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# On the growth behaviour of Hironaka quotients

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**Abstract.** We consider a finite analytic morphism  $\phi = (f, g) : (X, p) \rightarrow (\mathbb{C}^2, 0)$  where  $(X, p)$  is a complex analytic normal surface germ and  $f$  and  $g$  are complex analytic function germs. Let  $\pi : (Y, E_Y) \rightarrow (X, p)$  be a good resolution of  $\phi$  with exceptional divisor  $E_Y = \pi^{-1}(p)$ . We denote  $G(Y)$  the dual graph of the resolution  $\pi$ . We study the behaviour of the Hironaka quotients of  $(f, g)$  associated to the vertices of  $G(Y)$ . We show that there exists maximal oriented arcs in  $G(Y)$  along which the Hironaka quotients of  $(f, g)$  strictly increase and they are constant on the connected components of the closure of the complement of the union of the maximal oriented arcs.

**Mathematics Subject Classifications (2000).** 14B05, 14J17, 32S15, 32S45, 32S55, 57M45.

**Key words.** Normal surface singularity, Resolution of singularities, Hironaka quotients, Discriminant.

## 1 Introduction

Let  $\phi = (f, g) : (X, p) \rightarrow (\mathbb{C}^2, 0)$  be a finite analytic morphism which is defined on a complex analytic normal surface germ  $(X, p)$  by two complex analytic function germs  $f$  and  $g$ .

We say that  $\pi : (Y, E_Y) \rightarrow (X, p)$  is a *good resolution of  $\phi$*  if it is a resolution of the singularity  $(X, p)$  in which the total transform  $E_Y^\dagger = ((fg) \circ \pi)^{-1}(0)$  is a normal crossings divisor and such that the irreducible components of the exceptional divisor  $E_Y = \pi^{-1}(p)$  are non-singular. An irreducible component  $E_i$  of  $E_Y$  is called a *rupture component* if it is not a rational curve or if it intersects at least three other components of the total transform. By definition a *curveta*  $c_i$  of an irreducible component  $E_i$  of  $E_Y$  is a smooth curve germ that intersects transversally  $E_i$  at a smooth point of the total transform. To each irreducible component  $E_i$  of  $E$  we associate the rational number:

$$q_{E_i} = \frac{V_{f \circ \pi}(c_i)}{V_{g \circ \pi}(c_i)}$$

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where  $V_{f \circ \pi}(c_i)$  (resp.  $V_{g \circ \pi}(c_i)$ ) is the order of  $f \circ \pi$  (resp.  $g \circ \pi$ ) on  $c_i$ . This quotient is called the *Hironaka quotient* of  $(f, g)$  on  $E_i$ .

This set of rational numbers associated to a complex analytic normal surface germ has been first introduced in [6]. It is shown, in particular, that if  $(u, v)$  are local coordinates of  $(\mathbb{C}^2, 0)$  and if  $\pi$  is the minimal good resolution of  $\phi$ , then the subset of Hironaka quotients associated to the rupture components of  $E_Y$  are topological invariants of  $(\phi, u, v)$ . Another proof of this result is given in [12].

In this paper we study the growth behaviour of the Hironaka quotients of  $\phi = (f, g)$  in the dual graph of a good resolution  $R : (X', E_{X'}) \rightarrow (X, p)$  of the pair  $(X, \{fg = 0\})$ , obtained (see Section 2) using Hirzebruch-Jung's method. We also consider  $\rho : (\tilde{X}, E_{\tilde{X}}) \rightarrow (X, p)$ , the minimal good resolution of  $(X, p)$  such that the total transform of  $\{fg = 0\}$  (by  $\rho$ ) is a normal crossings divisor. By definition  $\rho$  is *the minimal resolution of  $\phi$* . But,  $X'$  dominates  $\tilde{X}$  by  $\beta : X' \rightarrow \tilde{X}$  which is a sequence of blowing-downs of some specific irreducible components of  $E_{X'}$ . Then, we obtain similar results, on the growth behaviour of the Hironaka quotients of  $(f, g)$ , for the minimal resolution of  $\phi$  and we can generalize them to any good resolution of  $\phi$ .

Let  $\pi : (Y, E_Y) \rightarrow (X, p)$  be a good resolution of  $\phi$ . The weighted dual graph associated to  $\pi$ , denoted  $G(Y)$ , is constructed as follows. To each irreducible component  $E_i$  of the exceptional divisor  $E_Y$  we associate a vertex  $(i)$  weighted by its Hironaka quotient  $q_{E_i}$ . When two irreducible components of  $E_Y$  intersect, we join their associated vertices by edges which number is equal to the number of intersection points. When  $k$  ( $k \geq 0$ ) irreducible components of the strict transform of  $\{fg = 0\}$  meet  $E_i$ , we add to the vertex  $(i)$   $k$  edges. If an edge represents the intersection point of an irreducible component of the strict transform of  $f$  (resp.  $g$ ) with  $E_i$ , we endow the edge with a going-out arrow (resp. a going-in arrow (it means a reverse arrow)). By convention the Hironaka quotient of a going-in arrow is 0 and the Hironaka quotient of a going-out arrow is infinite.

Moreover by construction the graph  $G(Y)$  is partially oriented as follows.

Let  $(e_{ij})$  be an edge which represents an intersection point  $E_i \cap E_j$ . When  $q_{E_i} = q_{E_j}$  the edge  $(e_{ij})$  is not oriented. When  $q_{E_i} < q_{E_j}$  then  $(e_{ij})$  is oriented from  $(i)$  to  $(j)$  and we say that the edge  $e_{ij}$  is positively oriented.

**Definition 1** *A maximal arc in  $G(Y)$  is a subgraph which is homeomorphic to a segment and which satisfies the following conditions:*

1. *it begins with a going-in arrow and ends with a going-out arrow,*
2. *it is a sequence of positively oriented edges,*
3. *the orientation of the edges induces a compatible positive orientation on the whole segment.*



Figure 1: The two possible shapes of a maximal arc in  $G(Y)$ .

We denote  $A(Y)$  the union of all maximal arcs in  $G(Y)$ .

**Remark 1** A vertex  $(i)$  of  $G(Y)$  is in  $A(Y)$  if and only if there exists at least one going-in arrow or edge arriving at  $(i)$  and at least one going-out arrow or edge leaving  $(i)$ .

Our main result is :

**Theorem 1** Let  $\pi : (Y, E_Y) \rightarrow (X, p)$  be a good resolution of  $\phi$ . The Hironaka quotients of  $\phi$  on the vertices of a connected component of the closure of  $G(Y) \setminus A(Y)$  are constant.

Moreover  $G(Y) \setminus A(Y)$  doesn't contain any arrow.

**Remark 2** A consequence of Theorem 1 is that  $G(Y) \setminus A(Y)$  does not contain any oriented edge.

Let  $(X, p)$  be an irreducible complex analytic surface germ (in particular,  $p$  is not necessarily an isolated singular point) and let  $\phi : (X, p) \rightarrow (\mathbb{C}^2, 0)$  be a finite analytic morphism defined on  $(X, p)$ . Theorem 1 will also be true for the resolutions of  $\phi$  which begin by the normalization  $\nu : (\bar{X}, \bar{p}) \rightarrow (X, p)$ . More precisely if  $\bar{R} : (\bar{Y}, E_{\bar{Y}}) \rightarrow (\bar{X}, \bar{p})$  is a good resolution of  $\phi \circ \nu$ , we apply theorem 1 to the finite morphism  $\phi \circ \nu$  and the resolution  $\bar{R}$ . Using the notation  $G(\bar{Y})$  for weighted dual graph associated to  $\bar{R}$ , we have:

**Theorem (generalized)** Let  $\phi : (X, p) \rightarrow (\mathbb{C}^2, 0)$  be a finite morphism defined on an irreducible complex analytic surface germ  $(X, p)$ . Let  $\nu \circ \bar{R} : (\bar{Y}, E_{\bar{Y}}) \rightarrow (X, p)$  be a good resolution of  $\phi$ . The Hironaka quotients of  $\phi$  on the vertices of a connected component of the closure of  $G(\bar{Y}) \setminus A(\bar{Y})$  are constant.

Moreover  $G(\bar{Y}) \setminus A(\bar{Y})$  doesn't contain any arrow.

One motivation to study Hironaka quotients is their relations with the Puiseux expansion of the branches of the discriminant of  $\phi$ . The first Puiseux exponents of the discriminant of  $\phi$  are the Hironaka quotients on the rupture vertices of the minimal resolution of  $\phi$  (see [6] and [12]). Moreover, as proved in [3], it is possible to express all the Puiseux exponents of the discriminant curve of a finite morphism  $\phi$  as some Hironaka quotients of the minimal resolution of finite morphisms  $\phi_i : (X, p) \rightarrow (\mathbb{C}^2, 0)$  defined by an iterative process which begins with  $\phi$ . It is illustrated in example 3.

We give two other examples to express the interest of Theorem 1. In example 1,  $(X, p) = (\mathbb{C}^2, 0)$  and Theorem 1 is applied to show the growth of the Hironaka quotients of  $\phi = (f, g)$ . This behaviour of Hironaka quotients can not be obtained using the previous results of [11], because  $\{f = 0\}$  and  $\{g = 0\}$  many branches with high contact. In example 2,  $(X, p)$  is singular and we have non trivial subgraphs  $A(\tilde{X})$  and  $\overline{G(\tilde{X}) \setminus A(\tilde{X})}$ .

Theorem 1 will be proved using Theorem 2 (Section 6.2) which precises the behaviour of Hironaka quotients in the Hirzebruch-Jung resolution of  $\phi$  described in section 2 (Definition 3). To prove Theorem 2, we relate the Hironaka quotients with the first Puiseux exponents of plane curve germs as follows.

Let  $(c, p)$  be a germ of irreducible curve on  $(X, p)$  which is not a branch of  $\{fg = 0\}$ . Let  $\pi : (Y, E_Y) \rightarrow (X, p)$  be a good resolution of  $\phi$ . Let  $(c_Y, z)$  be an irreducible component of the strict transform of  $(c, p)$  in  $(Y, E_Y)$ , in particular  $z \in E_Y$ . As explain in Section 3, the first Puiseux exponent  $q_c$  of the plane curve germ  $((\phi \circ \pi)(c), 0) \subset (\mathbb{C}^2, 0)$  has the following behaviour:

if  $z$  is a smooth point of the total transform  $E_Y^+$  and if  $E_i$  is the irreducible component of  $E_Y$  which contains  $z$ , then  $q_c$  is equal to the Hironaka quotient  $q_{E_i}$  of  $E_i$ .

if  $z \in E_i \cap E_j$  and  $q_{E_i} = q_{E_j}$ , then  $q_{E_i} = q_c = q_{E_j}$ .

if  $z \in E_i \cap E_j$  and  $q_{E_i} < q_{E_j}$ , then  $q_{E_i} < q_c < q_{E_j}$ .

This allows us to describe in Lemma 4 (in section 5), the growth behaviour of the Hironaka quotients associated to the minimal resolution of a quasi-ordinary normal surface germ.

In sections 2 we define the Hirzebruch-Jung resolution of  $\phi$  and we describe its topological properties used in sections 4 to 7.

In section 7 we show how Theorem 1 can be deduced from Theorem 2.

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## 2 The Hirzebruch-Jung's resolution of $(X, p)$ associated to $\phi$

Let  $\phi = (f, g) : (X, p) \rightarrow (\mathbb{C}^2, 0)$  be a finite analytic morphism which is defined on a complex analytic normal surface germ  $(X, p)$  by two complex analytic function germs  $f$  and  $g$ .

The discriminant curve of  $\phi$  is the image by  $\phi$  of the critical locus  $C(\phi)$  of  $\phi$ . We denote  $\Delta$  the union of the irreducible components of the discriminant curve which are not included in  $\{uv = 0\}$ .

We denote by  $r : (Z, E_Z) \rightarrow (\mathbb{C}^2, 0)$  the minimal embedded resolution of  $\Delta^+ = \Delta \cup \{uv = 0\}$  and  $G(Z)$  its dual graph constructed as described in the introduction where  $f$  is replaced by  $u$ ,  $g$  by  $v$  and we add an edge ended by a star for each irreducible component of the strict transform of  $\Delta$ . Moreover we weight the vertex associated to an irreducible component  $D_i$  of  $E_Z$  by its Hironaka quotient  $q_{D_i}$  defined as follows:

$$q_{D_i} = \frac{V_{u \circ r}(c_i)}{V_{v \circ r}(c_i)}$$

where  $c_i$  is a curvetta of  $D_i$ .

We construct as in [8] and [14] a Hirzebruch-Jung resolution of  $\phi : (X, p) \rightarrow (\mathbb{C}^2, 0)$ . Here, we begin with the minimal resolution  $r$  of  $\Delta^+$ . The pull-back of  $\phi$  by  $r$  is a finite morphism  $\phi_r : (Z', E_{Z'}) \rightarrow (Z, E_Z)$  which induces an isomorphism from  $E_{Z'}$  to  $E_Z$ . We denote  $r_\phi$  the pull-back of  $r$  by  $\phi$ ,  $r_\phi : (Z', E_{Z'}) \rightarrow (X, p)$ .

In general  $Z'$  is not normal. Let  $n : (\bar{Z}, E_{\bar{Z}}) \rightarrow (Z', E_{Z'})$  be its normalization.

**Remark 3** 1. *By construction, the discriminant locus of  $\phi_r \circ n$  is included in  $E_{\bar{Z}}^+ = r^{-1}(\Delta^+)$  which is the total transform of  $\Delta^+ = \Delta \cup \{uv = 0\}$  in  $Z$ .*

2. *Let  $E_{Z'}^0$  be the open set of the points of  $E_{Z'}$  which are smooth points in the total transform  $E_{Z'}^+ = \phi_r^{-1}(E_Z^+)$ . If  $z' \in E_{Z'}^0$ , there exists a small neighbourhood  $U'$  of  $z'$  in  $Z'$  which is a  $\mu$ -constant family of curves parametrized by  $U' \cap E_{Z'}$ . Of course  $U'$  can be chosen such that  $U' \cap E_{Z'}$  is a smooth disc in  $E_{Z'}^0$ . Therefore  $n^{-1}(U')$  is a finite disjoint union of smooth germs of surface.*

$$\begin{array}{ccc}
(Z', E_{Z'}) & \xrightarrow{r_\phi} & (X, p) \\
\downarrow \phi_r & & \downarrow \phi \\
(Z, E_Z) & \xrightarrow{r} & (\mathbb{C}^2, 0)
\end{array}$$

Figure 2: The first step to construct the Hirzebruch-Jung resolution of  $\phi$ .

3. The restriction of the map  $\phi_r \circ n$  to  $E_{\bar{Z}}$  induces a finite morphism from  $E_{\bar{Z}}$  to  $E_Z$  which is a regular covering on  $E_{\bar{Z}}^0 = n^{-1}(E_{Z'}^0)$ .

**Definition 2** A Hirzebruch-Jung singularity is a quasi-ordinary singularity of normal surface germ.

**Lemma 1** Let  $P$  be a double point of  $E_Z^+$  and  $\bar{P}$  a point of  $(\phi_r \circ n)^{-1}(P)$ . Then  $\bar{P}$  is a double point of  $E_{\bar{Z}}^+$ . Moreover, if  $\bar{P}$  is not a smooth point of  $\bar{Z}$  then  $\bar{P}$  is a Hirzebruch-Jung singularity of  $\bar{Z}$ .

*Proof.* Let  $P$  be a double point of  $E_Z^+$  and  $U(P)$  be a regular neighbourhood of  $P$  in  $Z$ . As  $Z$  is a smooth surface,  $\partial U(P)$  is a 3-dimensional sphere. Let us show that  $(\phi_r \circ n)^{-1}(P)$  is a union of double points of  $E_{\bar{Z}}^+$ . If  $\bar{P} \in (\phi_r \circ n)^{-1}(P)$ , let  $U$  be the connected component of  $(\phi_r \circ n)^{-1}(U(P))$  that contains  $\bar{P}$ . Let  $\phi_r \circ n|_U$  be the restriction of  $\phi_r \circ n$  to the boundary  $\partial U$  of  $U$ . So,  $\phi_r \circ n|_U : \partial U \rightarrow \partial U(P)$  is a finite ramified covering with ramification locus in  $\partial U(P) \cap E_Z^+$ . As  $Z$  is smooth and  $P$  is a double point of  $E_Z$ , then  $\partial U(P) \cap E_Z^+$  is a Hopf link in the 3-sphere  $\partial U(P)$ . Therefore  $\partial U$  is a lens space that contains the link  $\partial U \cap E_{\bar{Z}}^+$  included in two distinct irreducible components of  $E_{\bar{Z}}^+$ . Hence,  $\bar{P}$  is a double point of  $E_{\bar{Z}}^+$ . As the ramification locus of  $\phi_r \circ n|_U : U \rightarrow U(P)$  is included in a normal crossing divisor, if  $\bar{P}$  is not a smooth point of  $\bar{Z}$  it is a Hirzebruch-Jung singularity.

As explain in lemma 1, if  $\bar{z}$  is a singular point of  $\bar{Z}$ , then  $(\phi_r \circ n)(\bar{z})$  is a double point of  $E_Z^+$ . In particular, there are finitely many isolated singular points in  $\bar{Z}$ . The singularities of  $\bar{Z}$  are Hirzebruch-Jung singularities. More precisely, let  $\bar{z}_i, 1 \leq i \leq n$ , be the finite set of the singular points of  $\bar{Z}$  and  $(\bar{Z}_i, \bar{z}_i)$  a sufficiently small neighbourhood of  $\bar{z}_i$  in  $\bar{Z}$ . We have the following result (see [14] or [8] for a proof):

**Theorem.** The exceptional divisor of the minimal resolution of  $(\bar{Z}_i, \bar{z}_i)$  is a normal crossings divisor, each irreducible component of its exceptional divisor is a smooth rational curve, and its resolution dual graph is a bamboo (it means is homeomorphic to a segment).

**Remark 4** In  $\bar{Z}$  an irreducible component of the strict transform of  $\{fg = 0\}$  is not necessarily a curvetta of an irreducible component of the exceptional divisor. But the normalization morphism  $n$  has separated the irreducible components of  $\{f = 0\}$  from the ones of  $\{g = 0\}$ .

Let  $\bar{\rho}_i : (Z''_i, E_{Z''_i}) \rightarrow (\bar{Z}_i, \bar{z}_i)$  be the minimal resolution of the singularity  $(\bar{Z}_i, \bar{z}_i)$ . From [8] (corollary 1.4.3), see also [14] (paragraph 4), the spaces  $Z''_i$  and the maps  $\bar{\rho}_i$  can be glued for  $1 \leq i \leq n$ , in a suitable way to give a smooth space  $X'$  and a map  $\bar{\rho} : (X', E_{X'}) \rightarrow (\bar{Z}, E_{\bar{Z}})$  satisfying the following property :

**Proposition 1** The map  $r_\phi \circ n \circ \bar{\rho} : (X', E_{X'}) \rightarrow (X, p)$  is a good resolution of the singularity  $(X, p)$  in which the strict transform of  $\{fg = 0\}$  is a normal crossings divisor.

*Proof.* The resolution  $r$  separates the strict transform of  $\{u = 0\}$  from the one of  $\{v = 0\}$ . All the branches of the strict transform of  $\{g = 0\}$  (resp.  $\{f = 0\}$ ) by  $r_\phi$  meet  $E_{Z'}$  at the same point  $P'$  (resp.  $Q'$ ) and  $P' \neq Q'$  because  $\phi_r(P') \neq \phi_r(Q')$ . The normalization morphism  $n$  separates the irreducible components of  $f = 0$  and those of  $g = 0$ . In  $\bar{Z}$ , let  $\bar{P}$  be the intersection point of an irreducible component of the strict transform of  $\{g = 0\}$  (resp.  $\{f = 0\}$ ) with  $E_{\bar{Z}}$ . Let  $P = (\phi_r \circ n)(\bar{P})$ ,  $U(P)$  a regular neighbourhood of  $P$  in  $Z$  and  $U$  the connected component of  $(\phi_r \circ n)^{-1}(U(P))$  that contains  $\bar{P}$ .  $U(P)$  is a smooth surface germ that contains the double point  $P$ . Then from lemma 1,  $\bar{P}$  is either a smooth point of  $\bar{Z}$ , either a Hirzebruch-Jung singularity of  $\bar{Z}$ . In the second case,  $\bar{\rho}$  is a resolution of  $\bar{P}$ .

Let us denote  $R := r_\phi \circ n \circ \bar{\rho}$ .

As  $R$  is the composition of three well defined morphisms, we can use the following definition which is a relative (to  $\{fg = 0\}$ ) version of the classical Hirzebruch-Jung resolution of a normal germ of surface.

**Definition 3** The morphism  $R : (X', E_{X'}) \rightarrow (X, p)$  is the Hirzebruch-Jung resolution associated to  $\phi$ .

Now we can use the following result (for a proof see [4], Theorem 5.9, p.87):

**Theorem .** Let  $\rho : (\tilde{X}, E_{\tilde{X}}) \rightarrow (X, p)$  be the minimal resolution of  $\phi$ . There exists  $\beta : (X', E_{X'}) \rightarrow (\tilde{X}, E_{\tilde{X}})$  such that  $\rho \circ \beta = R$  and the map  $\beta$  consists in a composition of blowing-downs of irreducible components, of the successively obtained exceptional divisors, of self-intersection  $-1$ , genus 0 and which are not rupture components.

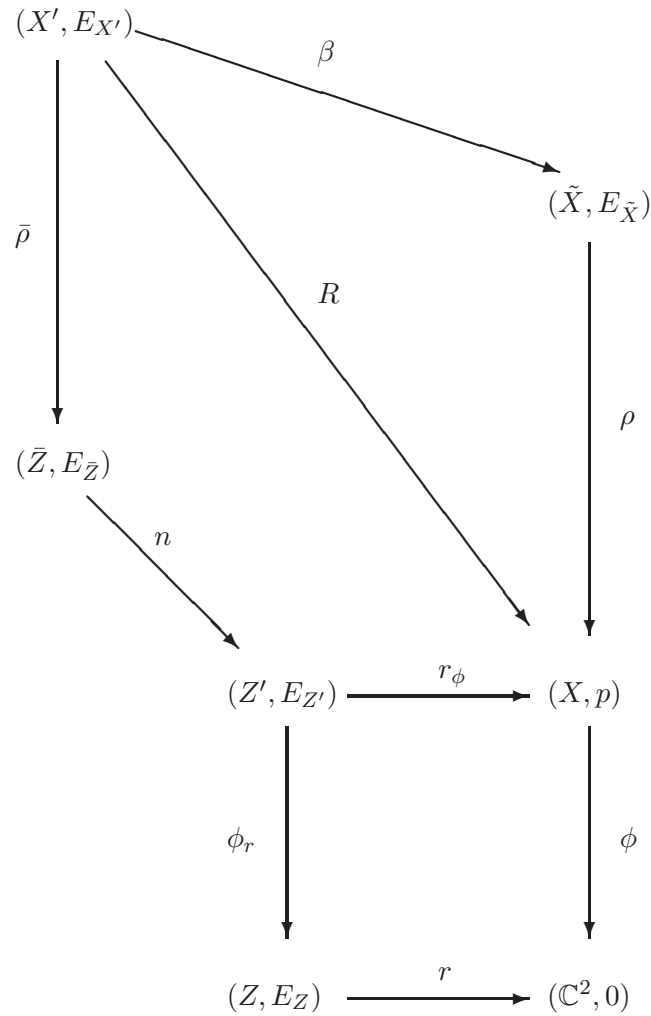


Figure 3: The commutative diagram of the morphisms involved in the Hizebruch-Jung resolution of  $\phi$ .

### 3 The quotients associated to the morphism $\phi$

Let  $(c, p)$  be a germ of irreducible curve on  $(X, p)$  which is not a branch of  $\{fg = 0\}$ . Let  $V_f(c)$  (resp.  $V_g(c)$ ) be the order of  $f$  (resp.  $g$ ) on  $c$ .

**Definition 4** *The contact quotient  $q_c$  of  $(c, p)$  associated to the morphism  $\phi = (f, g)$  is equal to:*

$$q_c = \frac{V_f(c)}{V_g(c)}.$$

The following remark relates the contact quotient of a germ  $(c, p)$  in  $(X, p)$  with the first Puiseux exponent of the direct image of  $(c, p)$  by  $\phi$ .



**Remark 5** For local coordinates  $(u, v)$  of  $(\mathbb{C}^2, 0)$  such that  $u \circ \phi = f$  and  $v \circ \phi = g$ , let  $u = a_0 v^{m/n} + a_1 v^{(m+1)/n} + \dots$ ,  $a_0 \neq 0$ , be a Puiseux expansion of  $\phi(c)$ . The definition of  $q_c$  implies that  $q_c$  is equal to the first Puiseux exponent of  $\phi(c)$ :

$$q_c = \frac{m}{n} = \frac{V_u(\phi(c))}{V_v(\phi(c))}.$$

When the strict transform of a germ  $(c, p)$  in a good resolution  $\pi : (Y, E_Y) \rightarrow (X, p)$  of  $(X, p)$  meets  $E_Y^+$  at a smooth point, then  $q_c$  is an Hironaka quotient. More precisely, we recall proposition 2.1 of [12].

**Proposition 2.1 of [12].** Let  $\pi : (Y, E_Y) \rightarrow (X, p)$  be a good resolution of  $\phi$  and  $E_i$  be an irreducible component of  $E_Y$  of Hironaka quotient  $q_{E_i}$ . We denote by  $E_i^0$  the smooth points of  $E_i$  in the total transform by  $\pi$  of  $\{fg = 0\}$ .

Let  $x$  be a point of  $E_i^0$  and  $(\xi_i, x)$  be an irreducible germ of curve at  $x$ . Then the contact quotient  $q_{\pi(\xi_i)}$  of  $(\pi(\xi_i), p)$  is equal to the Hironaka quotient on  $E_i$ :

$$q_{\pi(\xi_i)} = q_{E_i}.$$

Let  $r : (Z, E_Z) \rightarrow (\mathbb{C}^2, 0)$  be the minimal embedded resolution of  $\Delta^+ = \Delta \cup \{uv = 0\}$ . Let  $D_i$  be an irreducible component of  $E_Z$  and  $c_i$  a curvetta of  $D_i$ .

**Definition 5** The Hironaka quotient of  $D_i$ , denoted  $q_{D_i}$ , is equal to:

$$q_{D_i} = \frac{V_{u \circ r}(c_i)}{V_{v \circ r}(c_i)}.$$

**Remark 6** Let  $(\gamma, 0)$  be an irreducible curve germ in  $(\mathbb{C}^2, 0)$  which admits  $u = a_0 v^{m/n} + a_1 v^{(m+1)/n} + \dots$ ,  $a_0 \neq 0$  as Puiseux expansion. Let  $(C, z)$  be the strict transform by  $r$  of  $(\gamma, 0)$  in  $(Z, E_Z)$ . Then  $\frac{m}{n} = \frac{V_{u \circ r}(C)}{V_{v \circ r}(C)}$ .

Hence, if  $z$  is a smooth point of an irreducible component  $D_i$  of  $E_Z$ , we have  $\frac{m}{n} = q_{D_i}$ .

The following lemma is quite obvious, but very useful for computation of Hironaka quotients.

**Lemma 2** Let  $(c', p')$  be a germ of curve (at  $p'$ ). Let  $\alpha : (c', p') \rightarrow (c, p)$  be a holomorphic germ which is a ramified covering over  $(c, p)$  of generic degree  $k$  and ramification locus  $p'$ . We have:

$$q_c = \frac{V_f(c)}{V_g(c)} = \frac{V_{f \circ \alpha}(c')}{V_{g \circ \alpha}(c')}.$$

*Proof.* We have the following orders of functions:

$$V_{f \circ \alpha}(c') = k(V_f(c)) \text{ and } V_{g \circ \alpha}(c') = k(V_g(c)).$$

As in Section 2, we denote by  $\rho : (\tilde{X}, E_{\tilde{X}}) \rightarrow (X, p)$  the minimal resolution of  $\phi$  and by  $R : (X', E_{X'}) \rightarrow (X, p)$  the Hirzebruch-Jung resolution of  $\phi$ .

Using the above remark 5 and lemma 2 we obtain the following behaviour of the Hironaka quotients for the divisors and morphisms involved in the diagram of Figure 1.

**Lemma 3** *Let  $E'_i$  be an irreducible component of  $E_{X'}^+$ .*

*If  $(\phi_r \circ n \circ \bar{\rho})(E'_i)$  is an irreducible component  $D_i$  in  $E_Z^+$ , then  $q_{E'_i} = q_{D_i}$ .*

*If  $(\phi_r \circ n \circ \bar{\rho})(E'_i) \in D_i \cap D_j$  with  $q_{D_i} = q_{D_j}$ , then  $q_{E'_i} = q_{D_i} = q_{D_j}$ .*

*If  $(\phi_r \circ n \circ \bar{\rho})(E'_i) \in D_i \cap D_j$  with  $q_{D_i} < q_{D_j}$ , then  $q_{D_i} \leq q_{E'_i} \leq q_{D_j}$ .*

*When  $\beta(E'_i)$  is an irreducible component  $\tilde{E}_i$  of  $E_{\tilde{X}}$ , we have  $q_{E'_i} = q_{\tilde{E}_i}$ .*

## 4 The maximal arc for the minimal resolution of $\Delta^+$

As in Section 2,  $r : (Z, E_Z) \rightarrow (\mathbb{C}^2, 0)$  is the minimal embedded resolution of  $\Delta^+ = \Delta \cup \{uv = 0\}$ . Let  $G(Z)$  be its dual graph constructed as described in the introduction where  $f$  is replaced by  $u$ ,  $g$  by  $v$ . We add an edge ended by a star for each irreducible component of the strict transform of  $\Delta$ . Moreover we weight the vertex associated to an irreducible component  $D_i$  of  $E_Z$  by its Hironaka quotient  $q_{D_i}$  (see definition 5).

From remark 5, the quotient  $q_{D_i} = \frac{V_{u \circ r}(c_i)}{V_{v \circ r}(c_i)}$  is equal to the first Puiseux exponent of  $(r(c_i), 0)$ .

As  $\Delta^+$  is a plane curve germ,  $G(Z)$  is a tree. We consider the subgraph  $S(Z)$  of  $G(Z)$  which is the geodesic beginning with the (reverse) arrow associated to  $v$  and ending at the arrow associated to  $u$ . We orient this geodesic from  $v$  to  $u$ .

**Proposition 2** *The graph  $G(Z)$  admits an unique maximal arc which is equal to  $S(Z)$ .*

*Proof.* As  $G(Z)$  is a tree,  $S(Z)$  is homeomorphic to a segment. Notice that  $G(Z)$  has only two arrows (one associated to the strict transform of  $\{u = 0\}$  and the other to the strict transform of  $\{v = 0\}$ ), both of them contained in  $S(Z)$ . So  $\overline{G(Z)} \setminus S(Z)$  doesn't contain any arrow.

We number the irreducible components of  $E_Z$  corresponding to the vertices of  $S(Z)$  from  $v$  to  $u$ . Let  $(i)$  and  $(i+1)$  be two consecutive vertices on  $S(Z)$  which represent respectively the irreducible components  $D_i$  and  $D_{i+1}$ . We have to prove that  $q_{D_i} < q_{D_{i+1}}$ .

Let  $c_i$  (resp.  $c_{i+1}$ ) be a curvetta of  $D_i$  (resp.  $D_{i+1}$ ). The curve  $r(c_i)$  (resp.  $r(c_{i+1})$ ) admits a Puiseux expansion beginning by:

$$u = a_{i,0}v^{m_i/n_i} + a_{i,1}v^{(m_i+1)/n_i} + \dots \quad (\text{resp. } u = a_{i+1,0}v^{m_{i+1}/n_{i+1}} + a_{i+1,1}v^{(m_{i+1}+1)/n_{i+1}} + \dots)$$

The resolution of plane curve singularities computed by continued fraction expansion (for example see [9], ch. 6) implies that  $m_i/n_i < m_{i+1}/n_{i+1}$ . But  $m_i/n_i = q_{D_i}$  and  $m_{i+1}/n_{i+1} = q_{D_{i+1}}$ . So,  $S(Z)$  is a maximal arc as defined in the introduction.

It leaves to show that on a connected part  $T$  of  $\overline{G(Z) \setminus S(Z)}$  the Hironaka quotients are constant.

The intersection of  $T$  with  $S(Z)$  is composed of a unique vertex. Let us call it  $(i)$ . An irreducible component  $D_j$  associated to a vertex of  $T$  is obtained by a sequence of blowing-up of points which begins with the blow-up of a smooth point (in the total transform of  $\{uv = 0\}$ ) of  $D_i$ . From proposition 2.1 of [12],  $q_{D_j} = q_{D_i}$ .

Before describing the behaviour of the Hironaka quotients associated to  $(X', E_{X'})$ , we need to study the quotients associated to a resolution of a quasi-ordinary normal surface germ. We will use it to compute the Hironaka quotients on the irreducible components of  $E_{X'}$  created by  $\bar{\rho}$ .

## 5 Quotients associated to the minimal resolution of a quasi-ordinary normal surface germ

**Definition 6** *A germ  $(W, z)$  of normal surface is quasi-ordinary if there exists a finite morphism  $\Phi : (W, z) \rightarrow (\mathbb{C}^2, 0)$  such that the discriminant locus is the union of the two coordinate axes of  $\mathbb{C}^2$ .*

Let  $\Phi : (W, z) \rightarrow (\mathbb{C}^2, 0)$  be a finite morphism defined on a quasi-ordinary normal surface germ such that the discriminant locus is the union of the two coordinate axes of  $\mathbb{C}^2$ .

Let us denote  $(u, v)$  the coordinate of  $\mathbb{C}^2$ .

**Remark 7** *The link of  $W$  is connected because  $(W, z)$  is an irreducible germ of complex surface. As  $\{uv = 0\}$  is the discriminant locus of  $\Phi : (W, z) \rightarrow (\mathbb{C}^2, 0)$ , the topology of the situation implies that  $\Phi^{-1}(\{u = 0\})$  (resp.  $\Phi^{-1}(\{v = 0\})$ ) is an irreducible germ of curve in  $(W, z)$ .*

**Proposition 3** (see Theorem 1.4.2 of [8])  *$(W, z)$  has a minimal good resolution  $\rho_W : (\tilde{W}, E_{\tilde{W}}) \rightarrow (W, z)$  such that :*

*1) the dual graph of  $E_{\tilde{W}}$  is a bamboo and all the vertices represent a rational smooth curve.*

*Let  $k$  be the number of irreducible components of  $E_{\tilde{W}}$ . We orient the bamboo from the vertex (1) to the vertex ( $k$ ). The vertices are indexed by this orientation.*

II) the strict transform of  $\Phi^{-1}(\{v = 0\})$  (resp.  $\Phi^{-1}(\{u = 0\})$ ) is a curvetta of the irreducible component  $E_1^{\tilde{W}}$  (resp.  $E_k^{\tilde{W}}$ ) of  $E_{\tilde{W}}$ .

To obtain the dual graph  $G(\tilde{W})$  we add to the vertex (1) (resp. (k)) of the dual graph of  $E_{\tilde{W}}$  a reverse arrow indices by (v) which represents the strict transform (by  $\Phi \circ \rho_{\tilde{W}}$ ) of  $\{v = 0\}$  (resp. an arrow indices by (u) which represents the strict transform of  $\{u = 0\}$ ). We get a graph which has the following shape:



The total transform of  $\{uv = 0\}$  is  $E_{\tilde{W}}^+ = (\Phi \circ \rho_W)^{-1}(\{uv = 0\})$ . For all  $i$ ,  $1 \leq i \leq k$ , let  $x_i$  be a point of  $E_i^{\tilde{W}}$  which is smooth in  $E_{\tilde{W}}^+$  and  $(c_i, x_i)$  be a curvetta of  $E_i^{\tilde{W}}$ . Let  $\Gamma$  be the union of the plane curve germs  $\gamma_i = (\Phi \circ \rho_W)(c_i)$  and  $\Gamma^+ = \Gamma \cup \{uv = 0\}$ .

**Lemma 4** Let  $\frac{m_i}{n_i}$  be the first Puiseux exponent of  $\gamma_i$ . For all  $i$ ,  $1 \leq i < k$ , we have  $\frac{m_i}{n_i} < \frac{m_{i+1}}{n_{i+1}}$ .

**Remark 8** From section 3, the rational number  $\frac{m_i}{n_i} \in \mathbb{Q}_+$  is the Hironaka quotient  $q_{E_i^{\tilde{W}}}$  of  $\Phi$  on  $E_i^{\tilde{W}}$ .

*Proof of lemma 4.* We order the set  $Q = \{\frac{m_i}{n_i}\}$  of the first Puiseux exponent of the irreducible components of  $\Gamma$ . We obtain  $Q = \{s_1 < \dots < s_j < \dots < s_{k'}\}$  where  $k' \leq k$ ,  $s_j \in \mathbb{Q}_+$ . So, for all  $j$ ,  $1 \leq j \leq k'$ , there exists at least one index  $i(j)$  such that  $\frac{m_{i(j)}}{n_{i(j)}} = s_j$ .

The curve  $\gamma_{i(j)}$  admits a Puiseux expansion which begins by:

$$u = a_{i(j)0} v^{m_{i(j)}/n_{i(j)}} + a_{i(j)1} v^{(m_{i(j)}+1)/n_{i(j)}} + \dots, a_{i(j)0} \neq 0$$

We will say that the plane curve germ  $\gamma'_{i(j)}$ , having

$$u = a_{i(j)0} v^{m_{i(j)}/n_{i(j)}}$$

as Puiseux expansion, is the shadow of  $\gamma_{i(j)}$ . Let  $\Gamma'$  be the union of the curves  $\gamma'_{i(j)}$ ,  $1 \leq j \leq k'$ , and let  $r'$  be the minimal resolution of the plane curve germ  $\Gamma'^+ = \Gamma' \cup \{uv = 0\}$ :

$$r' : (M, E_M) \rightarrow (\mathbb{C}^2, 0).$$

The discriminant locus of  $\Phi$  is  $\{uv = 0\}$ . Here, we begin with the resolution  $r'$  followed by a Hirzebruch-Jung construction to obtain a good resolution of  $(W, z)$ . It is described in figure 4.

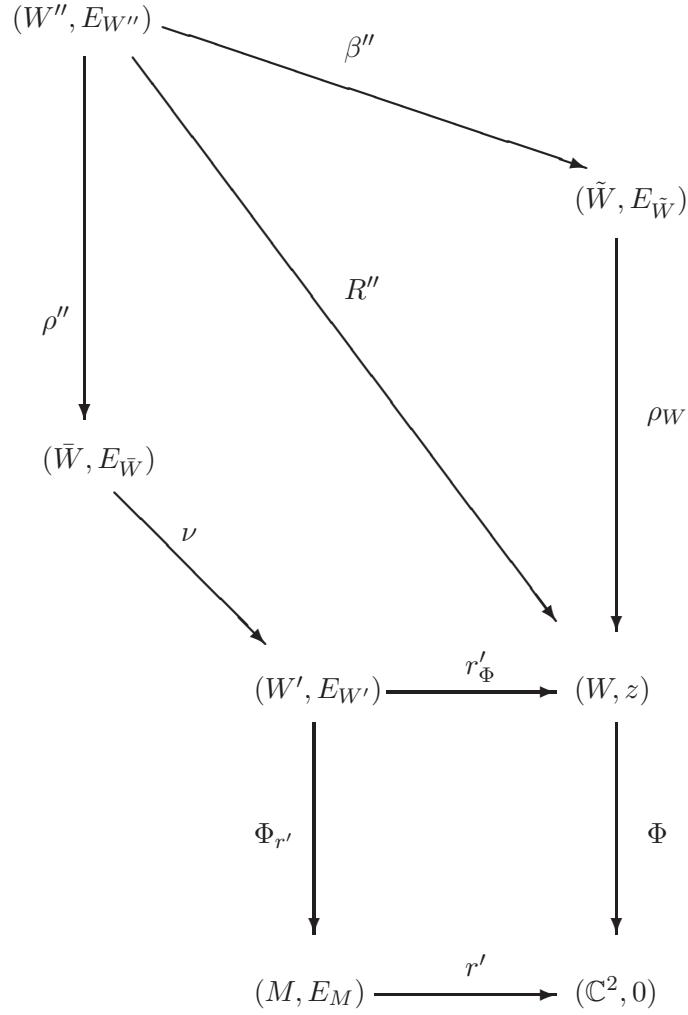


Figure 4: Diagram of the Hirzebruch-Jung's resolution of  $(W, z)$  constructed with the curve  $\Gamma'^+ = \Gamma' \cup \{uv = 0\}$

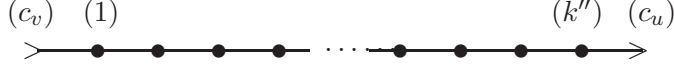
In figure 4,  $(W', E'_{W'})$  is the pull-back of  $\Phi$  by  $r'$ . As explained in Section 2, the normalization  $\nu : (\bar{W}, E_{\bar{W}}) \rightarrow (W', E'_{W'})$  followed by the minimal resolution

$$\rho'' : (W'', E_{W''}) \rightarrow (\bar{W}, E_{\bar{W}})$$

of the isolated singular points of  $\bar{W}$  is a good resolution  $R'' : (W'', E''_{W''}) \rightarrow (W, z)$  of  $(W, z)$ . There exist  $\beta'' : (W'', E''_{W''}) \rightarrow (\tilde{W}, E_{\tilde{W}})$  a composition of contraction of some irreducible components of  $E''_{W''}$  such that  $R'' = \rho_W \circ \beta''$ .

Step I) By the minimal resolution of  $\Gamma'^+ = \Gamma' \cup \{uv = 0\}$ , the dual graph  $G(M)$  (endowed with an arrow (resp. a reverse arrow) representing the strict transform  $c_u$ , of  $\{u = 0\}$  (resp.  $c_v$  of  $\{v = 0\}$ ) is a bamboo with  $k''$  ( $k' \leq k''$ ) vertices. We orient this bamboo from the reverse arrow associated to  $c_v$  to the arrow associated to  $c_u$ . The indices  $(l)$ ,  $1 \leq l \leq k''$ , of the vertices increase with this orientation.

The dual graph of  $E_M$  with the strict transforms  $c_u$  of  $\{u = 0\}$  and  $c_v$  of  $\{v = 0\}$  has the following shape:



The theory of the resolution of plane curve germ obtained by computation of the continued fraction expansion of the first Puiseux exponents  $s_j, 1 \leq j \leq k'$ , implies that the strict transform of a germ having  $s_j$  as first Puiseux exponent meets  $E_M$  at a point of the irreducible component  $D_{l(j)}$  which is a smooth point of  $r'^{-1}(\Gamma'^+)$ . Moreover, we have  $(l(1)) < (l(2)) < \dots < (l(j)) < \dots < (l(k'))$ .

Step II) As  $G(\tilde{W})$  is a bamboo and as  $\Phi^{-1}(\{u = 0\})$  (resp.  $\Phi^{-1}(\{v = 0\})$ ) is an irreducible germ of curve in  $(W, z)$ ,  $\Phi_{r'} \circ \nu$  induces an isomorphism of graph  $\nu_G : G(\tilde{W}) \rightarrow G(M)$ . Indeed:

The strict transform of  $\{v = 0\}$  (resp.  $\{u = 0\}$ ) being irreducible,  $(\Phi_{r'} \circ \nu)^{-1}(D_1)$  (resp.  $(\Phi_{r'} \circ \nu)^{-1}(D_{k''})$ ) is only one irreducible component of  $E_{\tilde{W}}$ . But there is no cycle in the graph  $G(W'')$  because  $G(\tilde{W})$  has no cycle. Moreover,  $\rho''$  is only a resolution of quasi-ordinary singular points. The graph  $G(W'')$  is obtained from  $G(\tilde{W})$  by replacing some edges by bamboos and  $G(\tilde{W})$  has no cycle. So, all the  $(\Phi_{r'} \circ \nu)^{-1}(D_l)$  are irreducible in  $E_{\tilde{W}}$  and two irreducible components of  $E_{\tilde{W}}$  has at most one common point. We can identify  $G(\tilde{W})$  with  $G(M)$  after putting, via  $\nu_G$ , the orientation and indices of  $G(M)$  on  $G(\tilde{W})$ .

Step III) By step II,  $G(\tilde{W})$  is, in particular, a bamboo. The graph  $G(W'')$  is obtained from  $G(\tilde{W})$  by replacing some edges by bamboos, it produces a new bamboo which is just an extension of  $G(\tilde{W})$ . We lift the orientation of  $G(\tilde{W})$  on  $G(W'')$  and we order the indices of the vertices of  $G(W'')$  with the help of this orientation. So,  $\beta'' : (W'', E_{W''}^0) \rightarrow (\tilde{W}, E_{\tilde{W}})$  induces a morphism between two oriented bamboos  $\beta_G'' : G(W'') \rightarrow G(\tilde{W})$ .

For all  $j, 1 \leq j \leq k'$ , let  $D_{l(j)}^0$  be the set of the smooth points (in  $r'^{-1}(\Gamma'^+)$ ) of the irreducible component  $D_{l(j)}$  of  $E_M$  which meets the strict transform of  $\gamma_{i(j)}$ .

It exists only one index  $l''(j)$  such that  $(\Phi_{r'} \circ \nu \circ \rho'')^{-1}(D_{l(j)}^0) = E_{l''(j)}''^0$  and the strict transform of  $\gamma_{i(j)}$  via  $(r' \circ \Phi_{r'} \circ \nu \circ \rho'')$  meets  $E_{l''(j)}''^0$  at a point  $z_{i(j)}$  which is smooth in the total transform of  $\Gamma^+$ . But  $\gamma_{i(j)}$  is the direct image (by  $\Phi \circ \rho_W$ ) of the chosen curvetta  $(c_{i(j)}, x_{i(j)})$  of  $\tilde{E}_{i(j)}$ . The commutation of the diagram implies that  $\beta''(E_{l''(j)}''^0) = \tilde{E}_{i(j)}$ . So,  $\gamma_{i(j)}$  is the only irreducible component of  $\Gamma$  such that  $\frac{m_{i(j)}}{n_{i(j)}} = s_j$ . So,  $k = k'$ , and  $j = i(j) = i$  for all  $j, 1 \leq j \leq k$ . This implies:

$$s_i = \frac{m_i}{n_i} < s_{i+1} = \frac{m_{i+1}}{n_{i+1}}$$

This ends the proof of Lemma 4.

## 6 Behaviour of the Hironaka quotients in each step of the Hirzebruch-Jung resolution

**Remark 9** *The computation of the Hironaka quotients of  $(f, g)$  in each step of the Hirzebruch-Jung resolution is based on the following principle: Lemma 2 and Remark 5 (of section 3), imply that the Hironaka quotient on an irreducible component  $E$  of the exceptional divisor  $E_{\bar{Z}}$  (resp.  $E_{X'}$ ) is equal to the contact quotient of the direct image, in  $(X, p)$ , of a curvetta of  $E$ .*

We will show how the Hironaka quotients associated to the irreducible components of  $E_Z$  enable us to describe the behaviour of the Hironaka quotients on the irreducible components of  $E_{\bar{Z}}$  (resp.  $E_{X'}$ ).

### 6.1 Hironaka quotients associated to $\bar{Z}$

Let  $\bar{E}_i$  be an irreducible component of  $E_{\bar{Z}}$ . Let  $\bar{E}_i^0$  be the open set of the smooth points of  $\bar{E}_i$  in the total transform  $E_{\bar{Z}}^+ = (r \circ \phi_r \circ n)^{-1}(\Delta^+)$ . By construction  $(\phi_r \circ n)(\bar{E}_i^0)$  is the set  $D_i^0$  of the smooth points, in the total transform  $E_Z^+ = r^{-1}(\Delta^+)$ , of an irreducible component  $D_i$  of  $E_Z$ .

**Proposition 4** *The Hironaka quotient on  $\bar{E}_i$  is equal to the Hironaka quotient on  $D_i$ .*

*Proof.* Let  $\bar{z} \in \bar{E}_i^0$  and let  $(\bar{c}_i, \bar{z})$  be a curvetta of  $\bar{E}_i$ . In section 3, we have seen that the Hironaka quotient  $q_{\bar{E}_i}$  is equal to the first Puiseux exponent of  $(\phi \circ r_\phi \circ n)(\bar{c}_i, \bar{z}) = (r \circ \phi_r \circ n)(\bar{c}_i, \bar{z})$ . Let  $z = (\phi_r \circ n)(\bar{z}) \in D_i^0$ . But, as  $(\gamma_i, z) = (\phi_r \circ n)(\bar{c}_i, \bar{z})$  is a germ of curve at a point of  $D_i^0$ , this implies that  $(r(\gamma_i), 0)$  has the same first Puiseux exponent than  $(r(c_i), 0)$  where  $(c_i, z)$  is a curvetta (at  $z$ ) of  $D_i$  and which is equal to the Hironaka quotient on  $q_{D_i}$  (see section 4). This proves that  $q_{D_i} = q_{\bar{E}_i}$ .

As described in the introduction, we construct the partially oriented dual graph  $G(\bar{Z})$  of  $\bar{Z}$ . Proposition 4 can also be stated in terms of dual graphs as follow:

**Remark 10** *Let  $n_G : G(\bar{Z}) \rightarrow G(Z)$  be the morphism of graphs induced on the dual graphs by  $\phi_r \circ n : (\bar{Z}, E_{\bar{Z}}) \rightarrow (Z, E_Z)$ . The Hironaka quotient of the vertex  $(i) \in G(\bar{Z})$  is equal to the Hironaka quotient of the vertex  $n_G(i) \in G(Z)$ . Hence,  $n_G$  is an orientations preserving morphism of graphs.*

### 6.2 Hironaka quotients associated to $X'$

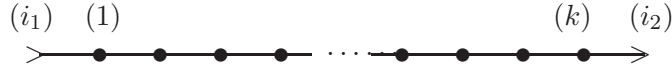
As in section 2, we consider the finite set  $\bar{z}_i, 1 \leq i \leq n$ , of the singular points of  $\bar{Z}$ . For each index  $i$ , we choose a sufficiently small neighbourhood  $(\bar{Z}_i, \bar{z}_i)$  of  $\bar{z}_i$  in  $\bar{Z}$  and we denote by  $\bar{\rho}_i : (Z_i'', E_{Z_i}'') \rightarrow (\bar{Z}_i, \bar{z}_i)$  the minimal resolution of the singularity  $(\bar{Z}_i, \bar{z}_i)$ . We have seen (Section 2) that  $\bar{z}_i = \bar{E}_{i_1} \cap \bar{E}_{i_2}$  is a double point of the total transform  $E_{\bar{Z}}^+ = (\phi_r \circ n)^{-1}(E_Z^+)$

( $\bar{E}_{i_1}$  or  $\bar{E}_{i_2}$  could be an irreducible component of the strict transform of the discriminant  $\Delta$ ). As  $(\bar{Z}_i, \bar{z}_i)$  is a quasi-ordinary singular point, the dual graph of  $E_{Z_i} = (\bar{\rho})^{-1}(\bar{z}_i)$  is a bamboo, let us denote by  $k$  the number of its irreducible components.

Let us denote by  $E'_{i_1}$  (resp.  $E'_{i_2}$ ) the irreducible component of  $E_{X'}$  such that  $(\bar{\rho})(E'_{i_1}) = \bar{E}_{i_1}$  (resp.  $(\bar{\rho})(E'_{i_2}) = \bar{E}_{i_2}$ ). One extremity of this bamboo represents the irreducible component  $E'_1$  of  $E_{X'}$  which meets  $E'_{i_1}$ . We index it by (1). To obtain the dual graph  $G(\bar{Z}_i)$ , we add to (1) a reverse arrow indexed by  $(i_1)$  which represents  $E'_{i_1}$ .

The other extremity of this bamboo represents the irreducible component  $E'_k$  of  $E_{X'}$  which meets  $E'_{i_2}$ . We index it by  $(k)$ . To obtain the dual graph  $G(\bar{Z}_i)$ , we add to  $(k)$  an arrow indexed by  $(i_2)$  which represents  $E'_{i_2}$ . We orient  $G(\bar{Z}_i)$  from (1) to  $(k)$  and we order the indices of the vertices with the help of this orientation.

The graph  $G(\bar{Z}_i)$  has the following shape:



**Theorem 2** *Let  $E'_j$  be an irreducible component of  $E_{X'}$  and let  $z'$  be a point of  $E_j^0$  which is the set of the smooth points of  $E'_j$  in the total transform  $(\bar{\rho})^{-1}(E_Z^+)$ . Let  $(c'_j, z')$  be a curvetta (at  $z'$ ) of  $E'_j$ .*

*If  $\bar{\rho}(z') \neq \bar{z}_i, \bar{z}_i \in \{\bar{z}_i, 1 \leq i \leq n\}$ , we have the following equality of the Hironaka quotients:  $q_{E'_j} = q_{\bar{E}_j} = q_{D_j}$ .*

*If  $\bar{\rho}(z') = \bar{z}_i$ , we have seen that  $\bar{z}_i = \bar{E}_{i_1} \cap \bar{E}_{i_2}$  is a double point of the total transform  $E_Z^+ = (\phi_r \circ n)^{-1}(E_Z^+)$ . We have two cases*

*I) either  $q_{\bar{E}_{i_1}} = q_{\bar{E}_{i_2}}$ , then  $q_{E'_j} = q_{\bar{E}_{i_1}} = q_{\bar{E}_{i_2}}$ ,*

*II) or  $q_{\bar{E}_{i_1}} < q_{\bar{E}_{i_2}}$ , then  $q_{\bar{E}_{i_1}} < q_{E'_j} < q_{\bar{E}_{i_2}}$ . More precisely, the dual graph of  $(\bar{\rho})^{-1}(\bar{z}_i)$  is a bamboo. We orient this bamboo from the vertex  $(i_1)$  to the vertex  $(i_2)$ . With this orientation, we order the indices  $(j)$  of the dual graph of  $(\bar{\rho})^{-1}(\bar{z}_i)$  from (1) to  $(k)$ . We have:*

$$q_{E'_{i_1}} = q_{\bar{E}_{i_1}} < q_{E'_1} < \dots < q_{E'_j} < \dots < q_{E'_k} < q_{\bar{E}_{i_2}} = q_{E'_{i_2}}.$$

*Proof of Theorem 2.*

Let us recall that  $q_{E'_j}$  is equal to the first Puiseux exponent of the plane curve germ  $\gamma_j$  where

$$\gamma_j = (r \circ \phi_r \circ n \circ \bar{\rho})(c'_j) = (\phi \circ r_\phi \circ n \circ \bar{\rho})(c'_j).$$

If  $\bar{\rho}(z') \neq \bar{z}_i, \bar{z}_i \in \{\bar{z}_i, 1 \leq i \leq n\}$ ,  $\bar{\rho}(E'_j)$  is an irreducible component  $\bar{E}_j$  of  $E_Z$  and  $(\phi_r \circ n \circ \bar{\rho})(E'_j) = D_j$  is an irreducible component of  $E_Z$ . Then, the first Puiseux exponent of  $(r \circ \phi_r \circ n \circ \bar{\rho})(c'_j, z')$  is equal to  $q_{\bar{E}_j} = q_{D_j}$ .



If  $\bar{\rho}(z') = \bar{z}_i$ , we have seen that  $\bar{z}_i = \bar{E}_{i_1} \cap \bar{E}_{i_2}$  is a double point of the total transform  $E_Z^+ = (\phi_r \circ n)^{-1}(E_Z^+)$ . Let  $D_{i_1} = (\phi_r \circ n)(\bar{E}_{i_1})$ ,  $D_{i_2} = (\phi_r \circ n)(\bar{E}_{i_2})$  and  $z = (\phi_r \circ n \circ \bar{\rho})(z') \in D_{i_1} \cap D_{i_2}$ . In Section 6.1 we proved that  $q_{D_{i_1}} = q_{\bar{E}_{i_1}}$  and  $q_{D_{i_2}} = q_{\bar{E}_{i_2}}$ .

I) If  $q_{\bar{E}_{i_1}} = q_{\bar{E}_{i_2}}$ , we have  $q_{D_{i_1}} = q_{D_{i_2}}$ .

As  $z = (\phi_r \circ n \circ \bar{\rho})(z') \in D_{i_1} \cap D_{i_2}$ , we deduce from lemma 3 that  $q_{E'_j} = q_{D_{i_1}} = q_{D_{i_2}}$ .

II) If  $q_{D_{i_1}} = q_{\bar{E}_{i_1}} < q_{\bar{E}_{i_2}} = q_{D_{i_2}}$ , we have to study the minimal resolution  $\bar{\rho}_i : (Z''_i, E_{Z''_i}) \rightarrow (\bar{Z}_i, \bar{z}_i)$  of the singularity  $(\bar{Z}_i, \bar{z}_i)$ .

For each irreducible component  $E'_j$  of  $(\bar{\rho})^{-1}(\bar{z}_i)$  we choose a curvetta  $(c'_j, z'_j)$  of  $E'_j$ . We take the following notation:  $D_{i_1} = (\phi_r \circ n \circ \bar{\rho})(E'_{i_1})$  and  $D_{i_2} = (\phi_r \circ n \circ \bar{\rho})(E'_{i_2})$ . So, we have  $z_i = (\phi_r \circ n \circ \bar{\rho})(z'_j) = D_{i_1} \cap D_{i_2}$ .

But,  $V_i = (\phi_r \circ n \circ \bar{\rho})(Z''_i)$  is a neighborhood of  $z_i$  in  $Z$ . Let us denote by  $\Phi_i$  the restriction of  $(\phi_r \circ n \circ \bar{\rho})$  on  $Z''_i$ . As  $Z$  is smooth and  $E_Z$  is a normal crossing divisor in  $Z$ , the morphism  $\Phi_i$  satisfies the hypothesis of Lemma 4 where the smooth plane curve germs  $(D_{i_1}, z_i)$  and  $(D_{i_2}, z_i)$  play the role of the two axes  $u = 0$  and  $v = 0$ . With this choice of axes at  $z_i$  in  $V_i$ , the first Puiseux exponents  $s_j$  of  $c_j = \Phi_i(c'_j)$  are strictly ordered  $s_1 < \dots < s_j < \dots < s_k$ .

The curve  $c_j$  admits a Puiseux expansion which begins by:

$$x = a_{j0}y^{m_j/n_j} + a_{j1}y^{(m_j+1)/n_j} + \dots, \quad a_{j0} \neq 0, \quad m_j/n_j = s_j.$$

We will say that the plane curve germ  $c_j^*$ , having

$$x = a_{j0}y^{m_j/n_j}$$

as Puiseux expansion, is the shadow of  $c_j$ . Let  $\gamma_j^* = r(c_j^*)$  and let  $t_j$  be the first Puiseux exponent of  $\gamma_j^*$ .

As  $q_{D_{i_1}} < q_{D_{i_2}}$ , the edge which represents  $z_i = D_{i_1} \cap D_{i_2}$  in  $G(Z)$  is an edge of the maximal arc  $S(Z)$  of Proposition 2, Section 4. Resolution of plane curve germs implies that  $q_{D_{i_1}} < t_1 < \dots < t_j < \dots < t_k < q_{D_{i_2}}$  and that  $t_j$  is also the first Puiseux exponent of  $r(c_j) = \gamma_j$ . But  $q_{E'_j}$  is equal to the first Puiseux exponent of  $\gamma_j$ . This ends the proof of Theorem 2.

## 7 Behaviour of the dual graphs in each step of the Hirzebruch-Jung resolution

### 7.1 Maximal arcs in the dual graph $G(\bar{Z})$ of the normalization

Let  $A(\bar{Z}) := n_G^{-1}(S(Z))$  be the inverse image of the maximal arc  $S(Z)$  of  $G(Z)$ .

**Theorem 3**  *$A(\bar{Z})$  is the union of all the maximal arcs of  $G(\bar{Z})$ . The Hironaka quotients of the vertices of a connected component of the closure of  $G(\bar{Z}) \setminus A(\bar{Z})$  are constant.*

*Moreover  $G(\bar{Z}) \setminus A(\bar{Z})$  doesn't contain any arrow.*

*Proof.* By remark 10,  $n_G$  preserves the Hironaka quotients. The definition of  $A(\bar{Z})$  implies that the Hironaka quotients of the vertices of a connected component of the closure of  $G(\bar{Z}) \setminus A(\bar{Z})$  are constant.

Again by remark 10, an edge of  $G(\bar{Z})$  is oriented if and only if it is an edge of  $A(\bar{Z})$ . All the going-in arrows (resp. going-out) arrows are in  $A(\bar{Z})$  because there are above (by  $n_G$ ) the unique going-in (resp. going-out) arrow of  $S(Z)$ . The image, by  $n_G$ , of a vertex of  $A(\bar{Z})$  is a vertex of  $S(Z)$ . In  $S(Z)$  a vertex has exactly one going-in edge and one going-out edge. As  $n_G$  preserves the orientation of the edges, a vertex of  $A(\bar{Z})$  has at least a going-in edge and a going-out edge. It allows us to show that each edge and each vertex of  $A(\bar{Z})$  belong to a maximal arc of  $A(\bar{Z})$ .

## 7.2 Maximal arcs in the dual graph $G(X')$ of the good resolution of the Hirzebruch-Jung singularities of $\bar{Z}$

Let  $A(X') := (\bar{\rho}_G)^{-1}(A(\bar{Z}))$  be the inverse image of  $A(\bar{Z})$  by the morphism of graphs  $\bar{\rho}_G : G(X') \rightarrow G(\bar{Z})$  induced by  $\bar{\rho}$ .

**Theorem 4**  *$A(X')$  is the union of all the maximal arcs of  $G(X')$ . The Hironaka quotients of the vertices of a connected component of the closure of  $G(X') \setminus A(X')$  are constant.*

*Moreover  $G(X') \setminus A(X')$  doesn't contain any arrow.*

*Proof.* The graph  $G(X')$  is obtained from  $G(\bar{Z})$  as follows.

If an edge of  $G(\bar{Z})$  represents a point which is a smooth point of  $\bar{Z}$ , then we keep this edge in  $G(X')$  and its extremities has the same Hironaka quotients.

If an edge  $(e_{ij})$  of  $G(\bar{Z})$  represents a Hirzebruch-Jung singular point of  $\bar{Z}$ , then in  $G(X')$  this edge is replaced by a bamboo. If  $(e_{ij})$  is not oriented, from point  $I$  of theorem 2, the Hironaka quotients are constant on the closure of the bamboo. So the closure of the bamboo is in the closure of  $G(X') \setminus A(X')$ . If  $(e_{ij})$  is oriented, from  $II$  of theorem 2, the bamboo has the same orientation and is included in  $A(X')$  by construction. In particular, the inverse image by  $\bar{\rho}_G$  of a maximal arc of  $A(\bar{Z})$  is a maximal arc of  $A(X')$ .

## 7.3 Maximal arcs in the dual graph $G(\tilde{X})$ of the minimal resolution of $\phi$

Let  $\beta_1 : (X', E_{X'}) \rightarrow (X_1, E_{X_1})$  be the contraction of an irreducible component of  $E_{X'}$  of self-intersection  $-1$  which is not a rupture component. A maximal sequence of such blowing-downs gives a morphism  $\beta : (X', E_{X'}) \rightarrow (\tilde{X}, E_{\tilde{X}})$ . Then the contraction of  $E_{\tilde{X}}$  denoted  $\rho : (\tilde{X}, E_{\tilde{X}}) \rightarrow (X, p)$  is the minimal resolution of  $\phi$  (see [4] (Theorem 5.9 p. 87) or [2] (Theorem 6.2 p. 86)).

**Lemma 5** *Let  $\beta_{1G} : G(X') \rightarrow G(X_1)$  be the morphism of graphs induced by  $\beta_1$ . The subgraph  $A(X_1) := \beta_{1G}(A(X'))$  is the union of the maximal arcs of  $G(X_1)$ . The Hironaka quotients of the vertices of a connected component of the closure of  $G(X_1) \setminus A(X_1)$  are constant.*

*Moreover  $G(X_1) \setminus A(X_1)$  doesn't contain any arrow.*

By finite iterations, the above lemma gives a proof of Theorem 1 for the minimal resolution of  $\phi$ .

*Proof of lemma 5.* By hypothesis,  $\beta_1$  is the contraction of an irreducible  $E'_i$  which meets one or two irreducible components of the total transform  $E_{X'}^+$ .

If  $E'_i$  meets only one irreducible component of  $E_{X'}^+$ , this neighbour is not an arrow because  $G(X')$  is a connected graph. Remark 1 implies that the only edge which arrives to  $(i)$  is not in a maximal arc.  $G(X_1)$  is obtained from  $G(X')$  by deleting the vertex  $(i)$  and the only non-oriented edge which meets  $(i)$ . In this case the restriction on  $A(X')$  of the morphism  $\beta_{1G}$  induces an isomorphism to  $A(X_1)$ .

If  $E'_i$  meets two irreducible components of  $E_{X'}^+$ , then two cases occur.

a) If  $(i)$  is not a vertex of  $A(X')$ , then from remark 1, the two edges which meet  $(i)$  are not oriented and the contraction of  $(i)$  deletes  $(i)$  and the two edges are replaced by a unique non oriented edge. In this case, the restriction on  $A(X')$  of the morphism  $\beta_{1G}$  induces an isomorphism to  $A(X_1)$ .

b) If  $(i)$  is a vertex of  $A(X')$ , notice first that  $(i)$  meets at most one arrow because  $G(X')$  is connected and the existence of  $\beta_1$  implies that  $R$  is not minimal. Let  $(e_{ij})$  be an edge which meets  $(i)$ .  $A(X_1)$  is obtained from  $A(X')$  by contracting  $(e_{ij})$ .

In this case the restriction of  $\beta_{1G}$  on  $G(X') \setminus A(X')$  induces an isomorphism to  $G(X_1) \setminus A(X_1)$ .

The contraction of  $A(X')$  in  $A(X_1)$  has one of the three following shapes:

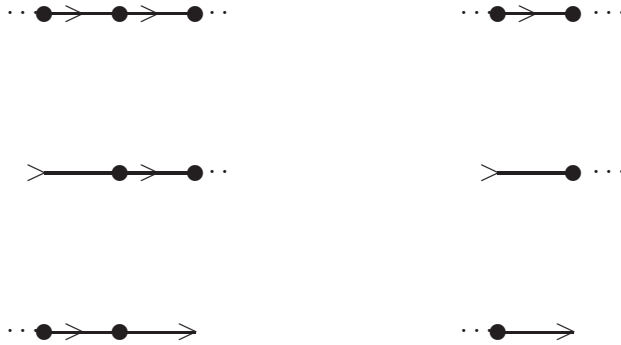


Figure 5: The shapes of the possible contractions of  $A(X')$  in  $A(X_1)$

#### 7.4 Maximal arcs in the dual graph $G(Y)$ of any good resolution of $\phi$

Let  $\pi : (Y, E_Y) \rightarrow (X, p)$  be a good resolution of  $\phi$ . As proved in [4] (Theorem 5.9 p. 87) and [2] (Theorem 6.2 p. 86), there exists a sequence of contractions of irreducible components of self-intersection  $-1$  which are not rupture components such that the following diagram commutes.

$$\begin{array}{ccc}
 (Y, E_Y) & \xrightarrow{\gamma} & (\tilde{X}, E_{\tilde{X}}) \\
 \pi \searrow & & \nearrow \rho \\
 & (X, p) &
 \end{array}$$

In section 7.3 we have proved theorem 1 for  $G(\tilde{X})$ . By iteration it is enough to prove theorem 1 when  $\gamma$  is a blowing-up of a point of  $E_{\tilde{X}}$ . As  $\tilde{X}$  is smooth, this blowing-up creates an irreducible component of the exceptional divisor of self-intersection  $-1$ . As in the proof of lemma 5, the subgraph  $A(Y) := \gamma_G^{-1}(A(\tilde{X}))$  is a union of maximal arcs and satisfies theorem 1.

## 8 Examples

In the minimal resolution of  $\Delta^+$  the strict transforms of the discriminant curve are represented by edges ended with a star.

### 8.1 Example 1

Let  $\phi = (f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  defined by

$$f(x, y) = (x^2 - y^3)y(y + x^5)(x + y + x^3) \text{ and } g(x, y) = x(y + 2x^5)(x + y).$$

The critical locus of  $\phi$  admits five irreducible components. Four of them are smooth and tangent to  $\{y = -x\}$ ,  $\{y = x\}$  and  $\{y = 0\}$  for two of them. The fifth one is tangent to  $\{x = 0\}$  and topologically equivalent to  $\{x^2 - y^3 = 0\}$ .

The graph  $G(Z)$  is in Figure 6.

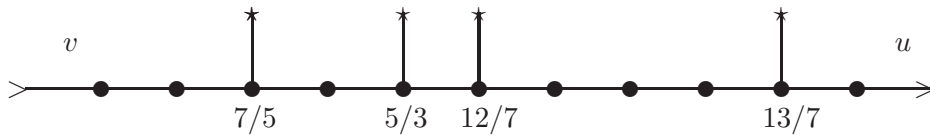


Figure 6: Graph of the minimal resolution  $r$  of  $\Delta^+$

Each vertex of  $G(Z)$  belongs to the maximal arc  $S(Z)$ .

The graph of the minimal resolution  $\rho$ , weighted with the Hironaka quotients of  $(f, g)$ , is in Figure 7.

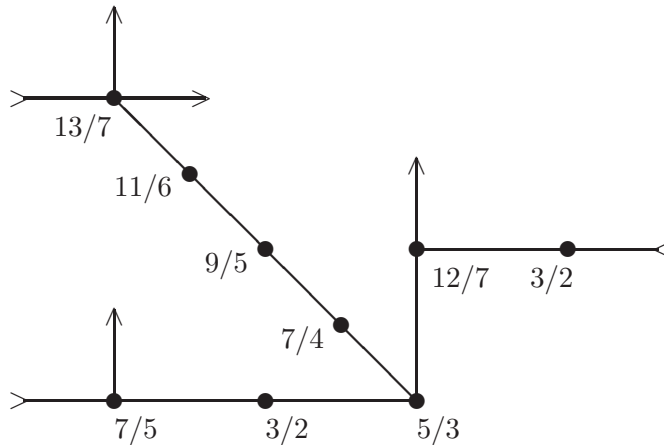


Figure 7: Graph of the minimal resolution  $\rho$

In this example, the subgraph  $A(\tilde{X})$  coincides with  $G(\tilde{X})$ .

Notice that the subgraphs  $A(\tilde{X})$  and  $G(\tilde{X})$  will have similar shapes when

$$f(x, y) = (x^2 - y^3)y(y + x^k)(x + y + x^l) \text{ and } g(x, y) = x(y + 2x^k)(x + y)$$

where  $k, l$  are integers strictly greater than 1.

## 8.2 Exemple 2

Let us consider the surface  $(X, 0)$  of equation:

$$z^3 = (y^3 - x^2)(y^3 - (x + y)^2)$$

and let  $\phi : (X, 0) \rightarrow (\mathbb{C}^2, 0)$  be the projection on the  $(x, y)$ -plane. Notice that this projection is not a generic one.

The discriminant locus of  $\phi$  is  $\Delta : (v^3 - u^2)(v^3 - (u + v)^2) = 0$ . The minimal resolution tree of  $\Delta^+$  is in Figure 8.

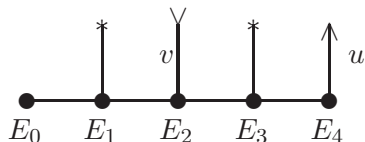


Figure 8: Graph of the minimal resolution  $r$  of  $\Delta^+$

The vertices  $E_2, E_3, E_4$  belong to the maximal arc  $S(Z)$ .

The set of Hironaka quotients associated to  $G(Z)$  is  $\left\{1, \frac{3}{2}, 2\right\}$  (1 for  $E_0, E_1, E_2$ ,  $3/2$  for  $E_3$  and 3 for  $E_4$ ).

The dual graph  $G(X')$  of the Hirzebruch-Jung good resolution  $\rho' : (X', E_{X'}) \rightarrow (X, 0)$  of  $\phi$  is in Figure 9. We represent the subgraph  $A(X')$  by a double-line joining the arrows associated to the strict transforms of  $\{f = 0\}$  and  $\{g = 0\}$ .

The dual graph  $G(\tilde{X})$  of the minimal good resolution  $\rho : (\tilde{X}, E_{\tilde{X}}) \rightarrow (X, 0)$  of  $\phi$ , is obtained from the one in Figure 9 by blowing-down  $E'_0, E'_4$  and 4 other vertices of self-intersection  $-1 : E'_{1,2}, E'_{2,3}, E'_{1,0}, E'_{3,0}$ .

In  $G(\tilde{X})$  each irreducible component of the exceptional divisor associated to the vertices of  $G(\tilde{X})$  is of genus zero and of self-intersection equal to  $-2$ , except the one intersected by the strict transform of  $\{g = 0\}$  which has self-intersection  $-3$ . We weight  $G(\tilde{X})$  with the Hironaka quotients of  $(f = u \circ \phi, g = v \circ \phi)$ . It is represented in Figure 10.

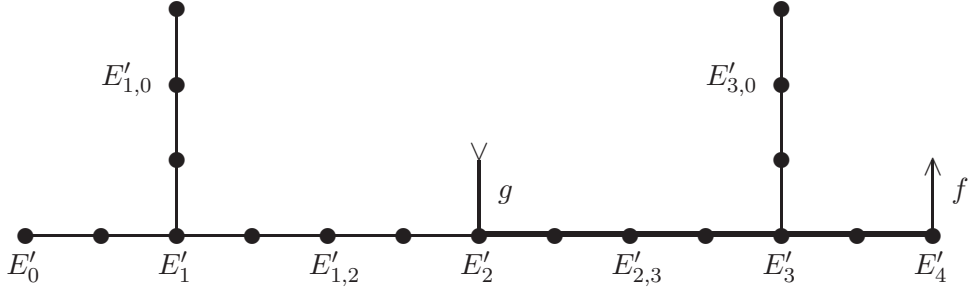


Figure 9: Graph of the Hirzebruch-Jung resolution of  $\phi$

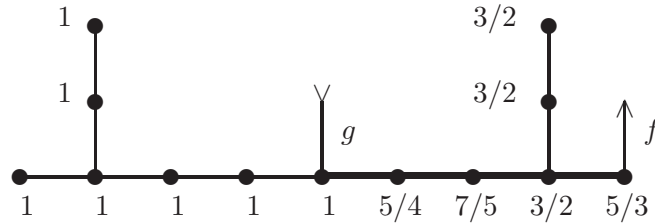


Figure 10: Graph of the minimal resolution of  $\phi$

In this example  $\overline{A(\tilde{X})}$  (respectively  $\overline{A(X')}$ ) is strictly included in  $G(\tilde{X})$  (respectively  $G(X')$ ) and  $G(\tilde{X}) \setminus \overline{A(\tilde{X})}$  (respectively  $G(X') \setminus \overline{A(X')}$ ) admits two connected components on which the Hironaka quotients are respectively equal to 1 and  $3/2$ .

### 8.3 Exemple 3

Let us consider the surface  $(X, 0)$  of the following equation:

$$z^2 = (y + x^3)(y + x^2)(x^{34} - y^{13}).$$

1. Let us first consider the case where  $\phi_1 : (X, 0) \rightarrow (\mathbb{C}^2, 0)$  is the projection on the  $(x, x + y)$ -plane. It is a generic projection.

The discriminant locus of  $\phi_1 = (f_1, g_1)$  is the curve  $\Delta_1$  which admits three components with Puiseux expansions given by :

$$v = u - u^2$$

$$v = u - u^3$$

$$v = u + u^{34/13}$$

Notice that the three components of  $\Delta_1$  admit 1 as first Puiseux exponent and respectively 2, 3,  $34/13$  as second Puiseux exponent.

The coordinate axes are transverse to the discriminant locus of  $\phi_1$ . Hence the maximal arc of the tree of the minimal embedded resolution of  $\Delta_1^+$  has a unique vertex of Hironaka quotient equal to one. Moreover the Hironaka quotients are constant in the tree  $G(Z)$  of the minimal embedded resolution of  $\Delta_1^+$ . The dual graph  $G(Z)$  is in Figure 11.

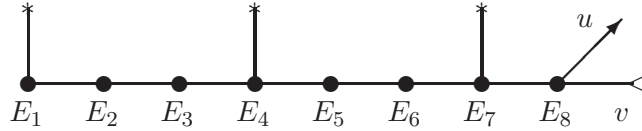


Figure 11: Graph of the minimal resolution of  $\Delta_1^+$

The Hironaka quotient associated to each irreducible component of  $E_Z$  is equal to one. Only the vertex  $E_8$  of  $G(Z)$  belongs to  $S(Z)$ .

The dual graph  $G(X')$  of  $R$  admits a cycle created by the normalization. The irreducible component  $E'_0$  is obtained by the resolution  $\bar{\rho}$ . The irreducible components of the exceptional divisor associated to the vertices of  $G(X')$  have a genus equal to zero. The subgraph  $\overline{G(X') \setminus A(X')}$  is connected of Hironaka quotient equal to one.

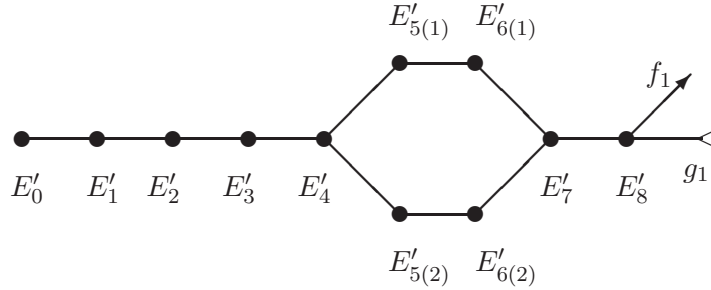


Figure 12: The graph of the Hirzebruch-Jung resolution of  $\phi_1$

The minimal good resolution  $\rho$  is obtained by blowing down  $E'_3$ . Its dual graph is in Figure 13.

2. Now let us consider the projection  $\phi_2 = (f_2, g_2)$  on the  $(x, y)$ -plane. Notice that  $\phi_1$  and  $\phi_2$  have the same critical locus.

The discriminant locus of  $\phi_2$  is the curve  $\Delta_2 : (v + u^3)(v + u^2)(u^{34} - v^{13}) = 0$ .

The first Puiseux exponents of the components of  $\Delta_2$  are 2, 3, 34/13.



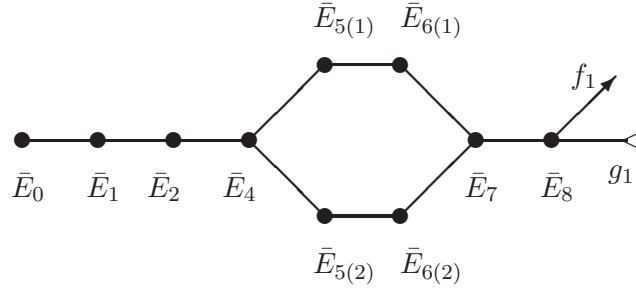


Figure 13: The graph of the minimal resolution of  $\phi_1$

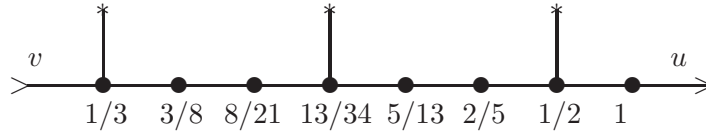


Figure 14: Graph of the minimal resolution of  $\Delta_2^+$

The tree  $G(Z)$  of the minimal embedded resolution of  $\Delta_2^+$  is in Figure 14.

Each vertex of  $G(Z)$  belongs to  $S(Z)$ .

The subgraph  $\overline{G(X') \setminus A(X')}$  admits a unique connected component corresponding to a full-torus. Its Hironaka quotient is equal to  $1/3$ .

The graph of the minimal resolution of  $\phi_2$  is obtained by blowing-down  $E'_3$  and  $E'_8$ .

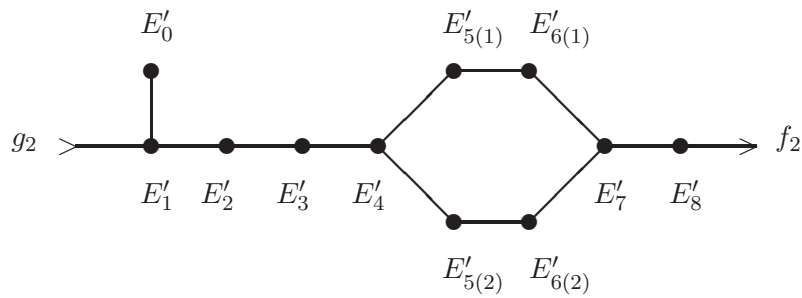


Figure 15: The graph of the Hirzebruch-Jung resolution of  $\phi_2$

The second Puiseux exponents of the components of  $\Delta_1$  are the Hironaka quotients of the rupture vertices of the minimal resolution of  $\phi_2$ . This comes from the fact that,

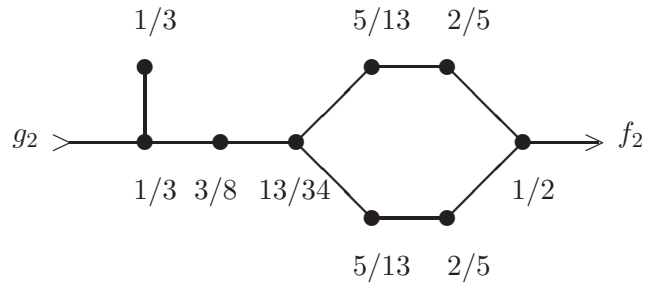


Figure 16: Graph of the minimal resolution of  $\phi_2$

in case 1 and 2, the functions  $f_1, g_1, f_2, g_2$  belong to the same pencils  $\Lambda$  generated by  $x$  and  $y$ . In case 1,  $f_1$  and  $g_1$  are generic elements of the pencil, and in case 2,  $g_2$  is not generic anymore.

As proved in [3] there exists an iterative process to compute the Puiseux exponents of the discriminant curve of a finite morphism  $(X, p) \rightarrow (\mathbb{C}^2, 0)$ .

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