Robustness and H_∞ control of MIMO systems

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Outline

1. Some definitions
   - Signal and system norms
2. What is the $\mathscr{H}_\infty$ performance?
   - The $\mathscr{H}_\infty$ norm definition
   - $\mathscr{H}_\infty$ norm as a measure of the system gain ?
   - How to compute the $\mathscr{H}_\infty$ norm?
   - $\mathscr{H}_\infty$ norm and stability issues
3. Introduction to LMIs
   - Background in Optimisation
   - LMI in control
4. How to define and solve an H_∞ control problem?
   - What is $\mathscr{H}_\infty$ control?
   - The Static State feedback case
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5. Why $\mathscr{H}_\infty$ control is adapted to control engineering?
   - Performance analysis using the sensitivity functions
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   - Introduction
   - Representation of uncertainties
   - Definition of Robustness analysis
   - Robustness analysis: the unstructured case
   - Robustness analysis: the structured case
   - Robust control design

Reference books

To be studied during the course

- S. Skogestad and I. Postlethwaite, Multivariable Feedback Control: analysis and design, John Wiley and Sons, 2005.
  `www.nt.ntnu.no/users/skoge/book`, chap 1 to 3 available
- K. Zhou, Essentials of Robust Control, Prentice Hall, New Jersey, 1998.
  `www.ece.lsu.edu/kemin`, book slides available
- J.C. Doyle, B.A. Francis, and A.R. Tannenbaum, Feedback control theory, Macmillan Publishing Company, New York, 1992.
  `https://sites.google.com/site/brucefranciscontact/Home/publications`, , book available
- Carsten Scherer's courses
  `http://www.dcsc.tudelft.nl/ cscherer/.` , Lecture slides available (MSc Course "Robust Control", MSc Course "Linear Matrix Inequalities in Control")
- + all the MATLAB demo, examples and documentation on the 'Robust Control toolbox' (`mathworks.com/products/robust`)

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Definition of LTI systems

Definition (LTI dynamical system)

Given matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_w}$, $C \in \mathbb{R}^{n_z\times n}$ and $D \in \mathbb{R}^{n_z\times n_w}$, a Linear Time Invariant (LTI) dynamical system (Σ_{LTI}) can be described as:

$$\Sigma_{LTI} : \begin{cases} \dot{x}(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \end{cases} \tag{1}$$

where $x(t)$ is the state which takes values in a state space $X \in \mathbb{R}^n$, $w(t)$ is the input taking values in the input space $W \in \mathbb{R}^{n_w}$ and $z(t)$ is the output that belongs to the output space $Z \in \mathbb{R}^{n_z}$.

The LTI system locally describes the real system under consideration and the linearization procedure allows to treat a linear problem instead of a nonlinear one. For this class of problem, many mathematical and control theory tools can be applied like closed loop stability, controllability, observability, performance, robust analysis, etc. for both SISO and MIMO systems. However, the main restriction is that LTI models only describe the system locally, then, compared to nonlinear models, they lack of information and, as a consequence, are incomplete and may not provide global stabilization.

Signal norms

Reader is also invited to refer to the famous book of Zhou et al., 1996, where all the following definitions and additional information are given.
All the following definitions are given assuming signals $x(t) \in \mathbb{C}$, then they will involve the conjugate (denoted as $x^*(t)$). When signals are real (i.e. $x(t) \in \mathbb{R}$), $x^*(t) = x^T(t)$.

Definition (Norm and Normed vector space)

- Let V be a finite dimension space. Then $\forall\ p \geq 1$, the application $||.||_p$ is a norm, defined as,

$$||v||_p = \left(\sum_i |v_i|^p\right)^{1/p} \tag{2}$$

- Let V be a vector space over $\mathbb{C}$ (or $\mathbb{R}$) and let $\|.\|$ be a norm defined on V. Then V is a normed space.

$\mathscr{L}_\star$ norms

Definition ($\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_\infty$ norms)

- The 1-Norm of a function $x(t)$ is given by,

$$\|x(t)\|_1 = \int_0^{+\infty} |x(t)|dt \tag{3}$$

- The 2-Norm (that introduces the energy norm) is given by,

$$\begin{aligned} \|x(t)\|_2 &= \sqrt{\int_0^{+\infty} x^*(t)x(t)dt} \\ &= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)X(j\omega)d\omega} \end{aligned} \tag{4}$$

  The second equality is obtained by using the Parseval identity.

- The ∞-Norm is given by,

$$\|x(t)\|_\infty = \sup_t |x(t)| \tag{5}$$

$$\|X\|_\infty = \sup_{Re(s)\geq 0} \|X(s)\| = \sup_\omega \|X(j\omega)\| \tag{6}$$

  if the signals that admit the Laplace transform, analytic in $Re(s) \geq 0$ (i.e. $\in \mathscr{H}_\infty$).

$\mathscr{L}_\infty$ and $\mathscr{H}_\infty$ spaces

Definition ($\mathscr{L}_\infty$ space)

$\mathscr{L}_\infty$ is the space of piecewise continuous bounded functions. It is a Banach space of matrix-valued (or scalar-valued) functions on $\mathbb{C}$ and consists of all complex bounded matrix functions $f(j\omega)$, $\forall \omega \in \mathbb{R}$, such that,

$$\sup_{\omega \in \mathbb{R}} \overline{\sigma}[f(j\omega)] < \infty \tag{7}$$

Definition ($\mathscr{H}_\infty$ and $\mathscr{RH}_\infty$ spaces)

$\mathscr{H}_\infty$ is a (closed) subspace in $\mathscr{L}_\infty$ with matrix functions $f(j\omega)$, $\forall \omega \in \mathbb{R}$, analytic in $Re(s) > 0$ (open right-half plane). The real rational subspace of $\mathscr{H}_\infty$ which consists of all proper and real rational stable transfer matrices, is denoted by $\mathscr{RH}_\infty$.

Example

In control theory

$$\begin{aligned} \frac{s+1}{(s+10)(s+6)} &\in \mathscr{RH}_\infty \\ \frac{s+1}{(s-10)(s+6)} &\notin \mathscr{RH}_\infty \\ \frac{s+1}{(s+10)} &\in \mathscr{RH}_\infty \end{aligned} \tag{8}$$

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$\mathcal{H}_\infty$ norm

Definition ($\mathcal{H}_\infty$ norm)

The $\mathcal{H}_\infty$ norm of a proper LTI system defined as on (1) from input $w(t)$ to output $z(t)$ and which belongs to $\mathcal{RH}_\infty$, is the induced energy-to-energy gain (induced $\mathcal{L}_2$ norm) defined as,

$$\begin{aligned} \|G(j\omega)\|_\infty &= \sup_{\omega\in\mathbb{R}} \overline{\sigma}(G(j\omega)) \\ &= \sup_{w(s)\in\mathcal{H}_2} \frac{\|z(s)\|_2}{\|w(s)\|_2} \\ &= \max_{w(t)\in\mathcal{L}_2} \frac{\|z\|_2}{\|w\|_2} \end{aligned} \tag{9}$$

Remark

$\mathcal{H}_\infty$ physical interpretations

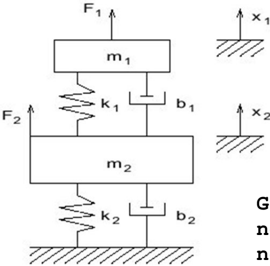
- *This norm represents the maximal gain of the frequency response of the system. It is also called the worst case attenuation level in the sense that it measures the maximum amplification that the system can deliver on the whole frequency set.*
- *For SISO (resp. MIMO) systems, it represents the maximal peak value on the Bode magnitude (resp. singular value) plot of $G(j\omega)$, in other words, it is the largest gain if the system is fed by harmonic input signal.*
- *Unlike $\mathcal{H}_2$, the $\mathcal{H}_\infty$ norm cannot be computed analytically. Only numerical solutions can be obtained (e.g. Bisection algorithm, or LMI resolution).*

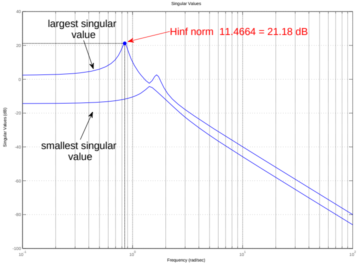
Characterization of the $\mathscr{H}_\infty$ norm as induced $\mathscr{L}_2$ norm

Finally, in the case of a transfer matrix $G(s)$: (m inputs, p outputs) u vector of inputs, y vector of outputs.

$$\underline{\sigma}(G(j\omega)) \leq \frac{\|z(\omega)\|_2}{\|d(\omega)\|_2} \leq \bar{\sigma}(G(j\omega))$$

Example of A two-mass/spring/damper system
2 inputs: F_1 and F_2 2 outputs: x_1 and x_2





`G=ss(A,B,C,D)` : LTI system
`normhinf(G)` : Compute Hinf norm
`norm(G,inf)` : Compute Hinf norm
`sigma(G)` : plot max and min SV

How to compute the $\mathscr{H}_\infty$ norm?

As said before, $\mathscr{H}_\infty$ norm cannot be computed analytically. Only numerical solutions can be obtained (e.g. Bisection algorithm, or LMI resolution).

Method 1: Since $\|G(j\omega)\|_\infty = \sup_{\omega\in\mathbb{R}} \overline{\sigma}(G(j\omega))$, the intuitive computation is to get the peak on the Bode magnitude plot, which can be estimated using a thin grid of frequency points, $\{\omega_1,\ldots,\omega_N\}$, and then:

$$\|G(j\omega)\|_\infty \approx \max_{1\le k\le N} \overline{\sigma}\{G(j\omega_k)\}$$

Method 2: Let the dynamical system $G=(A,B,C,D)\in\mathscr{RH}_\infty$:
$||G||_\infty < \gamma$ if and only if $\overline{\sigma}(D) < \gamma$ and the Hamiltonian H has no eigenvalues on the imaginary axis, where

$$H = \begin{pmatrix} A+BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I_n+DR^{-1}D^T)C & -(A+BR^{-1}D^TC) \end{pmatrix} \text{ and } R=\gamma^2 - D^TD$$

Use **norm(sys,inf)** or **hinfnorm(sys,tol)** in **Matlab**.

Method 3 (Bounded Real Lemma): A dynamical system $G=(A,B,C,D)$ is internally stable and with an $||G||_\infty < \gamma$ if and only if there exists a positive definite symmetric matrix $\mathbf{P}$ (i.e $\mathbf{P}=\mathbf{P}^T>0$ s.t

$$\begin{bmatrix} A^T\mathbf{P}+\mathbf{P}A & \mathbf{P}B & C^T \\ B^T\mathbf{P} & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0, \quad \mathbf{P}>0. \qquad (10)$$

Small Gain theorem

Consider the so called $M-\Delta$ loop.

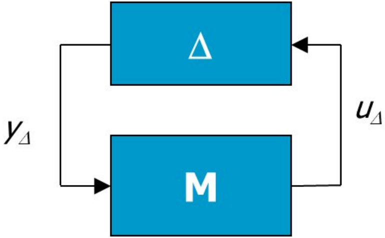


Figure: $M-\Delta$ form

Theorem

Suppose $M(s)$ in RH_∞ and γ a positive scalar. Then the system is well-posed and internally stable for all $\Delta(s)$ in RH_∞ such that $\|\Delta\|_\infty \leq 1/\gamma$ if and only if

$$\|M\|_\infty < \gamma$$

Input-Output Stability

Definition (BIBO stability)

A system G ($\dot{x} = Ax + Bu$; $y = Cx$) is BIBO stable if a bounded input $u(.)$ ($\|u\|_\infty < \infty$) maps a bounded output $y(.)$ ($\|y\|_\infty < \infty$).

Now, the quantification (for BIBO stable systems) of the signal amplification (gain) is evaluated as:

$$\gamma_{peak} = \sup_{0<\|u\|_\infty<\infty} \frac{\|y\|_\infty}{\|u\|_\infty}$$

and is referred to as the PEAK TO PEAK Gain.

Definition ($\mathscr{L}_2$ stability)

A system G ($\dot{x} = Ax + Bu$; $y = Cx$) is $\mathscr{L}_2$ stable if $\|u\|_2 < \infty$ implies $\|y\|_2 < \infty$.

Now, the quantification of the signal amplification (gain) is evaluated as:

$$\gamma_{energy} = \sup_{0<\|u\|_2<\infty} \frac{\|y\|_2}{\|u\|_2}$$

and is referred to as the ENERGY Gain, and is such that:

$$\gamma_{energy} = \sup \omega \|G(j\omega)\| := \|G\|_\infty$$

For a linear system, these stability definitions are equivalent (but not the quantification criteria).

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Brief on optimisation

Definition (Convex function)

A function $f:\mathbb{R}^m \to \mathbb{R}$ is convex if and only if for all $x,y \in \mathbb{R}^m$ and $\lambda \in [0\ 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{11}$$

Equivalently, f is convex if and only if its epigraph,

$$\mathbf{epi}(f) = \{(x,\lambda) | f(x) \leq \lambda\} \tag{12}$$

is convex.

Definition ((Strict) LMI constraint)

A Linear Matrix Inequality constraint on a vector $x \in \mathbb{R}^m$ is defined as,

$$F(x) = F_0 + \sum_{i=1}^{m} F_i x_i \succeq 0 (\succ 0) \tag{13}$$

where $F_0 = F_0^T$ and $F_i = F_i^T \in \mathbb{R}^{n\times n}$ are given, and symbol $F \succeq 0(\succ 0)$ means that F is symmetric and positive semi-definite ($\succeq 0$) or positive definite ($\succ 0$), i.e. $\{\forall u | u^T F u (>) \geq 0\}$.

Convex to LMIs

Example

Lyapunov equation. A very famous LMI constraint is the Lyapunov inequality of an autonomous system $\dot{x} = Ax$. Then the stability LMI associated is given by,

$$\begin{array}{rcl} x^T P x & > & 0 \\ x^T (A^T P + PA) x & < & 0 \end{array} \quad (14)$$

which is equivalent to,

$$F(P) = \begin{bmatrix} -P & 0 \\ 0 & A^T P + PA \end{bmatrix} \prec 0 \quad (15)$$

where $P = P^T$ is the decision variable. Then, the inequality $F(P) \prec 0$ is linear in P.

LMI constraints $F(x) \succeq 0$ are convex in x, i.e. the set $\{x | F(x) \succeq 0\}$ is convex. Then LMI based optimization falls in the convex optimization. This property is fundamental because it guarantees that the global (or optimal) solution x^* of the the minimization problem under LMI constraints can be found efficiently, in a polynomial time (by optimization algorithms like e.g. Ellipsoid, Interior Point methods).

LMI problem

Two kind of problems can be handled

Feasibility: The question whether or not there exist elements $x \in X$ such that $F(x) < 0$ is called a feasibility problem. The LMI $F(x) < 0$ is called feasible if such x exists, otherwise it is said to be infeasible.

Optimization: Let an objective function $f : S \rightarrow R$ where $S = \{x|F(x) < 0\}$. The problem to determine

$$V_{opt} = inf_{x \in S} f(x)$$

is called an optimization problem with an LMI constraint. This problem involves the determination of V_{opt}, the calculation of an almost optimal solution x (i.e., for arbitrary $\varepsilon > 0$ the calculation of an $x \in S$ such that $V_{opt} \leq f(x) \leq V_{opt} + \varepsilon$, or the calculation of a optimal solutions x_{opt} (elements $x_{opt} \in S$ such that $V_{opt} = f(x_{opt})$).

Examples of LMI problem

Stability analysis is a *feasability* problem.
LQ control is an optimization problem, formulated as:

LQ control

Consider a controllable system $\dot{x} = Ax + Bu$. Find a state feedback $u(t) = -Kx(t)$ s.t $J = \int_0^\infty (x^T Qx + u^T Ru)dt$ is minimum (given $Q > 0$ and $R > 0$) is an optimisation problem whose solution is obtained solving the Riccati equation:

$$Find\ P > 0,\ s.t.\ A^T P + PA - PBR^{-1}B^T P + Q = 0$$

and then the state feedback is given by:

$$u(t) = -R^{-1}B^T Px(t)$$

which is equivalent to: find $P > 0$ s.t

$$\begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & R \end{bmatrix} > 0$$

Semi-Definite Programming (SDP) Problem

LMI programming is a generalization of the Linear Programming (LP) to cone positive semi-definite matrices, which is defined as the set of all symmetric positive semi-definite matrices of particular dimension.

Definition (SDP problem)

A SDP problem is defined as,

$$\begin{array}{rl} \min & c^T x \\ \text{under constraint } F(x) & \succeq 0 \end{array} \tag{16}$$

where $F(x)$ is an affine symmetric matrix function of $x \in \mathbb{R}^m$ (e.g. LMI) and $c \in \mathbb{R}^m$ is a given real vector, that defines the problem objective.

SDP problems are theoretically tractable and practically:

- They have a polynomial complexity, i.e. there exists an algorithm able to find the global minimum (for a given a priori fixed precision) in a time polynomial in the size of the problem (given by m, the number of variables and n, the size of the LMI).
- SDP can be practically and efficiently solved for LMIs of size up to 100×100 and $m \leq 1000$ see ElGhaoui, 97. Note that today, due to extensive developments in this area, it may be even larger.

The state feedback design problem

Stabilisation

Let us consider a controllable system $\dot{x} = Ax + Bu$. The problem is to find a state feedback $u(t) = -Kx(t)$ s.t the closed-loop system is stable.

Using the Lyapunov theorem, this amounts at finding $\mathbf{P} = \mathbf{P}^T > 0$ s.t:

$$\begin{array}{rl} & (A - BK)^T \mathbf{P} + \mathbf{P}(A - BK) < 0 \\ \Leftrightarrow & A^T \mathbf{P} + \mathbf{P}A - K^T B^T P - PBK < 0 \end{array}$$

which is obviously not linear...

Solution; use of change of variables

First, left and right multiplication by $\mathbf{P}^{-1}$ leads to

$$\begin{array}{rl} & \mathbf{P}^{-1}A^T + A\mathbf{P}^{-1} - \mathbf{P}^{-1}K^T B^T - BK\mathbf{P}^{-1} < 0 \\ \Leftrightarrow & \mathbf{Q}A^T + A\mathbf{Q} + \mathbf{Y}^T B^T + B\mathbf{Y} < 0 \end{array}$$

with $\mathbf{Q} = \mathbf{P}^{-1}$ and $\mathbf{Y} = -K\mathbf{P}^{-1}$.

The problem to be solved is therefore formulated as an LMI ! and without any conservatism !

The Bounded Real Lemma

The $\mathscr{L}_2$-norm of the output z of a system Σ_{LTI} is uniformly bounded by γ times the $\mathscr{L}_2$-norm of the input w (initial condition $x(0)=0$).
A dynamical system $G=(A,B,C,D)$ is internally stable and with an $||G||_\infty < \gamma$ if and only is there exists a positive definite symmetric matrix $\mathbf{P}$ (i.e $\mathbf{P}=\mathbf{P}^T > 0$ s.t

$$\begin{bmatrix} A^T\,\mathbf{P}+\mathbf{P}\,A & \mathbf{P}\,B & C^T \\ B^T\,\mathbf{P} & -\gamma\,I & D^T \\ C & D & -\gamma\,I \end{bmatrix} < 0, \quad \mathbf{P} > 0. \tag{17}$$

The Bounded Real Lemma (BRL), can also be written as follows (see Scherer)

$$\left[\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right]^T \left[\begin{array}{cc|cc} 0 & \mathbf{P} & 0 & 0 \\ \mathbf{P} & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right] \prec 0 \tag{18}$$

Note that the BRL is an LMI if the only unknown (decision variables) are $\mathbf{P}$ and γ (or γ^2).

Quadratic stability

This concept is very useful for the stability analysis of uncertain systems.
Let us consider an uncertain system

$$\dot{x} = A(\delta)x$$

where δ is an parameter vector that belongs to an uncertainty set Δ.

Definition

The considered system is said to be quadratically stable for all uncertainties $\delta \in \Delta$ if there exists a (single) "Lyapunov function" $\boldsymbol{P} = \boldsymbol{P}^T > 0$ s.t

$$A(\delta)^T \boldsymbol{P} + \boldsymbol{P}A(\delta) < 0, \text{for all } \delta \in \Delta \tag{19}$$

This is a sufficient condition for ROBUST Stability which is obtained when $A(\delta)$ is stable for all $\delta \in \Delta$.

Interest of LMIs

LMIs allow to formulate complex optimization problems into "Linear" ones, allowing the use of convex optimization tools.
Usually it requires the use of different transformations, changes of variables ... in order to linearize the considered problmems: Congruence, Schur complement, projection lemma, Elimination lemma, S-procedure, Finsler's lemme ...

Examples of handled criteria

- stability
- H_∞, H_2, H_2/H_∞ performances
- robustness analysis: Small gain theorem, Polytopic uncertianties, LFT representations...
- Robust control and/or observer design
- pole placement
- stability, stabilization with input constraints
- Passivity constraints
- Time-delay systems

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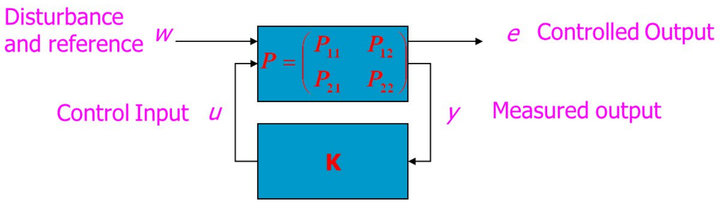
Towards $\mathcal{H}_\infty$ control: the General Control Configuration

This approach has been introduced by Doyle (1983). The formulation makes use of the general control configuration.



P is the generalized plant (contains the plant, the weights, the uncertainties if any) ; K is the controller. The closed-loop transfer matrix from w to z is given by:

$$T_{Zw}(s) = F_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

where $F_l(P,K)$ is referred to as a lower Linear Fractional Transformation.

Problem definition

The overall control objective is to minimize some norm of the transfer function from w to z , for example, the $\mathscr{H}_\infty$ norm.

Definition ($\mathscr{H}_\infty$ optimal control problem)

$\mathscr{H}_\infty$ control problem: Find a controller $K(s)$ which based on the information in y, generates a control signal u which counteracts the influence of w on z, thereby minimizing the closed-loop norm from w to z.

Definition ($\mathscr{H}_\infty$ suboptimal control problem)

Given γ a pre-specified attenuation level, a $\mathscr{H}_\infty$ sub-optimal control problem is to design a stabilizing controller that ensures :

$$\|T_{zw}(s)\|_\infty \quad = \quad \max_\omega \bar{\sigma}(T_{zw}(j\omega)) \leq \gamma$$

The optimal problem aims at finding γ_{min} (done using **`hinfsyn`** in MATLAB).

A first case: the state feedback control problem

Let consider the system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= Cx(t) + D_{11} w(t) + D_{12} u(t) \end{aligned} \quad (20)$$

The objective is to find a state feedback control law $u = -Kx$ s.t:

$$\|T_{zw}(s)\|_\infty \leq \gamma$$

The method consists in applying the Bounded Real Lemma to the closed-loop system, and then try to obtain some convex solutions (LMI formulation).
This is achieved if and only is there exists a positive definite symmetric matrix **P** (i.e $\mathbf{P} = \mathbf{P}^T > 0$ s.t

$$\begin{bmatrix} (A - B_2 K)^T \mathbf{P} + \mathbf{P}(A - B_2 K) & \mathbf{P} B_1 & C^T \\ \star & -\gamma I & D^T \\ \star & \star & -\gamma I \end{bmatrix} < 0, \quad \mathbf{P} > 0. \quad (21)$$

Use of change of variables

First, left and right multiplication by $diag(\mathbf{P}^{-1}, I_n, I_n)$, and use $\mathbf{Q} = \mathbf{P}^{-1}$ and $\mathbf{Y} = -K\mathbf{P}^{-1}$. It leads to

$$\begin{bmatrix} A\mathbf{Q} + B_2\mathbf{Y} + \mathbf{Q}A^T + \mathbf{Y}^T B_2^T & B_1 & \mathbf{Q}C^T - \mathbf{Y}^T D_{12}^T \\ \star & -\gamma I & D_{11}^T \\ \star & \star & -\gamma I \end{bmatrix} < 0, \quad \mathbf{Q} > 0. \quad (22)$$

The state feedback controller is then: $K = -\mathbf{Y}\mathbf{Q}^{-1}$

The Dynamic Output feedback case

It will be shown how to formulate such a control problem using "classical" control tools. The procedure will be 2-steps:

- **Get P:** Build the General Control Configuration scheme s.t. the closed-loop system matrix does correspond to the tackled H_∞ problem (for instance the mixed sensitivity problem). Use of **`Matlab, sysic`** tool.
  A state space representation of P, the generalized plant, is needed.
- **Compute K:** Use an optimisation algorithm that finds the controller K solution of the considered problem.
  The calculation of the controller, solution of the $\mathscr{H}_\infty$ control problem , can then be done using the Riccati approach or the LMI approach of the $\mathscr{H}_\infty$ control problem (Zhou et al., 1996; Skogestad & Postlwaite 1996)

Notations:

$$P\left\{\begin{array}{l}\dot{x}=Ax+B_1w+B_2u\\ z=C_1x+D_{11}w+D_{12}u\\ y=C_2x+D_{21}w+D_{22}u\end{array}\right. \quad \Rightarrow P=\left[\begin{array}{c|cc}A & B_1 & B_2\\ \hline C_1 & D_{11} & D_{12}\\ C_2 & D_{21} & D_{22}\end{array}\right]$$

with
- $x\in\mathbb{R}^n$: { plant state variables $\cup$ state variables of weights}
- $w\in\mathbb{R}^{n_w}$: external inputs
- $u\in\mathbb{R}^{n_u}$ control inputs
- $z\in\mathbb{R}^{n_z}$: controlled outputs
- $y\in\mathbb{R}^{n_y}$ measured outputs (inputs of the controller)

Problem formulation

Let $K(s)$ be a dynamic output feedback LTI controller defined as

$$K(s) : \begin{cases} \dot{x}_K(t) &= A_K\, x_K(t) + B_K\, y(t), \\ u(t) &= C_K\, x_K(t) + D_K\, y(t). \end{cases}$$

where $x_K \in \mathbb{R}^n$, and A_K, B_K, C_K and D_K are matrices of appropriate dimensions.
Remark. *The controller will be considered here of the same order (same number of state variables) n than the generalized plant, which here, in the $\mathscr{H}_\infty$ framework, the order of the optimal controller.*
With $P(s)$ and $K(s)$, the closed-loop system $N(s)$ is:

$$N(s) : \begin{cases} \dot{x}_{cl}(t) &= \mathscr{A}_{CL}\, x_{cl}(t) + \mathscr{B}_{CL}\, w(t), \\ z(t) &= \mathscr{C}_{CL}\, x_{cl}(t) + \mathscr{D}_{CL}\, w(t), \end{cases} \tag{23}$$

where $x_{cl}^T(t) = \left[x^T(t)\ x_K^T(t)\right]$ and

$$\begin{cases} \mathscr{A}_{CL} = \begin{pmatrix} A + B_2\, D_K\, C_2 & B_2\, C_K \\ B_K\, C_2 & A_K \end{pmatrix}, \\ \mathscr{B}_{CL} = \begin{pmatrix} B_1 + B_2\, D_K\, D_{21} \\ B_K\, D_{21} \end{pmatrix}, \\ \mathscr{C}_{CL} = \begin{pmatrix} C_1 + D_{12}\, D_K\, C_2, & D_{12}\, C_K \end{pmatrix}, \\ \mathscr{D}_{CL} = B_1 + B_2\, D_K\, D_{21}. \end{cases}$$

The aim is of course to find matrices A_K, B_K, C_K and D_K s.t. the $\mathscr{H}_\infty$ norm of the closed-loop system (23) is as small as possible, i.e. $\gamma_{opt} = \min \gamma$ s.t. $||N(s)||_\infty < \gamma$.

The LMI approach for $\mathscr{H}_\infty$ control design- Solvability

Assumption: (A,B_2) stabilizable and (C_2,A) detectable: necessary for the existence of stabilizing controllers.
The solution is base on the use of the Bounded Real Lemma, and some relaxations that leads to an LMI problem to be solved Scherer & Wieland, 2004.
when we refer to the $\mathscr{H}_\infty$ control problem, we mean: Find a controller K for system P such that, given γ_∞,

$$||\mathscr{F}_l(P,K)||_\infty < \gamma_\infty \tag{24}$$

The minimum of this norm is denoted as γ_∞^* and is called the optimal $\mathscr{H}_\infty$ gain. Hence, it comes,

$$\gamma_\infty^* = \min_{(A_K,B_K,C_K,D_K) s.t. \sigma \mathscr{A}_{CL} \subset \mathbb{C}^-} ||T_{zw}(s)||_\infty \tag{25}$$

As presented in the previous sections, this condition is fulfilled thanks to the BRL. As a matter of fact, the system is internally stable and meets the quadratic $\mathscr{H}_\infty$ performances iff. $\exists\ \mathscr{P} = \mathscr{P}^T \succ 0$ such that,

$$\begin{bmatrix} \mathscr{A}_{CL}^T\mathscr{P} + \mathscr{P}\mathscr{A}_{CL} & \mathscr{P}\mathscr{B}_{CL} & \mathscr{C}_{CL}^T \\ \mathscr{B}_{CL}^T\mathscr{P} & -\gamma_2^2 I & \mathscr{D}_{CL}^T \\ \mathscr{C}_{CL} & \mathscr{D}_{CL} & -I \end{bmatrix} < 0 \tag{26}$$

where $\mathscr{A}_{CL}$, $\mathscr{B}_{CL}$, $\mathscr{C}_{CL}$, $\mathscr{D}_{CL}$ are given in (23). Since this inequality is not an LMI and not tractable for SDP solver, relaxations have to be performed (indeed it is a BMI), as proposed in Scherer et al. 1997.

The LMI approach for $\mathcal{H}_\infty$ control design- Problem solution

Theorem (LTI/$\mathcal{H}_\infty$ solution Scherer & Wieland, 2004)

A dynamical output feedback controller of the form $K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *that solves the* $\mathcal{H}_\infty$ *control problem, is obtained by solving the following LMIs in (**X**, **Y**,* $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{B}}$, $\widetilde{\boldsymbol{C}}$ *and* $\widetilde{\boldsymbol{D}}$*), while minimizing* γ_∞*,*

$$\begin{array}{l} \begin{bmatrix} M_{11} & (*)^T & (*)^T & (*)^T \\ M_{21} & M_{22} & (*)^T & (*)^T \\ M_{31} & M_{32} & M_{33} & (*)^T \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \prec 0 \\ \begin{bmatrix} \boldsymbol{X} & I_n \\ I_n & \boldsymbol{Y} \end{bmatrix} \succ 0 \end{array} \tag{27}$$

where,

$$\begin{array}{ll} M_{11} = A\boldsymbol{X} + \boldsymbol{X}A^T + B_2\widetilde{\boldsymbol{C}} + \widetilde{\boldsymbol{C}}^T B_2^T & M_{21} = \widetilde{\boldsymbol{A}} + A^T + C_2^T \widetilde{\boldsymbol{D}}^T B_2^T \\ M_{22} = \boldsymbol{Y}A + A^T\boldsymbol{Y} + \widetilde{\boldsymbol{B}}C_2 + C_2^T \widetilde{\boldsymbol{B}}^T & M_{31} = B_1^T + D_{21}^T \widetilde{\boldsymbol{D}}^T B_2^T \\ M_{32} = B_1^T \boldsymbol{Y} + D_{21}^T \widetilde{\boldsymbol{B}}^T & M_{33} = -\gamma_\infty I_{n_u} \\ M_{41} = C_1\boldsymbol{X} + D_{12}\widetilde{\boldsymbol{C}} & M_{42} = C_1 + D_{12}\widetilde{\boldsymbol{D}}C_2 \\ M_{43} = D_{11} + D_{12}\widetilde{\boldsymbol{D}}D_{21} & M_{44} = -\gamma_\infty I_{n_y} \end{array} \tag{28}$$

Controller reconstruction

Once **A**, **B**, **C**, **D**, **X** and **Y** have been obtained, the reconstruction procedure consists in finding non singular matrices M and N s.t. $M\ N^T = I - X\ Y$ and the controller K is obtained as follows

$$\begin{cases} D_K &= \widetilde{\mathbf{D}} \\ C_K &= (\widetilde{\mathbf{C}} - D_c C_2 \mathbf{X}) M^{-T} \\ B_K &= N^{-1}(\widetilde{\mathbf{B}} - \mathbf{Y} B_2 D_c) \\ A_K &= N^{-1}(\widetilde{\mathbf{A}} - \mathbf{Y}A\mathbf{X} - \mathbf{Y}B_2 D_c C_2 \mathbf{X} - N B_c C_2 \mathbf{X} - \mathbf{Y} B_2 C_c M^T) M^{-T} \end{cases} \tag{29}$$

where M and N are defined such that $MN^T = I_n - XY$ (that can be solved through a singular value decomposition plus a Cholesky factorization).

Remark. *Note that other relaxation methods can be used to solve this problem, as suggested by Gahinet & Apkarian(1994).*

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   - How to compute the $\mathscr{H}_\infty$ norm?
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   - Background in Optimisation
   - LMI in control
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   - The Dynamic Output feedback case
   - The LMI approach for $\mathscr{H}_\infty$ control design
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   - Performance analysis using the sensitivity functions
   - Performance specifications in view of $\mathscr{H}_\infty$ control design
   - The mixed sensitivity $\mathscr{H}_\infty$ control design
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   - Introduction
   - Representation of uncertainties
   - Definition of Robustness analysis
   - Robustness analysis: the unstructured case
   - Robustness analysis: the structured case
   - Robust control design

Introduction

Objectives of any control system Skogestad & Postlethwaite, 2005

shape the response of the system to a given reference and get (or keep) a stable system in closed-loop, with desired performances, while minimising the effects of disturbances and measurement noises, and avoiding actuators saturation, this despite of modelling uncertainties, parameter changes or change of operating point.

This is formulated as:

- Nominal stability (NS): The system is stable with the nominal model (no model uncertainty)
- Nominal Performance (NP): The system satisfies the performance specifications with the nominal model (no model uncertainty)
- Robust stability (RS): The system is stable for all perturbed plants about the nominal model, up to the worst-case model uncertainty (including the real plant)
- Robust performance (RP): The system satisfies the performance specifications for all perturbed plants about the nominal model, up to the worst-case model uncertainty (including the real plant).

A 1 d-o-f control scheme

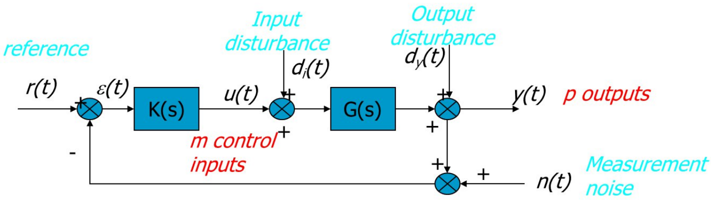


Figure: Complete control scheme

The output & the control input satisfy the following equations :

$$\begin{aligned}(I_p + G(s)K(s))y(s) &= (GKr + d_y - GKn + Gd_i) \\ (I_m + K(s)G(s))u(s) &= (Kr - Kd_y - Kn - KGd_i)\end{aligned}$$

BUT : $K(s)G(s) \neq G(s)K(s)$!!

Definition of the sensitivity functions: MIMO case

Definitions

Output and Output complementary sensitivity functions:

$$S_y = (I_p + GK)^{-1}, \quad T_y = (I_p + GK)^{-1} GK, \quad S_y + T_y = I_p$$

Input and Input complementary sensitivity functions:

$$S_u = (I_m + KG)^{-1}, \quad T_u = KG(I_m + KG)^{-1}, S_u + T_u = I_m$$

Properties

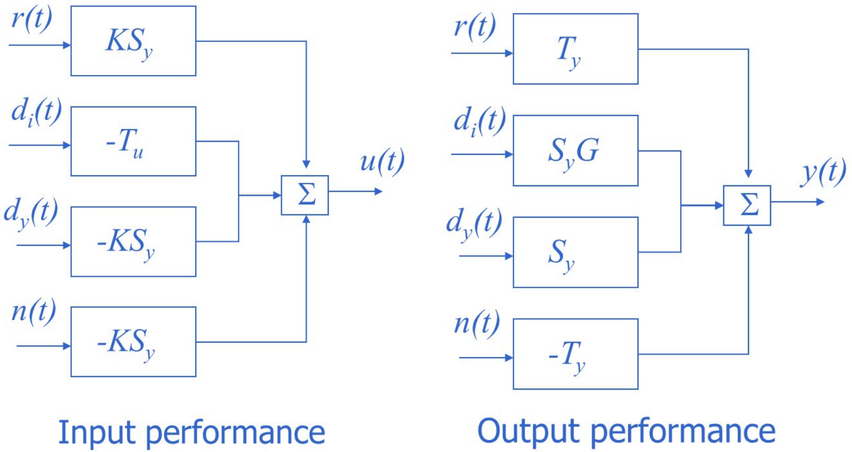
$$T_y = GK(I_p + GK)^{-1}$$
$$T_u = (I_m + KG)^{-1} KG$$
$$S_u K = K S_y$$

The SISO case

"Output" Sensitivity function	$S(s) = \frac{1}{1+GK(s)}$
Complementary Sensitivity function	$T(s) = \frac{GK(s)}{1+GK(s)}$
"Controller" Sensitivity function	$KS(s) = \frac{1}{1+GK(s)}$
"Input" Sensitivity function	$SG(s) = \frac{G}{1+GK(s)}$

MIMO Input/Output performances

Defining two new 'sensitivity functions':
Plant Sensitivity: $S_yG = S_y(s).G(s)$
Controller Sensitivity: $KS_y = K(s).S_y(s)$



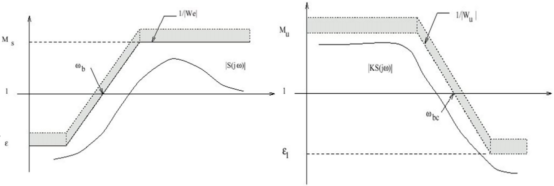
Weighting functions for performance specifications

Method

Define WEIGHTS $W_\star(s)$ with the objective

$$\Leftrightarrow \|W_\star S_\star\|_\infty \leq 1$$

for the selected sensitivity functions.



Definition

SISO systems: weights are usually simple first or second order filters.
MIMO systems: weights are diagonal matrices made of individual weights for inputs/outputs of interest, as for instance for a 2 input systems: $W_u(s) = \begin{pmatrix} W_u^1(s) & 0 \\ 0 & W_u^2(s) \end{pmatrix}$ where each $W_u^i(s)$ is defined according to the actuator specification

The mixed sensitivity $\mathcal{H}_\infty$ control design - Problem definition

The corresponding $\mathcal{H}_\infty$ suboptimal control problem is therefore to find a controller $K(s)$ such that $\|T_{ew}(s)\|_\infty = \left\| \begin{array}{c} W_e S \\ W_u K S \end{array} \right\|_\infty \leq \gamma$ with

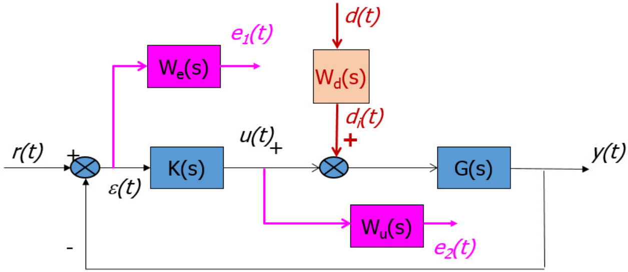
$$\begin{aligned} T_{ew}(s) = F_l(P,K) &= P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ &= \begin{bmatrix} W_e \\ 0 \end{bmatrix} + \begin{bmatrix} -W_e G \\ W_u \end{bmatrix} K(I + GK)^{-1} I \\ &= \begin{bmatrix} W_e S \\ W_u K S \end{bmatrix} \end{aligned}$$

in Matlab

```
% Generalized plant P is found with function sysic
systemnames = 'G We Wu';
inputvar = '[ r(1);u(1)]';
outputvar = '[We; Wu; r-G]';
input_to_G = '[u]';
input_to_We = '[r-G]';
input_to_Wu = '[u]';
sysoutname = 'P';
cleanupsysic = 'yes';
sysic;
% Find H-infinity optimal controller
nmeas=1; nu=1;
[K,CL,GAM,INFO] = hinfsyn(P,nmeas,nu,'DISPLAY','ON');
gopt
```

What about disturbance attenuation ?

New scheme



This corresponds to the closed-loop system.

$$T_{ew} = \begin{bmatrix} W_e S_y & W_e S_y G W_d \\ W_u K S_y & W_u T_u W_d \end{bmatrix}$$

More generally...

To include multiple objectives in a SINGLE $\mathscr{H}_\infty$ control problem, there are 2 ways:

1. add some external inputs (reference, noise, disturbance, uncertainties ...)
2. add new controlled outputs

Of course both ways increase the dimension of the problem to be solved....thus the complexity as well. Moreover additional constraints appear that are not part of the objectives ....

General rule: first think simple !!

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Introduction

- A control system is robust if it is insensitive to differences between the actual system and the model of the system which was used to design the controller
- How to take into account the difference between the actual system and the model ?
- A solution: using a model set BUT : very large problem and not exact yet

A method: these differences are referred as model uncertainty.
The approach

1. determine the uncertainty set: mathematical representation
2. check Robust Stability
3. check Robust Performance

Lots of forms can be derived according to both our knowledge of the physical mechanism that cause the uncertainties and our ability to represent these mechanisms in a way that facilitates convenient manipulation.
Several origins :

- Approximate knowledge and variations of some parameters
- Measurement imperfections (due to sensor)
- At high frequencies, even the structure and the model order is unknown (100
- Choice of simpler models for control synthesis
- Controller implementation

Two classes: parametric uncertainties / neglected or unmodelled dynamics

Example 1: uncertainties

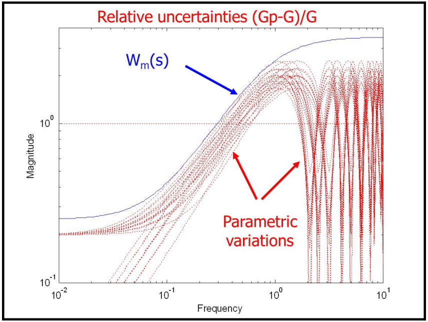
Let consider the example from (Sokestag & Postlewaite, 1996).

$$\tilde{G}(s) = \frac{k}{1+\tau s}e^{-sh}, \quad 2 \leq k,\ h,\ \tau \leq 3$$

Let us choose the nominal parameters as, $k = h = \tau = 2.5$ and G the according nominal model. We can define the 'relative' uncertainty, which is actually referred as a MULTIPLICATIVE UNCERTAINTY, as

$$\tilde{G}(s) = G(s)(I + W_m(s)\Delta(s))$$

with $W_m(s) = \frac{3.5s+0.25}{s+1}$
and $\|\Delta\|_\infty \leq 1$



Example 2: parametric uncertainties in state space equations

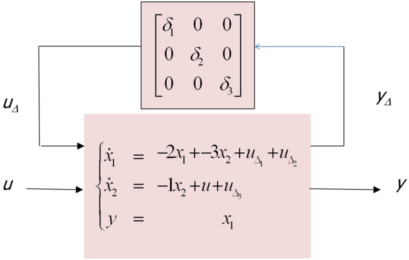
Let us consider the following uncertain system:

$$G:\begin{cases}\dot{x}_1 &= (-2+\delta_1)x_1+(-3+\delta_2)x_2\\ \dot{x}_2 &= (-1+\delta_3)x_2+u\\ y &= x_1\end{cases} \tag{30}$$

In order to use an LFT, let us define the uncertain inputs:

$$u_{\Delta_1}=\delta_1 x_1,\;\; u_{\Delta_2}=\delta_2 x_2,\;\; u_{\Delta_3}=\delta_3 x_2$$

Then the previous system can be rewritten in the following LFR:



where Δ and y_Δ are given as:

$$\Delta=\begin{bmatrix}\delta_1 & 0 & 0\\ 0 & \delta_2 & 0\\ 0 & 0 & \delta_3\end{bmatrix},\quad y_\Delta=\begin{pmatrix}x_1\\ x_2\\ x_2\end{pmatrix}$$

and N given by the state space representation:

$$N:\begin{cases}\dot{x}_1 &= -2x_1-3x_2+u_{\Delta_1}+u_{\Delta_2}\\ \dot{x}_2 &= -x_2+u+u_{\Delta_3}\\ y &= x_1\end{cases}$$

(

Towards LFR (LFT)

The previous computations are in fact the first step towards an unified representation of the uncertainties: the Linear Fractional Representation (LFR).
Indeed the previous schemes can be rewritten in the following general representation as:

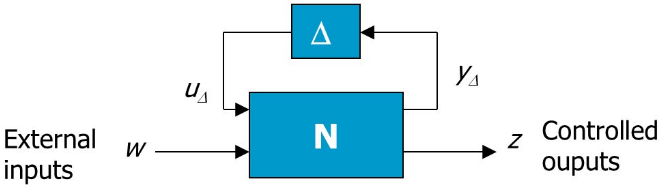


Figure: $N\Delta$ structure

This LFR gives then the transfer matrix from w to z, and is referred to as the upper Linear Fractional Transformation (LFT) :

$$F_u(N,\Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

This LFT exists and is well-posed if $(I - N_{11}\Delta)^{-1}$ is invertible

LFT definition

In this representation N is known and $\Delta(s)$ collects all the uncertainties taken into account for the stability analysis of the uncertain closed-loop system.
$\Delta(s)$ shall have the following structure:

$$\Delta(s) = diag\left\{\Delta_1(s), \cdots, \Delta_q(s), \delta_1 I_{r_1}, \cdots, \delta_r I_{r_r}, \varepsilon_1 I_{c_1}, \cdots, \varepsilon_c I_{c_c}\right\}$$

with $\Delta_i(s) \in \mathscr{RH}_\infty^{k_i \times k_i}$, $\delta_i \in \mathbb{R}$ and $\varepsilon_i \in \mathbb{C}$.
Remark: $\Delta(s)$ includes

- q full block transfer matrices,
- r real diagonal blocks referred to as 'repeated scalars' (indeed each block includes a real parameter δ_i repeated r_i times),
- c complex scalars ε_i repeated c_i times.

Constraints: The uncertainties must be normalized, i.e such that:

$$\|\Delta\|_\infty \leq 1, \quad |\delta_i| \leq 1, \quad |\varepsilon_i| \leq 1$$

Uncertainty types

We have seen in the previous examples the two important classes of uncertainties, namely:

- UNSTRUCTURED UNCERTAINTIES: we ignore the structure of Δ, considered as a full complex perturbation matrix, such that $\|\Delta\|_\infty \leq 1$.
  We then look at the maximal admissible norm for Δ, to get Robust Stability and Performance.
  This will give a global sufficient condition on the robustness of the control scheme.
  This may lead to conservative results since all uncertainties are collected into a single matrix ignoring the specific role of each uncertain parameter/block.
- STRUCTURED UNCERTAINTIES: we take into account the structure of Δ, (always such that $\|\Delta\|_\infty \leq 1$).
  The robust analysis will then be carried out for each uncertain parameter/block.
  This needs to introduce a new tool: the Structured Singular Value. We then can obtain more fine results but using more complex tools.

The analysis is provided in what follows for both cases.
In **`Matlab`**this analysis is provided in the tools **`robuststab`** and **`robustperf`**.

Robustness analysis: problem formulation

Since the analysis will be carried you for a closed-loop system, N should be defined as the connection of the plant and the controller. Therefore, in the framework of the H_∞ control, the following extended General Control Configuration is considered:

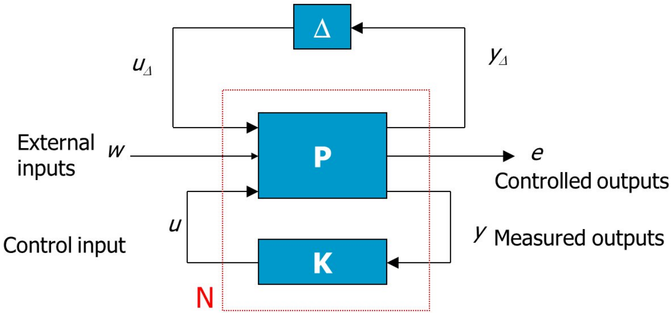


Figure: $P-K-\Delta$ structure

and N is such that

$$N = F_l(P,K)$$

Robust analysis: problem definition

In the global $P-K-\Delta$ General Control Configuration, the transfer matrix from w to z (i.e the closed-loop uncertain system) is given by:

$$z = F_u(N,\Delta)w,$$

with $F_u(N,\Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$.
and the objectives are then formulated as follows:

Nominal stability (NS): N is internally stable

Nominal Performance (NP): $\|N_{22}\|_\infty < 1$ and NS

Robust stability (RS): $F_u(N,\Delta)$ is stable $\forall\Delta,\ \|\Delta\|_\infty < 1$ and NS

Robust performance (RP): $\|F_u(N,\Delta)\|_\infty < 1\ \forall\Delta,\ \|\Delta\|_\infty < 1$ and NS

Towards Robust stability analysis

Robust Stability= with a given controller K, we determine wether the system remains stable for all plants in the uncertainty set.
According to the definition of the previous upper LFT, when N is stable, the instability may only come from $(I-N_{11}\Delta)$. Then it is equivalent to study the $M-\Delta$ structure, given as:

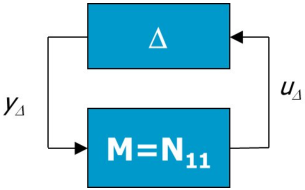


Figure: $M-\Delta$ structure

This leads to the definition of the Small Gain Theorem

Theorem (Small Gain Theorem)

Suppose $M \in RH_\infty$. Then the closed-loop system in Fig. 5 is well-posed and internally stable for all $\Delta \in RH_\infty$ such that :

$$\|\Delta\|_\infty \leq \delta (resp. < \delta) \quad if\ and\ only\ if \quad \|M(s)\|_\infty < 1/\delta (resp. \|M(s)\|_\infty \leq 1)$$

Definition of the uncertainty types

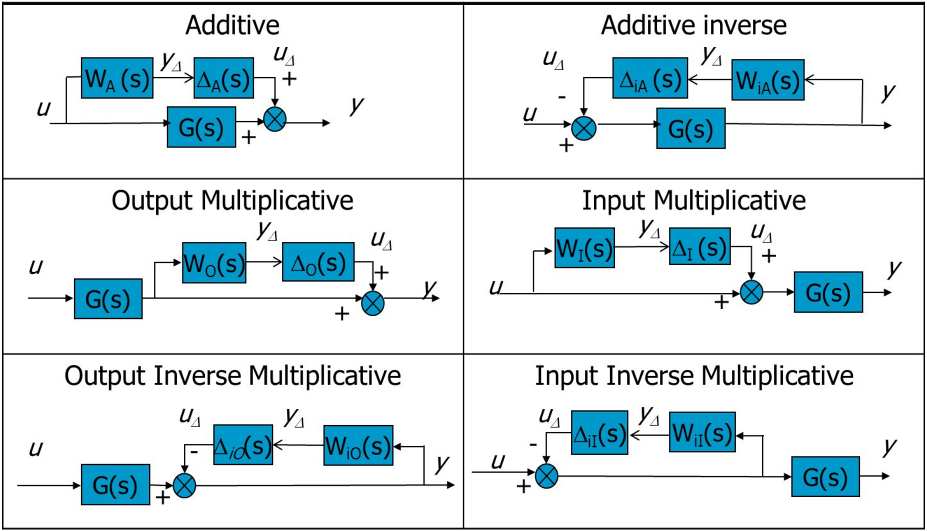


Figure: 6 uncertainty representations

General results

Theorem (Small Gain Theorem)

Consider the different uncertainty types, and assume that NS is achieved, i.e $M \in RH_\infty$ for each type. Then the closed-loop system is robustly stable, i.e. internally stable for all $\Delta_k \in RH_\infty$ (for $k =$A, 0, I, iO, iI) such that :

Additive :	$\left\|W_A K S_y\right\|_\infty \leq 1$
Additive Inverse:	$\left\|W_{iA} S_y\right\|_\infty \leq 1$
Output Multiplicative:	$\left\|W_O T_y\right\|_\infty \leq 1$
Input Multiplicative:	$\left\|W_I T_u\right\|_\infty \leq 1$
Output Inverse Multiplicative:	$\left\|W_{iO} S_y\right\|_\infty \leq 1$
Input Inverse Multiplicative:	$\left\|W_{iI} S_u\right\|_\infty \leq 1$

This gives some robustness templates for the sensitivity functions. However this may be conservative.

A first insight in Robust Performance

Objective: applying the Small Gain Theorem to these unstructured uncertainty representations.

Let us consider the following simple control scheme as:

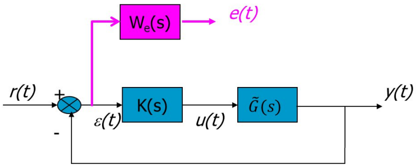
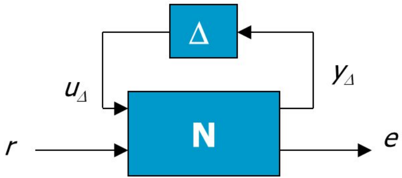


Figure: Control scheme

We wish to get:



Case of **Output Multiplicative** uncertainties: $\tilde{G}(s) = (I + W_O(s)\Delta_O(s))G(s)$.
Computing the $N - \Delta$ form gives

$$N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} = \begin{pmatrix} -W_O T_y & W_O T_y \\ -W_e S_y & W_e S_y \end{pmatrix}$$

The objectives are then formulated as follows:

- NS: N is internally stable
- NP: $\|W_e S_y\|_\infty < 1$ and NS
- RS: $\|W_O T_y\|_\infty < 1$ and NS
- RP: $\|F_u(N,\Delta)\|_\infty < 1\ \forall \Delta,\ \|\Delta\|_\infty < 1$,
  Sufficient condition: NS and $\bar{\sigma}(W_O T_y) + \bar{\sigma}(W_e S_y) < 1,\ \forall \omega$

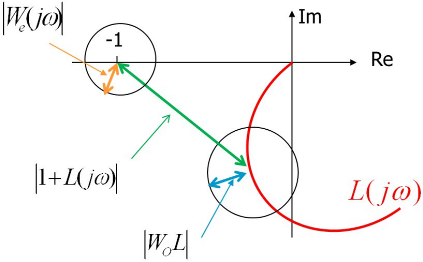
Illustration on the SISO case

Here Robust Performance is analyzed through the Nyquist plot. For illustration, let us consider the case of Multiplicative uncertainties (Input and Output case are identical for SISO systems), i.e

$$\tilde{G} = G(I + W_m\Delta_m)$$

Then the loop transfer function is given as:

$$\tilde{L} = \tilde{G}K = GK(I + W_m\Delta_m) = L + W_m L\Delta_m;$$



First NP is achieved when:
$|W_e S| < 1 \quad \forall\omega, \quad \Leftrightarrow \quad |W_e| < |1+L|, \ \forall\omega.$
Therefore RP is achieved if

$$\begin{aligned} & \left|W_e\tilde{S}\right| < 1, \quad \forall\tilde{S}, \forall\omega \\ \Leftrightarrow \quad & |W_e| < \left|1+\tilde{L}\right|, \quad \forall\tilde{L}, \forall\omega \end{aligned}$$

Since $\left|1+\tilde{L}\right| \geq |1+L| - |W_m L\Delta_m|$, a sufficient condition is actually:

$$\begin{aligned} & |W_e| + |W_m L| < |1+L|, \quad \forall\omega \\ \Leftrightarrow \quad & |W_e S| + |W_m T| < 1, \quad \forall\omega \end{aligned}$$

The structured case

$$\underline{\Delta} = \{\text{diag}\{\Delta_1, \cdots, \Delta_q, \delta_1 I_{r_1}, \cdots, \delta_r I_{r_r}, \varepsilon_1 I_{c_1}, \cdots, \varepsilon_c I_{c_c}\} \in \mathbb{C}^{k\times k}\} \quad (32)$$

with $\Delta_i \in \mathbb{C}^{k_i \times k_i}$, $\delta_i \in \mathbb{R}$, $\varepsilon_i \in \mathbb{C}$,
where $\Delta_i(s), i = 1, \ldots, q$, represent full block complex uncertainties, $\delta_i(s), i = 1, \ldots, r$, real parametric uncertainties, and $\varepsilon_i(s), i = 1, \ldots, c$, complex parametric uncertainties.
Taking into account the uncertainties leads to the following General Control Configuration,

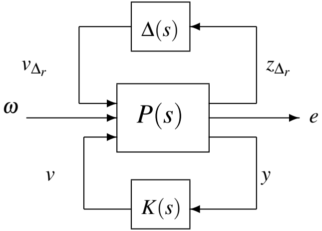


Figure: General control configuration with uncertainties

where $\Delta \in \underline{\Delta}$.

The structured singular value or μ-analysis

To handle parametric uncertainties, we need to introduce μ, the structured singular value, defined as:

Definition (μ)

For $M \in \mathbb{C}^{n\times n}$, the structure singular value is defined as:

$$\mu_{\underline{\Delta}}(M) := \frac{1}{\min\{\overline{\sigma}(\Delta) : \Delta \in \underline{\Delta},\ \det(I - \Delta M) \neq 0\}}$$

In other words, it allows to find the smallest structured Δ which makes $det(I - M\Delta) = 0$.

Theorem (The structured Small Gain Theorem)

Let $M(s)$ be a MIMO LTI stable system and $\Delta(s)$ a LTI uncertain stable matrix, (i.e. $\in \mathscr{RH}_\infty$). The system in Fig. 5 is stable for all $\Delta(s)$ in (32) if and only if:

$$\forall \omega \in \mathbb{R} \quad \mu_{\underline{\Delta}}(M(j\omega)) \leq 1, \text{ with } M(s) := N_{zv}(s)$$

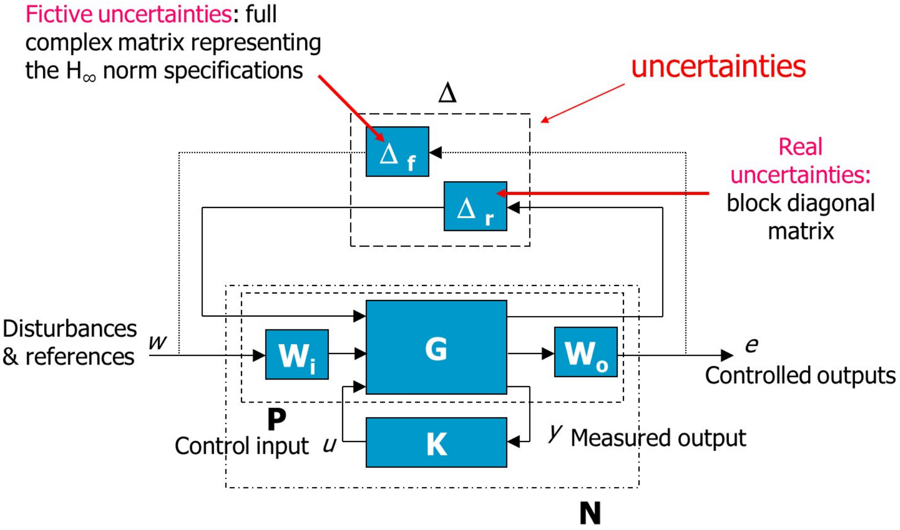
More generally both following statements are equivalent

- *For $\overline{\mu} \in \mathbb{R}$, $N(s)$ and $\Delta(s)$ belong to $\mathscr{RH}_\infty$, and*

$$\forall \omega \in \mathbb{R}, \quad \mu_{\underline{\Delta}}(M(j\omega)) \leq \overline{\mu}$$

- *the system represented in figure 5 is stable for any uncertainty $\Delta(s)$ of the form (32) such that :*

Build the whole control scheme



Introduction of a fictive block

Usually only real parametric uncertainties (given in Δ_r) are considered for RS analysis. RP analysis also needs a fictive full block complex uncertainty, as below,

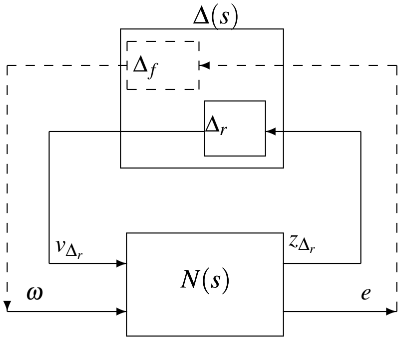


Figure: $N\Delta$

where $N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$, and the closed-loop transfer matrix is:

$$T_{ew}(s) = N_{22}(s) + N_{21}(s)\Delta(s)(I - N_{11}(s))^{-1}N_{12}(s) \quad (33)$$

Robust analysis theorem

For RS, we shall determine how large Δ (in the sense of H_∞) can be without destabilizing the feedback system. From (33), the feedback system becomes unstable if $det(I - N_{11}(s) = 0$ for some $s \in \mathbb{C}, \Re(s) \geq 0$. The result is then the following.

Theorem (Skogestad & Postlethwaite, 2005)

Assume that the nominal system N_{ew} and the perturbations Δ are stable. Then the feedback system is stable for all allowed perturbations Δ such that $||\Delta(s)||_\infty < 1/\beta$ if and only if $\forall \omega \in \mathbb{R}, \;\; \mu_{\underline{\Delta}}(N_{11}(j\omega)) \leq \beta$.

Assuming nominal stability, RS and RP analysis for structured uncertainties are therefore such that:

$$\begin{array}{lcl} \text{NP} & \Leftrightarrow & \overline{\sigma}(N_{22}) = \mu_{\underline{\Delta_f}}(N_{22}) \leq 1, \;\; \forall \omega \\ \text{RS} & \Leftrightarrow & \mu_{\Delta_r}(N_{11}) < 1, \, \forall \omega \\ \text{RP} & \Leftrightarrow & \mu_{\Delta}(N) < 1, \, \forall \omega, \, \Delta = \begin{bmatrix} \Delta_f & 0 \\ 0 & \Delta_r \end{bmatrix} \end{array}$$

Finally, let us remark that the structured singular value cannot be explicitly determined, so that the method consists in calculating an upper bound and a lower bound, as closed as possible to μ.

Summary

The steps to be followed in the RS/RP analysis for structured uncertainties are then:

- Definition of the real uncertainties Δ_r and of the weighting functions
- Evaluation of $\mu(\mathrm{N}_{22})_{\underline{\Delta}_f}, \mu(\mathrm{N}_{11})_{\underline{\Delta}_r}$ and $\mu(\mathrm{N})_{\underline{\Delta}}$
- Computation of the admissible intervals for each parameter

Remark: The Robust Performance analysis is quite conservative and requires a tight definition of the weighting functions that do represent the performance objectives to be satisfied by the uncertain closed-loop system. Therefore it is necessary to distinguish the weighting functions used for the nominal design from the ones used for RP analysis.

Brief overview

In order to design a robust control, i.e. a controller for which the synthesis actually accounts for uncertainties, some of the methods are:

- **Unstructured uncertainties:** Consider an uncertainty weight (unstructured form), and include the Small Gain Condition through a new controlled output. For example, robustness face to Ouptut Multiplicative Uncertainties can be considered into the design procedure adding the controlled output $e_y = W_O y$, which, when tracking performance is expected, leads to the condition $\| W_O T_y \|_\infty \leq 1$.
- **Structured uncertainties:** the design of a robust controller in the presence of such uncertainties is the $\mu - synthesis$. It is handled through an interactive procedure, referred to as the DK iteration. This procedure is much more involved than a "simple" H_∞ control design and often leads to an increase of the order of the controller (which is already the sum of the order of the plant and of the weighting functions).
- Use other mathematical representation of parametric uncertainties, Scherer & Wieland, 2004, as for instance the **polytopic model**. In that case the set of uncertain parameters is assumed to be a polytope (i.e. a convex) set. The stability issue in that framework is referred to as the 'Quadratic stability' i.e find a single Lyapunov function for the uncertainty set. While in the general case this is an unbounded problem, in the polytopic case (or in the affine case), the stability is to be analyzed only at the vertices of the polytope, which is a finite dimensional problem.
  This approach can then be applied to find a single controller, valid over the potyopic set. Note that this approach gives rise to the LPV design for polytopic systems, as described next.

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