamma$ as for instance in Remark 2.8. We have $\operatorname{div}(\mu - \Gamma) = 0$ in $\mathbb{R}^2$ and by Poincaré's lemma there exists $U \in BV(\mathbb{R}^2; \mathbb{R}^{N-1})$ such that $\mu = \Gamma - DU^\perp$. So the relaxed functional reads

$$\mathcal{E}^\alpha(U) = \mathcal{R}^\alpha(\mu) \quad \text{for } \mu = \Gamma - DU^\perp,\, U \in BV(\mathbb{R}^2; \mathbb{R}^{N-1}). \tag{4.6}$$

The relaxed irrigation problem $(I_\alpha) \equiv \min_{BV} \mathcal{E}^\alpha(U)$ can thus be described in the following equivalent way, according to (4.5): let $q = \varphi$ be any matrix valued test function (with columns $q_i = \varphi_i$ for $1 \leq i \leq N-1$), then we have

$$(I_\alpha) \equiv \min_{U \in BV(\mathbb{R}^2;\mathbb{R}^{N-1})} \sup \left\{ \int_{\mathbb{R}^2} \sum_{i=1}^{N-1} (Du_i^\perp - \Gamma_i) \cdot \varphi_i \,, \quad \varphi \in C_c^\infty(\mathbb{R}^2; K^\alpha) \right\}.$$

Notice that with respect to the similar formulation proposed in [18], there is here the presence of an additional "drift" term, moreover the constraints set $K^\alpha$ is somewhat different.

We compare now the functional $\mathcal{E}^\alpha(U)$ with the convex envelope $(\mathcal{F}^\alpha)^{**}(U)$ in $BV(\mathbb{R}^2; \mathbb{R}^{N-1})$, where $\mathcal{F}^\alpha(U) := \mathcal{F}^\alpha(\Gamma - DU^\perp)$ if $\Gamma - DU^\perp = \Lambda$ canonically represents an acyclic graph, and $\mathcal{F}^\alpha(U) = +\infty$ elsewhere in $BV(\mathbb{R}^2; \mathbb{R}^{N-1})$. Observe first that $(\mathcal{F}^\alpha)^{**}(U) = \mathcal{E}^\alpha(U)$ whenever $\Lambda = \Gamma - DU^\perp$ canonically represents a graph connecting $P_1, ..., P_N$, so that $\mathcal{E}^\alpha(U) \leq (\mathcal{F}^\alpha)^{**}(U)$ by convexity of $\mathcal{E}^\alpha(U)$. Moreover, we have $(\mathcal{F}^\alpha)^{**}(U) \leq (\mathcal{F}^1)^{**}(U)$, since $\mathcal{F}^\alpha(U) \leq \mathcal{F}^1(U)$. For $\alpha > 0$, denoting $\mu = \Gamma - DU^\perp$, we deduce

$$(\mathcal{F}^1)^{**}(U) = \sum_{i=1}^{N-1} |\mu_i|(\mathbb{R}^d) \leq (N-1)^{1-\alpha} \left( \sum_{i=1}^{N-1} |\mu_i|^{1/\alpha} \right)^\alpha (\mathbb{R}^d) \leq (N-1)^{1-\alpha} \mathcal{E}^\alpha(U),$$

and analogously we have $(\mathcal{F}^1)^{**}(U) \leq (N-1)\mathcal{E}^0(U)$. We obtain (cf. Proposition 3.1 of [18])

**Lemma 4.5** *We have $\mathcal{E}^\alpha(U) \leq (\mathcal{F}^\alpha)^{**}(U) \leq (N-1)^{1-\alpha}\mathcal{E}^\alpha(U)$ for any $U \in BV(\mathbb{R}^2; \mathbb{R}^{N-1})$ and any $0 \leq \alpha < 1$.*

# 5 Numerical identification of optimal structures

## 5.1 Local optimization by Γ-convergence

In this section, we plan to illustrate the use of Proposition 3.3 to identify numerically local minima of the Steiner problem. We base our numerical approximation on a standard discretization of (3.7). Let $\Omega = (0, 1)^2$ and assume that the convex hull of the given points $P_1, \ldots, P_N \in \mathbb{R}^2$ is contained in $\Omega$. As a standard consequence, the associated Steiner

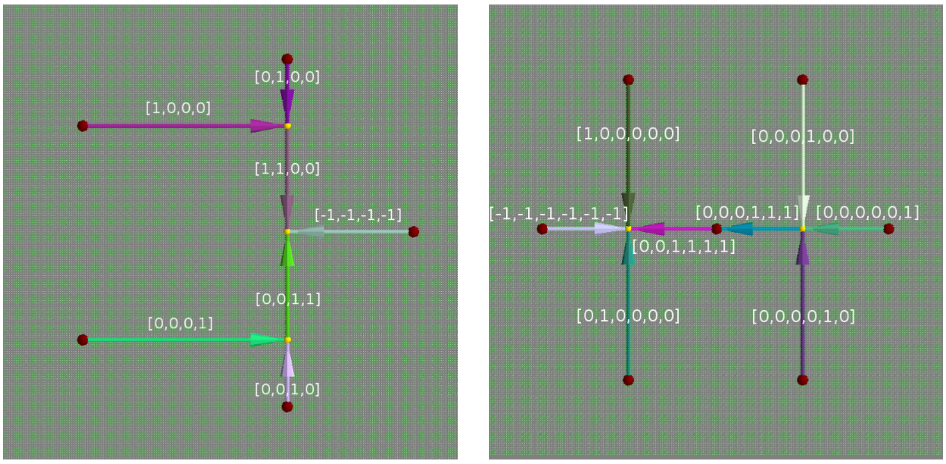


Figure 1: Rectilinear Steiner trees and associated vectorial drifts for five and seven points

tree is also contained in $\Omega$. Consider a Cartesian grid covering $\Omega$ of step size $h = \frac{1}{M}$ where $M > 1$ is a fixed integer. Dividing every square cell of the grid into two triangles, we define a triangular mesh $\mathcal{T}$ associated to $\Omega$ and replace each point $P_i$ with the closest grid point.

Fix now $\Gamma_i$ an oriented vectorial measure absolutely continuous with respect to $\mathcal{H}^1$ as in Remark 2.8. Assume for simplicity that $\Gamma_i$ is supported on $\gamma_i$ a union of vertical and horizontal segments contained in $\Omega$ and covered by the grid associated to the discrete points $\{(kh, lh),\, 0 \le k, l < M\}$. Notice that such a measure can be easily constructed by considering for instance the oriented $\ell^1$-spanning tree of the given points.

To mimic Proposition 3.3, we define the function space

$$H_i^h \equiv P_1(\mathcal{T}, \Omega \setminus \gamma_i) \cap BV(\Omega)$$

to be the set of functions which are globally continuous on $\Omega \setminus \gamma_i$ and piecewise linear on every triangle of $\mathcal{T}$. Moreover, we require that every function of $H_i^h$ has a jump through $\gamma_i$ of amplitude $+1$ in the orthogonal direction of the orientation of $\Gamma_i$. Observe that $H_i^h$ is a finite dimensional space of dimension $M^2$: one element $u_i^h$ can be described by $M^2 + n_i$ parameters and $n_i$ linear constraints describing the jump condition where $n_i$ is the number of grid points covered by $\gamma_i$.

Then, we define the discrete version of (3.7) by

$$G_h^i(u_i^h) = F_h^i(u_i^h) = \int_{\Omega\setminus\gamma_i} f_h^i(u_i^h) = \int_{\Omega\setminus\gamma_i} |Du_i^h|^2 + \frac{1}{h^2} W(u_i^h), \tag{5.1}$$

if $u \in L^1(\Omega)$ is in $H_i^h$ and extend $G_h^i$ by letting $G_h^i(u) = +\infty$ otherwise. Then we define

$$\mathcal{G}_h^0(u_i^h) = \int_{\Omega\setminus\gamma_i} h \sup_{1\le i\le N-1} f_h^i(u_i^h)$$

and

$$\mathcal{G}_h^\alpha(u_i^h) = \int_{\Omega\setminus\gamma_i} h\left(\sum_{i=1}^{N-1} f_h^i(u_i^h)^{1/\alpha}\right)^\alpha.$$

By a similar strategy we used to prove Theorem 3.9, we can establish that we still have

$$\mathcal{G}_h^0 \xrightarrow{\Gamma} c_0\mathcal{F}^0 \qquad \text{and} \qquad \mathcal{G}_h^\alpha \xrightarrow{\Gamma} c_0\mathcal{F}^\alpha$$

with respect to the strong topology of $L^1(\mathbb{R}^2;\mathbb{R}^{N-1})$. Observe that an exact evaluation of the integrals involved in (5.1) is required to obtain this $\Gamma$-convergence result (an approximation formula can also be used but then a theoretical proof of convergence would require to study the interaction of the order of approximation with the $\Gamma$-convergence). We point out that this constraint is not critical from a computational point of view since every function $u_h^i$ of finite energy has a constant gradient on every triangle of the mesh. On the other hand, the potential integral can be evaluated formally to obtain an exact estimate of this term whith respect to the degrees of freedom which describe a function of $H_i^h$.

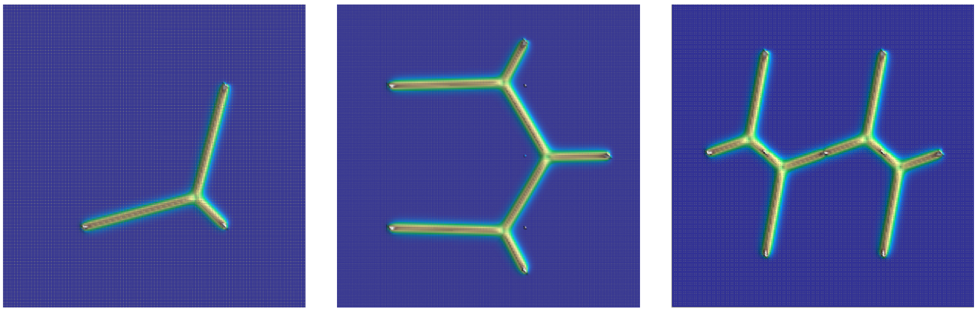

Figure 2: Local minimizers obtained by the $\Gamma$-convergence approach applied to 3, 5 and 7 points

Based on these results we performed two different numerical experiments.

We first approximated the optimal Steiner trees associated to the vertices of a triangle, a regular pentagon and a regular hexagon with its center. To obtain the results of figure 2 we discretized the problem on a grid of size $200 \times 200$. In the case of the triangle we used the associated spanning tree to define the measures $(\Gamma_i)_{i=1,2}$. In the case of the pentagon and of the hexagon we used the rectilinear Euclidean Steiner trees computed by the Geosteiner's library (see [33] for instance) to initiate the vectorial measures. We refer to figure 1 for an illustration of both singular vector fields. We solved the resulting finite dimensional problem using an interior point solver. Notice that in order to deal with the non smooth cost function $\mathcal{G}_h^0(u_i^h)$ we had to introduce standard gap variables to get a smooth non convex constrained optimization problem. Using [17], we have been able to recover the locally optimal solutions depicted in figure 2 in less than five minutes on a standard computer. Whereas the results obtained for the triangle and the pentagon describe globally optimal Steiner trees, the one obtained for the hexagon and its center is only a local minimizer.

In a second experiment we focus on simple irrigation problems to illustrate the versatility of our approach. We applied exactly the same approach to the pentagon setting minimizing the functional $\mathcal{G}_h^\alpha(u_i^h)$. We illustrate our results in figure 3 in which we recover the solutions of Gilbert-Steiner problems for different values of $\alpha$. Observe that for small values of $\alpha$, as expected by Proposition 2.6, we recover an irrigation network close to an optimal Steiner tree.

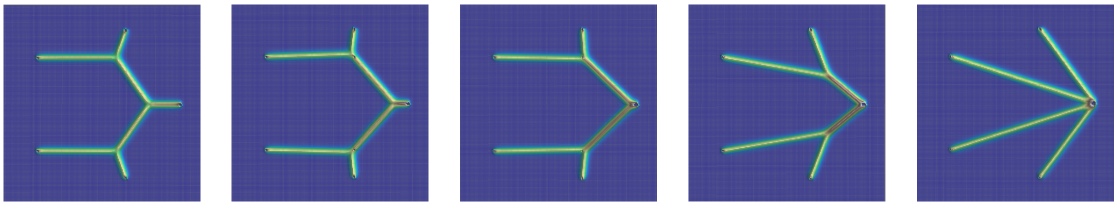

Figure 3: Gilbert-Steiner solutions associated to parameters $\alpha = 0.2$, 0.4, 0.6, 0.8 and 1 (from left to right)

### 5.2 Convex relaxation and multiple solutions

The convex relaxation of Steiner problem $(I_0)$ obtained following [18] reads in our discrete setting as:

$$\min_{(u_i^h)_{1\le i<N}} \sup_{(\varphi_i^h)_{1\le i<N}\in K^0} \frac{h^2}{2}\sum_{t\in\mathcal{T}}\sum_{i=1}^{N-1}(\nabla u_i^h)_t\cdot(\varphi_i^h)_t \tag{5.2}$$

where

$$K^0 = \left\{ (\varphi_i^h)_{1\le i<N} \in (\mathbb{R}^{2\mathcal{T}})^{N-1}\,|\forall J\subset\{1,\dots,N-1\}, \forall t\in\mathcal{T}, \left|\sum_{j\in J}(\varphi_j^h)_t\right| \le 1 \right\} \tag{5.3}$$

and $\forall 1\le i<N$, $u_i^h \in H_i^h$. Applying conic duality (see for instance Lecture 2 of [9]), we obtain that the optimal vector $(u_i^h)$ solves the following minimization problem

$$\min_{(u_i^h)_{1\le i<N}\in L,\,(\psi_J^h)_{J\subset\{1,\dots,N-1\}}\in(\mathbb{R}^{2\mathcal{T}})^{2^{N-1}}} \frac{h^2}{2}\sum_{t\in\mathcal{T}}\sum_{J\subset\{1,\dots,N-1\}}|(\psi_J^h)_t| \tag{5.4}$$

where $L$ is the set of discrete vectors $(u_i^h)_{1\le i<N}$ which satisfy $\forall i=1,\dots,N-1$, $\forall t\in\mathcal{T}$:

$$(\nabla u_i^h)_t = \sum_{J\subset\{1,\dots,N-1\},\, i\in J}(\psi_J^h)_t. \tag{5.5}$$

We solved this convex linearly constrained minimization problem using the conic solver of the library Mosek [28] on a grid of dimension $300\times 300$. Observe that this convex formulation is also well adapted to the, now standard, large scale algorithms of proximal type. We studied four different test cases: the vertices of an equilateral triangle, a square, a pentagon and finally an hexagon and its center as in previous section. As illustrated in the left picture of figure 4, the convex formulation is able to approximate the optimal structure in the case of the triangle. Due to the symmetries of the problems, the three last examples do not have unique solutions. Thus, the result of the optimization is expected to be a convex combination of all solutions whenever the relaxation is sharp, as it can be observed on the second and fourth case of figure 4. Notice that we do not expect this behaviour to hold for any configuration of points. Indeed the numerical solution in the third picture of figure 4 is not supported on a convex combination of global solutions since the density in the middle point is not 0. Whereas the local Γ-convergence approach of previous section was only able to produce a local minimum in the case of the hexagon and its center, the convexified formulation gives a relatively precise idea of the set of optimal configurations (see the last picture of figure 4 where we can recognize within the figure the two global solutions).

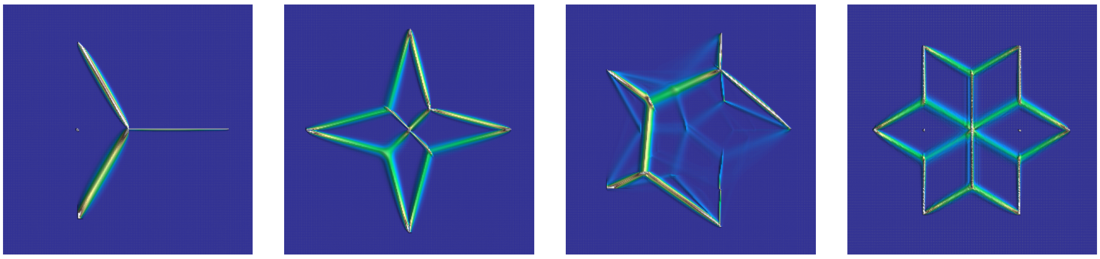

Figure 4: Results obtained by convex relaxation for $3, 4$, 5 and 7 given points

# 6 Generalizations

In this article we have focused on the optimization of one dimensional structures in the plane in specific, classical cases. A first possible generalization is to consider the same problems with respect to more general norms, for instance anisotropic ones: given $|\cdot|_a$ an anisotropic norm on $\mathbb{R}^d$ let respectively $\Phi^a_\alpha(\tau\otimes g) = |\tau|_a \cdot |g|_{1/\alpha}$ and $\Phi^a_0(\tau\otimes g) = |\tau|_a \cdot |g|_\infty$ for $\tau \in \mathbb{R}^d$, $|\tau| = 1$ and $g \in \mathbb{R}^{N-1}_+$, the positive orthant of $\mathbb{R}^{N-1}$. Consider a convex positively 1-homogeneous extension $\Psi_\alpha$ and $\Psi_0$ of resp. $|\cdot|_{1/\alpha}$ and $|\cdot|_\infty$ to the whole of $\mathbb{R}^{N-1}$ and form the functionals, for $0 \le \alpha \le 1$

$$\mathcal{F}^\alpha_a(\Lambda) = \int_L \Phi^a_\alpha(\tau\otimes g)d\mathcal{H}^1 = \int_L |\tau|_a \Psi_\alpha(g) d\mathcal{H}^1 = ||\Lambda||_{a,\alpha}\,. \tag{6.1}$$

where $\Phi^a_\alpha(\tau\otimes g) = |\tau|_a\Psi_\alpha(g)$ denotes the corresponding extension of $\Phi^a_\alpha$ to $\mathbb{R}^2 \otimes \mathbb{R}^{N-1}$. Then minimizers of $\mathcal{F}^\alpha_a$ solve the anisotropic irrigation problem (resp. the anisotropic Steiner tree problem in case $\alpha = 0$), in particular, if $|\cdot|_a = |\cdot|_1$, $\mathcal{F}^0_a$ corresponds to the rectilinear Steiner tree problem in $\mathbb{R}^d$. For $d = 2$, following [14, 30, 2] one may reproduce the Γ-convergence and convex relaxation approach developed here to numerically study the anisotropic problem (6.1). A further step in this direction would consist in considering size or $\alpha$-mass minimization problems in suitable homology and/or oriented cobordims classes for one dimensional chains in manifolds endowed with a finsler metric.

Another generalization concerns the convex relaxation and the variational approximation of (STP) and $(I_\alpha)$ in the higher dimensional case $d \geq 3$. This is done in the companion paper [11], where we obtain a full $\Gamma$-convergence result by using functionals of Ginzburg-Landau type in the spirit of [1] and [32]. Moreover, as in the present paper, we introduce appropriate "local" convex envelopes, discuss calibration principles and show some numerical simulations.

In parallel to previous theoretical generalizations, we are currently developing numerical approaches adapted to these new formulations. On the one hand, we are studying a large scale approach to solve problems analogous to the conic convexified formulation of section 5.2. Such an extension is definitely required to approximate realistic problems in dimension three and higher. On the other hand, we want to focus on refinement techniques which may decrease dramatically the number of degrees of freedom involved in the optimization process. Observe for instance that very few parameters are required to describe exactly a drift as the ones given in Figure 1. Based on such observations, a sequential localized approach may provide a very precise description of, at least locally, optimal structures.

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