Strong Structural Input and State Observability of LTV Network Systems with Multiple Unknown Inputs

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Abstract: This paper studies linear time-varying (LTV) network systems affected by multiple unknown inputs. The goal is to reconstruct both the initial state and the unknown input. The main result is a characterization of strong structural input and state observability, i.e., the conditions under which both the whole network state and the unknown input can be reconstructed for all system matrices that share a common zero/non-zero pattern. This characterization is in terms of strong structural observability of a suitably-defined linear time-invariant (LTI) subsystem.

Keywords: Network Systems, Cyber-Physical Security, Linear time-varying (LTV) systems, Input and State observability (ISO), Strong structural observability, Uniform observability

1. INTRODUCTION

Network systems are increasingly becoming ubiquitous. These have found application in robotic networks, energy distribution systems, infrastructure networks and others. However, failure of one of the subsystems could lead to failure of the entire network, whereas monitoring every individual subsystem requires large amount of resources which is undesirable. Kalman in his seminal paper (Kalman (1959)) introduced the concept of *observability* which allows to obtain the complete information of a network with minimum resources.

Network systems and in particular cyber-physical systems are also prone to failure due to attack by external unknown agents (Pasqualetti et al. (2015)). Therefore, it is important to not only estimate the state in the presence of the unknown input (Kitanidis (1987)) but also estimate the unknown input (Yong et al. (2016)). System theorists refer to this as *Input and State Observability (ISO)* problem. The study of ISO in linear network systems can be characterized using algebraic techniques. These involve matrix rank test tools (Trentelman et al. (2002)) and hence are ill-suited for large networks. Therefore, it makes sense to look for ISO results that depend on the *structure* of the system. A linear system (under state space representation) is said to be structured if the system matrices have coefficients that are either a fixed zero or a parameter (i.e., these coefficients are allowed to take any value in $\mathbb{R}$). For structured systems if a property holds for *almost all* choices of parameters, then the property is said to be *structural* (see Lin (1974)) whereas if a property holds for *all non-zero* choices of parameters, then the property is said to be *strongly structural (s-structural)* (see Mayeda and Yamada (1979)).

For studying structural ISO and s-structural ISO in LTI framework, there exist graph-theoretic techniques as shown in Boukhobza et al. (2007) and Kibangou et al. (2016) respectively. However, to the best of our knowledge no such techniques are available for LTV setup. In this paper, we would be seeking graphical characterizations for *strongly structural* ISO for discrete-time LTV systems. To this end, our main result shows the equivalence between s-strucural ISO of an LTV system over sufficiently long intervals and s-structural observability of a relevant LTI subsystem. This equivalence allows one to study s-structural ISO for LTV systems using techniques for s-structural observability as given in Chapman and Mesbahi (2013), Trefois and Delvenne (2015) and Weber et al. (2014) where a linear time algorithm for verifying s-structural observability is given.

The organization of this paper is as follows. We introduce basic notations and problem statement in Section 2, whereas we present a couple of algebraic characterizations in Section 3. The decomposition of a LTV system into two subsystems is shown in Section 4. The main result is given in Section 5 and we discuss future directions in Section 6.

2. PROBLEM FORMULATION

2.1 Notations

$\mathbb{R}$ and $\mathbb{Z}$ denote the set of real numbers and integers respectively. For $a, b \in \mathbb{Z}$, $a \leq b, [a, b]$ denotes a discrete interval. $e_{j;N}$ denotes the j^{th} vector of the canonical basis of $\mathbb{R}^N$. If the length is clear from context, we would represent the same as just e_j. Let $P, Q \in \mathbb{Z}$, then $\mathbf{0}_P$ denotes a zero vector of length P whereas $0_{P\times Q}$ denotes a zero matrix of P rows and Q columns, while I represents an identity matrix. $A =$ diag $(A_1, A_2, \ldots, A_n)$ denotes a block diagonal matrix with A_i; $i = 1, 2, \ldots, n$, being blocks along the diagonal. $[A]_{i,j}$ denotes the entry in matrix A corresponding to its i^{th} row and j^{th} column. Two matrices A and B, with the same dimensions are said to be *consistent* if their zero/non-zero positions coincide. $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. $\{A_k, B_k\}_{k_0}^{k_1}$ denotes a sequence of matrices A_k and B_k with $k = k_0, k_0+1, \cdots, k_1$.

2.2 Problem Statement

Consider a linear network system with N nodes, represented by a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ where $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{E} = \{(j,i) \in \mathcal{V} \times \mathcal{V} \mid [A_\mathcal{G}]_{i,j} = 1\}$; $A_\mathcal{G}$ being the adjacency matrix of $\mathcal{G}$. Some of the nodes in $\mathcal{G}$ are attacked by P external malicious agents. A scheme of this sort could be used to depict attacks on multiple nodes, including deception attacks (Teixeira et al. (2010)), false data injection (Liu et al. (2011)), fault diagnosis and detection (Patton et al. (1989)).We denote by $\mathcal{A} = \{i_1, i_2, \ldots, i_R\} \subseteq \mathcal{V}$ the set of *attacked* nodes in $\mathcal{G}$ with $|\mathcal{A}| = R \leq N$. Let $\mathcal{I}$ be the set of *malicious* agents with $|\mathcal{I}| = P$. The interaction between the *malicious* agents and *all* the nodes in $\mathcal{G}$ can be captured with a bipartite graph $\mathcal{F} = \{\mathcal{V}, \mathcal{I}, \mathcal{E}_F\}$ where $\mathcal{E}_F = \{(j,i) \in \mathcal{I} \times \mathcal{V} \mid [A_B]_{i,j} = 1\}$; A_B being the biadjacency matrix of $\mathcal{F}$. Similarly, we introduce a binary matrix A_C to denote the zero/non-zero pattern of the observation matrix. The dynamics of a LTV network system over $[k_0, k_1]$, in the presence of multiple unknown inputs are given as follows:

$$\begin{cases} x_{k+1} = W_k x_k + B_k u_k \\ y_k = C_k x_k \end{cases} \tag{1}$$

with state vector $x_k \in \mathbb{R}^N$, unknown input vector $u_k \in \mathbb{R}^P$ and output vector $y_k \in \mathbb{R}^M$. Furthermore, $W_k \in \mathbb{R}^{N \times N}$, $B_k \in \mathbb{R}^{N \times P}$ and $C_k \in \mathbb{R}^{M \times N}$.
Let $\mathcal{W}, \mathcal{B}$ and $\mathcal{C}$ represent the sets of all matrices *consistent* with $A_\mathcal{G}$, A_B and A_C respectively. We assume that $\forall k \in \mathbb{Z}$, i) $W_k \in \mathcal{W}$, ii) $B_k \in \mathcal{B}$, iii) $C_k \in \mathcal{C}$. The assumption $W_k \in \mathcal{W}$ $\forall k \in \mathbb{Z}$ implies that the topology of $\mathcal{G}$ remains fixed throughout. However, the entries corresponding to the non-zero positions of W_k can vary with time. We restrict our focus to the case where each unknown input affects exactly one node of the system and each node is attacked at most by a single unknown input. We exclude cases where a node is subject to an attack by a linear combination of some (possibly all) unknown inputs. Therefore, number of attacked nodes is the same as number of malicious agents, i.e., $R = P$. In the context of network systems, it is natural to think of states as being local variables that are distributed in space. For instance, in a power network, each state corresponds to individual generating stations and hence local measurements are dependent *only* on local states. Hence, we can assume that some states can be directly measured up to a multiplicative constant. These states are called *observed nodes* and they are denoted by $\mathcal{O}$ (Fig. 1 depicts the setup).
Let $\mathcal{O} = \{j_1, j_2, \ldots, j_M\} \subseteq \mathcal{V}$ be the set of observed nodes.

As a consequence, the biadjacency matrix A_B and the matrix A_C are given by:

A1:

$A_B = [e_{i_1;N} \; e_{i_2;N} \; . \; . \; e_{i_P;N}]$, with all the i_k being distinct,

$A_C^T = [e_{j_1;N} \; e_{j_2;N} \; . \; . \; e_{j_M;N}]$, with all the j_r being distinct,

where $k \in \{1, 2, \ldots, P\}$ and $r \in \{1, 2, \ldots, M\}$. ■

Our goal is to provide conditions under which it is possible to jointly estimate both the initial conditions and the sequence of multiple unknown inputs from measurements of a subset of state vertices. In particular, we would like to determine the aforesaid conditions, that depend only on the structure of the graph $\mathcal{G}$ or equivalently for all choices of non-zero entries in $\mathcal{W}, \mathcal{B}$ and $\mathcal{C}$.

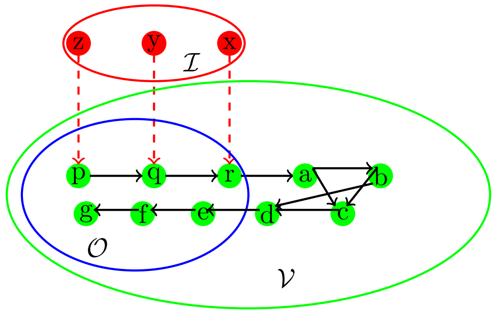

Fig. 1. Graph representation of a network system subject to attacks by external agents $\mathcal{I} = \{x, y, z\}$. $\mathcal{O}$ and $\mathcal{V}$ are the sets of observed nodes and state nodes respectively, while the set of attacked nodes is $\mathcal{A} = \{p, q, r\}$.

3. PRELIMINARIES

3.1 Definitions

Recalling some of the classical definitions (Rugh (1996)), we have the following:

Definition 1. The system $\{W_k, C_k\}_{k_0}^{k_1}$ is *observable over* $[k_0, k_1]$ if any initial state x_{k_0} is uniquely determined by the corresponding zero-input response $\{y_{k_0}, y_{k_0+1}, \ldots, y_{k_1}\}$. ■

Along similar lines, we define ISO as follows:

Definition 2. The system $\{W_k, B_k, C_k\}_{k_0}^{k_1}$ is ISO over the interval $[k_0, k_1]$ if the initial condition $x_{k_0} \in \mathbb{R}^N$ and the unknown inputs sequence $\{u_{k_0}, u_{k_0+1}, \ldots, u_{k_1-1}\}$ can be uniquely recovered from $\{y_{k_0}, y_{k_0+1}, \ldots, y_{k_1}\}$. ■

Definition 2 requires strong observability (i.e., recovering the initial state x_{k_0} even in the presence of unknown inputs) along with invertibility of delay 1 (i.e., recovering the multiple unknown inputs up to u_{k_1-1} from the outputs up to y_{k_1}). This is of particular importance in designing unbiased minimum-variance filters that estimate both state and unknown input (see Gillijns and De Moor (2007), Hsieh (2000)).

A stronger notion of observability (resp. ISO) in LTV systems is that of uniform δ-step observability (resp. ISO), which requires the system to be observable (resp. ISO) over every time window of length δ. Following Levine (1996) (see page 435) we define *uniform δ-step ISO* as follows:

Definition 3. The system $\{W_k, B_k, C_k\}_{k \in \mathbb{Z}}$ is uniformly δ-step ISO if $\{W_k, B_k, C_k\}_{k_0}^{k_0+\delta}$ is ISO over $[k_0, k_0+\delta]$ $\forall k_0 \in \mathbb{Z}$. ■
Similar to Definition 3, one can also define *uniform δ-step observability.*

3.2 Kalman-like Algebraic Characterization for ISO

Let $y_{k_0:k_1}$ and $u_{k_0:k_1-1}$ be the vector of concatenated outputs and unknown inputs over $[k_0, k_1]$, respectively. From

(1), we get the following relation: $y_{k_0:k_1} = \Theta_{k_0,k_1} x_{k_0} + \Gamma_{k_0,k_1} u_{k_0:k_1-1} = \Psi_{k_0,k_1}[x_{k_0}^T u_{k_0:k_1-1}^T]^T$, where $\Theta_{k_0,k_1}, \Gamma_{k_0,k_1}$ and Ψ_{k_0,k_1} represent the observability matrix, invertibility matrix and input and state observability (ISO) matrix respectively over the interval $[k_0, k_1]$. These are defined as follows:

$$\Theta_{k_0,k_1} = \begin{bmatrix} C_{k_0} \\ C_{k_0+1} W_{k_0} \\ C_{k_0+2} W_{k_0+1} W_{k_0} \\ \cdot \\ \cdot \\ \cdot \\ C_{k_1} W_{k_1-1} \cdots W_{k_0} \end{bmatrix}, \tag{2}$$

$$\Gamma_{k_0,k_1} = \begin{bmatrix} 0 & \cdots\cdots & 0 \\ C_{k_0+1} B_{k_0} & 0 \cdots\cdots & 0 \\ \cdot & \cdots\cdots & \cdot \\ \cdot & \cdots\cdots & \cdot \\ C_{k_1} W_{k_1-1} \cdots W_{k_0+1} B_{k_0} & \cdots\cdots & C_{k_1} B_{k_1-1} \end{bmatrix},$$

$\Psi_{k_0,k_1} = [\Theta_{k_0,k_1} \Gamma_{k_0,k_1}]$.

From Definition 1 it is well-known that the system $\{W_k, C_k\}_{k_0}^{k_1}$ is *observable over $[k_0, k_1]$* if and only if $\text{rank}(\Theta_{k_0,k_1}) = N$. Similarly, Ψ_{k_0,k_1} along with Definition 2 imediately gives rise to:

Lemma 4. The system $\{W_k, B_k, C_k\}_{k_0}^{k_1}$ is ISO over $[k_0, k_1]$ if and only if $\text{rank}(\Psi_{k_0,k_1}) = N + (k_1 - k_0)P$. ■

We briefly summarize some of the necessary conditions for a system $\{W_k, B_k, C_k\}_{k_0}^{k_1}$ to be ISO over a given interval.

Proposition 5. The following conditions are necessary for the system $\{W_k, B_k, C_k\}_{k_0}^{k_1}$ to be ISO over $[k_0, k_1]$:
i) $\text{rank}(\Theta_{k_0,k_1}) = N$,
ii) $\text{rank}\ (C_{k_1} B_{k_1-1}) = P$,
iii) $M \geq P$.
iv) $N \geq P$.
In case $N > P$, then the following conditions are also necessary:
v) $M > P$,
vi) $k_1 - k_0 \geq \left\lceil \frac{N-M}{M-P} \right\rceil$. ■

Proof: Item i) asks that the first N columns of Ψ_{k_0,k_1} be linearly independent. Item ii) asks that the last P columns of Ψ_{k_0,k_1} be linearly independent, while items iii) and iv) are necessary conditions for item ii). To see the necessity of items v) and vi), notice that, in order for Ψ_{k_0,k_1} to be full column rank, it is necessary that Ψ_{k_0,k_1} has at least as many rows as columns, i.e.,

$$M(k_1 - k_0 + 1) \geq N + (k_1 - k_0)P. \tag{3}$$

From Eq.(3), since $(k_1 - k_0 + 1) > 0$, it follows that $M \geq P + \frac{N-P}{(k_1-k_0+1)}$. If $N > P$, this implies that $M > P$. Then, under $M > P$, item vi) immediately follows from Eq. (3). ■

The particular case $P = N$ (i.e., all nodes are attacked) yields a system that is ISO if and only if $\mathcal{O} = \mathcal{V}$ (i.e., all nodes are observed). In this paper, we focus on the non-trivial case of $N > P$. Therefore, from Prop. 5, $M > P$ is a necessary condition for ISO.

Notice that under A1, item ii) in Prop. 5 leads to the following remark

Remark 6. A necessary condition for ISO is $\{i_1, i_2,.., i_P\} \subset \{j_1, j_2.., j_M\}$, namely, $\mathcal{A} \subset \mathcal{O}$, i.e., all the attacked nodes must be observed. ■

3.3 Alternative Algebraic Characterization

In addition to Kalman rank condition, there exists an alternative characterization of observability (see Theorem 6.4.1 in Murota (2000)). Notice that the problem of reconstructing x_{k_0} from $y_{k_0:k_1}$ is equivalent to the problem of reconstructing $x_{k_0}, x_{k_0+1}, \ldots, x_{k_1}$. To see this, consider the following argument: reconstructing $x_{k_0}, x_{k_0+1}, \ldots, x_{k_1}$ is sufficient for reconstructing x_{k_0} On the other hand, under the assumption that $W_k\ \forall k \in [k_0, k_1 - 1]$ is known, if x_{k_0} can be reconstructed, then $x_{k_0+1}, \ldots, x_{k_1}$ can also be reconstructed.
The relationship between the states and outputs can be expressed via a system of linear equations as follows. From Eq. (1) and setting $u(k) = \mathbf{0}_P$, we have: $\forall k \in [k_0, k_1 - 1]$, $W_k x_k - x_{k+1} = \mathbf{0}_N$ and $\forall k \in [k_0, k_1]\ C_k x_k = y_k$.
This can be rewritten as: $\mathcal{Q}_{k_0,k_1} x_{k_0:k_1} = \begin{bmatrix} y_{k_0:k_1} \\ \mathbf{0}_{(k_1-k_0-1)N} \end{bmatrix}$, where

$$\mathcal{Q}_{k_0,k_1} = \begin{bmatrix} C_{k_0} & 0 & \ldots & \ldots & 0 \\ 0 & C_{k_0+1} & \ldots & \ldots & 0 \\ . & \ldots & \ldots & \ldots & . \\ . & \ldots & \ldots & \ldots & . \\ 0 & \ldots & \ldots & \ldots & C_{k_1} \\ W_{k_0} & -I_N & \ldots & \ldots & 0 \\ 0 & W_{k_0+1} & -I_N & \ldots & 0 \\ . & \ldots & \ldots & \ldots & . \\ . & \ldots & \ldots & \ldots & . \\ 0 & \ldots & \ldots & W_{k_1-1} & -I_N \end{bmatrix}.$$

Therefore, observability is equivalent to uniqueness of solution of the above system of linear equations and hence we have the following proposition.

Proposition 7. The system $\{W_k, C_k\}_{k_0}^{k_1}$ is observable over $[k_0, k_1]$ if and only if $\text{rank}(\mathcal{Q}_{k_0,k_1}) = (k_1 - k_0 + 1)N$. ■

One can do similar reasoning for ISO as well. Here, both the state equation and output equation at each time instant can be expressed as a linear combination of $x_{k_0}, x_{k_0+1}, \ldots, x_{k_1}$ as well as $u_{k_0}, u_{k_0+1}, \ldots, u_{k_1-1}$ in the following manner: from Eq. (1) we have: $\forall k \in [k_0, k_1 - 1]$, $W_k x_k - x_{k+1} + B_k u_k = \mathbf{0}_N$ and $\forall k \in [k_0, k_1]$, $C_k x_k = y_k$. Rewriting this in a more compact form leads to:

$$\mathcal{J}_{k_0,k_1} \begin{bmatrix} u_{k_0:k_1-1} \\ x_{k_0:k_1} \end{bmatrix} = \begin{bmatrix} y_{k_0:k_1} \\ \mathbf{0}_{(k_1-k_0)N} \end{bmatrix},$$

where

$$\mathcal{J}_{k_0,k_1} = \begin{bmatrix} 0 & \mathcal{Q}_{k_0,k_1} \\ \mathbf{B} & \end{bmatrix}$$

and $\mathbf{B}_{k_0,k_1} =$ diag $(B_{k_0}, B_{k_0+1} \ldots B_{k_1-1})$.
ISO is equivalent to uniqueness of solution of the above system of linear equations. Therefore, we can state the following proposition:

Proposition 8. The system $\{W_k, B_k, C_k\}_{k_0}^{k_1}$ is ISO over $[k_0, k_1]$ if and only if $\text{rank}\ (\mathcal{J}_{k_0,k_1}) = (k_1 - k_0)P + (k_1 - k_0 + 1)N$. ■

4. ISO AS OBSERVABILITY OF A SUITABLE SUBSYSTEM

In the LTI setup, Proposition 4 in Kibangou et al. (2016) shows the equivalence between ISO of a system and observability of a relevant subsystem. The proof therein relied on PBH rank tests which do not hold for LTV system. Nonetheless, the central idea is that this approach enables one to tackle the ISO problem by exploiting the results on observability. Here, we present a similar result for LTV systems not relying on PBH tests but on Prop 8. Moreover, while Kibangou et al. (2016) considered the particular case of *single* input, we extend the same for *multiple* inputs under A1.

As stated in Remark 6 we know that observing all the attacked nodes is a necessary condition for ISO. Therefore, from now, we assume that all the attacked nodes are observed. Under this assumption and A1, we can relabel the nodes in the following manner: $i_1 = j_1 = 1, i_2 = j_2 = 2, ..., i_P = j_P = P$.

Hence, we can rewrite assumption A1 as follows:
A2:

$$A_B = [e_{1;N} \; e_{2;N} \; \dots \; e_{P;N}],$$

$$A_C^T = [e_{1;N} \; e_{2;N} \; \dots \; e_{P;N} \; e_{j_{P+1};N} \; \dots \; e_{j_M;N}]. \blacksquare$$

We would like to decompose the system $\{W_k, B_k, C_k\}_{k_0}^{k_1}$ into two subsytems. To this end, we define matrices Q_N and $\bar{Q}_N$ as follows:

$$Q_N = [e_{1;N} \; e_{2;N} \; \dots \; e_{P;N}],$$

$$\bar{Q}_N = [e_{P+1;N} \; e_{P+2;N} \; \dots \; e_{N;N}].$$

Along similar lines, we define Q_M and $\bar{Q}_M$. Postmultiplying a matrix with Q_N selects the first P columns of the said matrix, while doing so with $\bar{Q}_N$ selects the last $N-P$ columns. Note that the same also holds for Q_M and $\bar{Q}_M$.

The fact that the identity of the nodes being attacked does not change with time, allows us to decompose the state vector in the following manner:

$$x_k = \begin{bmatrix}\hat{x}_k^T & \tilde{x}_k^T\end{bmatrix}^T,$$

where $\hat{x}_k \in \mathbb{R}^{\text{P}}$ denotes the states that are directly affected by the unknown inputs whereas $\tilde{x}_k \in \mathbb{R}^{\text{N-P}}$ denotes the remaining states. This enables us to decompose the system $\{W_k, B_k, C_k\}_{k_0}^{k_1}$ into two subsystems as follows:

$$\begin{cases} \hat{x}_{k+1} = \hat{W}_k \hat{x}_k + Q_N^T W_k \bar{Q}_N \tilde{x}_k + Q_N^T B_k u_k \\ \hat{y}_k = \hat{C}_k \hat{x}_k \end{cases} \quad (4)$$

$$\begin{cases} \tilde{x}_{k+1} = \tilde{W}_k \tilde{x}_k + \bar{Q}_N^T W_k B_k \hat{x}_k \\ \tilde{y}_k = \tilde{C}_k \tilde{x}_k \end{cases} \quad (5)$$

where $\hat{C}_k$ (resp. $\hat{W}_k$) contains the first P rows and the P columns from C_k (resp. W_k). Hence, $\hat{C}_k \in \mathbb{R}^{\text{P} \times \text{P}}$, $\hat{W}_k \in \mathbb{R}^{\text{P} \times \text{P}}$. Similarly, $\tilde{C}_k$ (resp. $\tilde{W}_k$) is obtained by removing the first P rows and first P columns from C_k (resp. W_k). Therefore, $\hat{W}_k = Q_N^T W_k Q_N$, $\hat{W}_k \in \mathbb{R}^{\text{P} \times \text{P}}$, $\hat{C}_k = Q_N^T C_k Q_N$, $\hat{C}_k \in \mathbb{R}^{\text{P} \times \text{P}}$, $\tilde{W}_k = \bar{Q}_N^T W_k \bar{Q}_N$, $\tilde{W}_k \in \mathbb{R}^{\text{(N-P)} \times \text{(N-P)}}$, $\tilde{C}_k = \bar{Q}_M^T C_k \bar{Q}_N$, $\tilde{C}_k \in \mathbb{R}^{\text{(M-P)} \times \text{(N-P)}}$.

$\hat{C}_k$ is a diagonal matrix with no zeros along the diagonal. Therefore, from Eq. (4) (second line) it can be seen that $\hat{x}_k$ are directly observed. Hence, Eq. (4) represents a system with *known state but two unknown inputs, namely,* $\tilde{x}_k$ *and* u_k, while Eq. (5) represents a system with *unknown state but known input.* If we assume that the system $\{\tilde{W}_k, \tilde{C}_k\}_{k_0}^{k_1}$ is observable over $[k_0, k_1]$, then one of the two unknown inputs in Eq. (4), namely $\tilde{x}_k$, is known and hence we can also compute u_k. This leads us to the following proposition.

Proposition 9. Under A2, the system $\{W_k, B_k, C_k\}_{k_0}^{k_1}$ is ISO over $[k_0, k_1]$ if and only if the system $\{\tilde{W}_k, \tilde{C}_k\}_{k_0}^{k_1}$ is observable over $[k_0, k_1]$. $\blacksquare$

Proof: Let Π_1 and Π_2 represent row and column permutation matrices respectively, defined as follows: For column permutations, first put at the beginning the first P columns of C_k

$$\mathcal{J}\Pi_2 = \begin{bmatrix} R_1 & 0 & R_3 \\ R_2 & \mathbf{B}_{k_0,k_1} & R_4 \end{bmatrix}$$

where, $R_1 = \operatorname{diag}(C_{k_0}Q_N, \dots, C_{k_1}Q_N)$,

$$R_2 = \begin{bmatrix} W_{k_0}Q_N & -Q_N & 0_{N\times P} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0_{N\times P} & \dots & \dots & W_{k_1-1}Q_N & -Q_N \end{bmatrix},$$

$R_3 = \operatorname{diag}(C_{k_0}\bar{Q}_N, \dots, C_{k_1}\bar{Q}_N)$ and

$$R_4 = \begin{bmatrix} W_{k_0}\bar{Q}_N & -I_N\bar{Q}_N & 0 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & W_{k_1-1}\bar{Q}_N & -I_N\bar{Q}_N \end{bmatrix}.$$

For row permutations, consider the following steps: we first arrange the $(k_1 - k_0 + 1)$ row blocks corresponding to the first P rows of C_k, then the $(k_1 - k_0)$ row blocks corresponding to the first P rows of B_k and W_k, and finally the remaining rows of C_k and W_k, so as to obtain

$$\Pi_1\mathcal{J}\Pi_2 = \begin{bmatrix} \mathbf{I}_{(k_1-k_0+1)P} & 0 & 0 \\ P_1 & \mathbf{I}_{(k_1-k_0)P} & P_2 \\ 0 & 0 & \tilde{\mathbf{C}} \\ P_3 & 0 & \tilde{\mathbf{W}} \end{bmatrix},$$

where

$$P_1 = \begin{bmatrix} \hat{W}_{k_0} & -Q_N & 0 & \dots \\ \dots & \ddots & \ddots & \dots \\ 0 & \dots & \hat{W}_{k_1-1} & -Q_N \end{bmatrix},$$

$$P_2 = \begin{bmatrix} Q_N^T W_{k_0}\bar{Q}_N & 0 & \dots & \dots \\ \dots & \ddots & \dots & \dots \\ 0 & \dots & Q_N^T W_{k_1-1}\bar{Q}_N & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} \bar{Q}_N^T W_{k_0}Q_N & 0 & \dots & \dots \\ \dots & \ddots & \dots & \dots \\ 0 & \dots & \bar{Q}_N^T W_{k_1-1}Q_N & 0 \end{bmatrix},$$

$\tilde{\mathbf{C}} = \operatorname{diag}(\tilde{C}_{k_0}, \dots, \tilde{C}_{k_1})$ and

$$\tilde{\mathbf{W}} = \begin{bmatrix} \tilde{W}_{k_0} & -I_{N-1} & \dots & \dots & 0 \\ 0 & \tilde{W}_{k_0+1} & -I_{N-1} & \dots & 0 \\ . & \dots & \dots & \dots & . \\ . & \dots & \dots & \dots & . \\ 0 & \dots & \dots & \tilde{W}_{k_1-1} & -I_{N-1} \end{bmatrix}.$$

Let $\bar{\mathcal{J}} = \Pi_1 \mathcal{J} \Pi_2$,

$$\hat{\mathcal{J}} = \begin{bmatrix} \mathbf{I}_{(k_1-k_0)P} & P_2 \\ 0 & \tilde{\mathcal{J}} \end{bmatrix}, \qquad \tilde{\mathcal{J}} = \begin{bmatrix} \tilde{\mathbf{C}} \\ \tilde{\mathbf{W}} \end{bmatrix}.$$

Notice that $\bar{\mathcal{J}}$ is block lower triangular with the blocks over the diagonal $\mathbf{I}_{(k_1-k_0+1)P}$ and $\hat{\mathcal{J}}$. This implies rank $(\bar{\mathcal{J}}) = (k_1 - k_0 + 1)P +$ rank $(\hat{\mathcal{J}})$. $\hat{\mathcal{J}}$ is block upper triangular with blocks over the diagonal $\mathbf{I}_{(k_1-k_0)P}$ and $\tilde{\mathcal{J}}$. Therefore, the following holds:

rank $(\bar{\mathcal{J}}) = (k_1 - k_0 + 1)P + (k_1 - k_0)P +$ rank $(\tilde{\mathcal{J}})$.

From Prop. 8 we know that $\{W_k, B_k, C_k\}_{k_0}^{k_1}$ is ISO over $[k_0, k_1]$ if and only if rank $(\bar{\mathcal{J}}) = (k_1 - k_0)P + (k_1 - k_0 + 1)N$, which in turn is equivalent to rank $(\tilde{\mathcal{J}}) = (k_1 - k_0 + 1)(N - P)$. From Prop. 7, the latter corresponds to observability of $\{\tilde{W}_k, \tilde{C}_k\}_{k_0}^{k_1}$ over $[k_0, k_1]$. ■

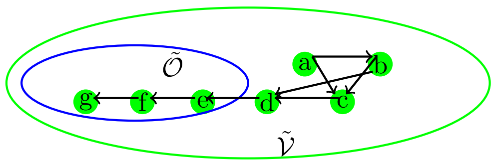

Fig. 2. Subsystem for the system in Fig. 1

5. MAIN RESULT

5.1 S-Structural ISO

Insofar, ISO has been characterized in a purely algebraic manner i.e., in terms of rank conditions of matrices namely,Ψ_{k_0,k_1} and $\mathcal{J}_{k_0,k_1}$. This approach suffers from two drawbacks namely, exact knowledge of all the coefficients of the said matrices is required and secondly it becomes computationally heavy as the size of the network grows. Therefore, here we seek s-structural results i.e., the focus is on finding conditions such that the system is ISO regardless of the choice of non-zero coefficients in the system matrices. Let $\{\mathcal{W}, \mathcal{B}, \mathcal{C}\}_{\text{LTV}}$ represent the family of *all* LTV systems as given in Eq. (1) and respecting the structure $\mathcal{W}, \mathcal{B}, \mathcal{C}$. *S-structural ISO* asks that *every* member in the family of time-varying systems represented by $\{\mathcal{W}, \mathcal{B}, \mathcal{C}\}_{\text{LTV}}$ be ISO.

Definition 10. Let $k_1, k_0 \in \mathbb{Z}$ and $k_1 > k_0$, $\{\mathcal{W}, \mathcal{B}, \mathcal{C}\}_{\text{LTV}}$ is *strongly structurally ISO on* $[k_0, k_1]$ if for *every system*$\{W_k, B_k, C_k\}_{k_0}^{k_1}$ with $W_k \in \mathcal{W}, B_k \in \mathcal{B}$ and $C_k \in \mathcal{C}$; rank $(\mathcal{J}_{k_0,k_1}) = (k_1 - k_0 + 1)N + (k_1 - k_0)P$. ■

Definition 11. $\{\mathcal{W}, \mathcal{B}, \mathcal{C}\}_{\text{LTV}}$ is *s-structurally uniform δ-step ISO* if for *every system* $\{W_k, B_k, C_k\}_{k \in \mathbb{Z}}$ with $W_k \in \mathcal{W}, B_k \in \mathcal{B}$ and $C_k \in \mathcal{C}$; rank$(\Psi_{k_0,k_0+\delta}) = N + \delta P$, $\forall\, k_0 \in \mathbb{Z}$. ■

Let $\{\mathcal{W}, \mathcal{C}\}_{\text{LTV}}$ represent the family of *all* LTV systems as given in Eq. (1) *but without the unknown input* u_k. Therefore, analogous to Definition 10 and Definition 11, one can define *s-structural observability on* $[k_0, k_1]$ and *s-structural uniform δ-step observability* respectively.

5.2 S-Structural ISO over an interval

We define the sets of matrices $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{C}}$ as $\tilde{\mathcal{W}} = \{\bar{Q}_N^T W \bar{Q}_N \mid W \in \mathcal{W}\}$ and $\tilde{\mathcal{C}} = \{\bar{Q}_M^T C \bar{Q}_N \mid C \in \mathcal{C}\}$ respectively. Let $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTV}}$ represent the family of *all* LTV systems as given in Eq. (5) *but without the unknown input* $\hat{x}_k$. Since we are interested in obtaining s-structural ISO results, notice that an immediate corollary of Prop. 9 is as follows:

Proposition 12. Under A2, $\{\mathcal{W}, \mathcal{B}, \mathcal{C}\}_{\text{LTV}}$ is s-structurally ISO over $[k_0, k_1]$ if and only if $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTV}}$ is s-structurally observable over $[k_0, k_1]$. ■

The central idea is that Prop. 12 rephrases the s-structural ISO as an equivalent problem in s-structural observability. Furthermore, it is also pertinent to ask if we can draw an equivalence with s-structural observability of a suitable family of LTI systems. The relevance stems from the fact that LTI systems are either observable in at most N steps or they are never. We will focus our attention on intervals of length at least N, with the following assumption.

A3: Let $k_0, k_1 \in \mathbb{Z}, k_0 + N \leq k_1$. ■

Let $\{\mathcal{W}, \mathcal{B}, \mathcal{C}\}_{\text{LTI}}$ represent the corresponding family of linear time-invariant systems (i.e., whose matrices have the same zero/non-zero structure as given by $\mathcal{W}, \mathcal{B}$ and $\mathcal{C}$). Similarly, let $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTI}}$ represent the LTI analogue of $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTV}}$. Then, based on Corollary IV.2 in Reissig et al. (2014) we have the following:

Lemma 13. **[Corollary IV.2 Reissig et al. (2014)]** Under A3, $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTV}}$ is s-structurally observable over $[k_0, k_1]$ if and only if $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTI}}$ is s-structurally observable over $[k_0, k_1]$. ■

From Prop. 12 and Lemma 13 the following is immediate:

Proposition 14. Under A2 and A3, $\{\mathcal{W}, \mathcal{B}, \mathcal{C}\}_{\text{LTV}}$ is s-structurally ISO over $[k_0, k_1]$ if and only if $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTI}}$ is s-structurally observable over $[k_0, k_1]$. ■

The beauty of Prop. 14 lies in the rephrasing of s-structural ISO problem in LTV setup as an equivalent s-structural observability problem of its LTI counterpart. However, notice that it is specific to *some* interval $[k_0, k_1]$ that satisfies A3.

5.3 S-Structural Uniform N-step ISO

It is well-known that, for LTI systems, notion of observability is independent of time-interval. That is, if an LTI system is observable over *a* time window of *some* length, then it is also observable over *every* time window of length *at least* N. Hence, if $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTI}}$ is s-structurally observable over $[k_0, k_1]$ then it is s-structurally observable over every sufficiently large interval. Hence, Lemma 13 can be rewritten as follows:

Lemma 15. $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTI}}$ is s-structurally observable if and only if $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTV}}$ is s-structurally observable over every $[k_0, k_1]$ satisfying A3. ■

Therefore, we can rewrite Prop. 12 and Prop. 14 as follows:

Theorem 16. Under A2, $\{\mathcal{W}, \mathcal{B}, \mathcal{C}\}_{\text{LTV}}$ is s-structurally ISO over every $[k_0, k_1]$ satisfying A3 if and only if $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\text{LTI}}$ is s-structurally observable. ■

Notice that Theorem 16 differs from Prop. 14 in the sense that it concerns *every* $[k_0, k_1]$ that satisfies A3.

Letting $\delta = N$, Definition 3 asks for ISO (resp. observability) in *exactly* N steps, while under A3, Prop. 9 asks for the same in *at least* N steps. The following lemma gives the equivalence between the two.

Lemma 17. $\{W_k, B_k, C_k\}_{k\in\mathbb{Z}}$ is uniformly δ-step ISO (resp. observable) if and only if $\{W_k, B_k, C_k\}_{k\in\mathbb{Z}}$ is ISO (resp. observable) over $[k_0, k_0+\eta]$ $\forall k_0 \in \mathbb{Z}$, $\forall \eta \geq \delta$. ■

Proof: If a system is uniformly δ-step observable, then it is also observable over $[k_0, k_0+\eta]$ $\forall k_0 \in \mathbb{Z}$, $\forall \eta \geq \delta$. For ISO, one needs to prove that all inputs up to $\eta - 1$ are reconstructed and not only those up to $\delta - 1$. This can be done by using δ-step ISO over successive time windows. Also, notice that if a system is observable (resp. ISO) over $[k_0, k_0+\eta]$ $\forall k_0 \in \mathbb{Z}$, $\forall \eta \geq \delta$, then in particular it is also observable (resp. ISO) over $[k_0, k_0+\delta]$ $\forall k_0 \in \mathbb{Z}$. ■ From Lemma 17, Theorem 16 can be rewritten as follows:

Proposition 18. $\{\mathcal{W}, \mathcal{B}, \mathcal{C}\}_{\mathrm{LTV}}$ is s-structurally uniformly N-step ISO if and only if $\{\tilde{\mathcal{W}}, \tilde{\mathcal{C}}\}_{\mathrm{LTI}}$ is s-structurally observable. ■

Prop. 18 rephrases the s-structurally uniform N-step ISO problem of an LTV system as that of s-structural observability problem of a suitable LTI subsystem. This can then be analysed using a graphical approach as shown in Chapman and Mesbahi (2013), the notion of zero-forcing sets as shown in Trefois and Delvenne (2015) or a linear-time algorithm as given in Weber et al. (2014).

6. CONCLUSION

We have shown that for discrete-time LTV network systems, under appropriate assumptions, ISO problem over an interval can be rephrased as an observability problem of a suitable subsystem over the same interval. Moreover, we have studied how to extend these results to a family of systems via *s-structural* results. An interesting direction of future work would be to study ISO for LTV network systems wherein the topology varies. Another line of work would be to investigate what happens when the identity of the attacked nodes is not known a priori, e.g. random attacks.

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