Values of globally bounded G-functions

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Abstract

In this paper we define and study a filtration $(\mathbf{G}_s)_{s\geq 0}$ on the algebra of values at algebraic points of analytic continuations of G-functions: $\mathbf{G}_s$ is the set of values at algebraic points in the disk of convergence of all G-functions $\sum_{n=0}^{\infty} a_n z^n$ for which there exist some positive integers b and c such that $d_{bn}^s c^{n+1} a_n$ is an algebraic integer for any n, where $d_n = \mathrm{lcm}(1, 2, \ldots, n)$.

We study the situation at the boundary of the disk of convergence, and using transfer results from analysis of singularities we deduce that constants in $\mathbf{G}_s$ appear in the asymptotic expansion of such a sequence (a_n).

1 Introduction

The motivation of this paper is to define an arithmetic weight (or degree) on the set $\mathcal{P}$ of periods (in the sense of Kontsevich-Zagier [16], say); we refer to [21] for an apparently unrelated geometric approach. Conjecturally $\mathcal{P}[1/\pi]$ is the set of values at algebraic points of analytic continuations of G-functions (see [13, end of §2.2]); we shall focus on these values.

Throughout this paper we fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$; all algebraic numbers and all convergents series are considered in $\mathbb{C}$. We deal with fine properties of G-functions $\sum_{n=0}^{\infty} a_n z^n$, which are defined as follows.

Definition 1. *A G-function f is a formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that the coefficients a_n are algebraic numbers and there exists $C > 0$ such that, for any $n \geq 0$:*

(i) the maximum of the moduli of the conjugates of a_n is $\leq C^{n+1}$.

(ii) there exists a sequence of (rational) integers $D_n > 0$, with $D_n \leq C^{n+1}$, such that $D_n a_m$ is an algebraic integer for all $m \leq n$.

(iii) $f(z)$ satisfies a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

We denote by $\mathcal{G}$ the ring of G-functions. For any integer $n \geq 0$, let $d_n := \mathrm{lcm}(1, 2, \ldots, n)$. For any integer $s \geq 0$, we denote by $\mathcal{G}_s$ (as in [14]) the set of G-functions for which there exist some positive integers b and c such that $d_{bn}^s c^{n+1} a_n$ is an algebraic integer for any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. The series in $\mathcal{G}_0$ are exactly those G-functions which are *globally bounded* in the sense of Christol [7], see also [17, pp. 1-2]; the set $\mathcal{G}_0$ is a ring and it contains all algebraic functions holomorphic at $z = 0$. It is clear that $\mathcal{G}_s$ is a $\overline{\mathbb{Q}}$-vector space stable under derivation, and that $\mathcal{G}_s\mathcal{G}_t \subset \mathcal{G}_{s+t}$ for any $s, t \in \mathbb{N}$. For instance the s-th polylogarithm $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} z^n/n^s$ belongs to $\mathcal{G}_s$. Any hypergeometric G-function $f(z) = {}_{p+1}F_p(z)$ with rational parameters belongs to $\mathcal{G}_s$ for some s; the least such integer s can be computed in terms of the parameters using a criterion of Christol [7] (see also [10]). More generally we have proved [14] that any G-function coming from geometry belongs to $\mathcal{G}_\infty := \cup_{s\in\mathbb{N}}\mathcal{G}_s$, so that conjecturally $\mathcal{G}_\infty = \mathcal{G}$. To sum up, $(\mathcal{G}_s)_{s\geq 0}$ is a total filtration of the $\overline{\mathbb{Q}}$-algebra $\mathcal{G}_\infty$, which is conjecturally equal to $\mathcal{G}$.

In this paper we define and study an analogous filtration on the $\overline{\mathbb{Q}}$-algebra $\mathbf{G}$ of values of G-functions (see [13]), namely the set of values of any (analytic continuation of a) G-function at any algebraic point. Recall that each G-function f has a positive radius of convergence R_f, which is infinite if and only if f is a polynomial. It will be considered as a function holomorphic on the open disk of radius R_f, and extended to a continuous function on the closed disk minus finitely many points (at which $|f(z)|$ tends to infinity). In other words, if $|\alpha| = R_f < \infty$ then $f(\alpha)$ stands for the limit of $f(z)$ as $z \to \alpha$ with $|z| < R_f$.

For any $s \geq 0$, we denote by $\mathbf{G}_s$ the set of all numbers $f(\alpha)$ with $f \in \mathcal{G}_s$ and $\alpha \in \overline{\mathbb{Q}}$ such that $|\alpha| \leq R_f$ and $f(\alpha)$ is finite. Of course, we may assume $\alpha = 1$ in this definition since $f(\alpha z) \in \mathcal{G}_s$. Therefore $\mathbf{G}_s$ is a $\overline{\mathbb{Q}}$-vector space; it contains $\mathrm{Li}_s(1) = \zeta(s)$ if $s \geq 2$, where ζ is Riemann zeta function (see §2 for other examples). Letting $\mathbf{G}_\infty = \cup_{s\geq 0}\mathbf{G}_s$, we shall prove in Corollary 1 below that the $\overline{\mathbb{Q}}$-vector spaces $\mathbf{G}_s$ define a total filtration of $\mathbf{G}_\infty$, with $\mathbf{G}_s \cdot \mathbf{G}_t \subset \mathbf{G}_{s+t}$ for any s, t.

We have proved [13, Theorem 1] that any $\xi \in \mathbf{G}$ can be written as $f(1)$ where $f \in \mathcal{G}$ has coefficients in $\mathbb{Q}(i)$ and R_f can be chosen arbitrarily large; therefore the above-mentioned conjecture $\mathcal{G}_\infty = \mathcal{G}$ (which follows from standard conjectures, see [14]) implies $\mathbf{G}_\infty = \mathbf{G}$.

Let us point out that in the definition of $\mathbf{G}_s$ we consider only values $f(\alpha)$ with $|\alpha| \leq R_f$: analytic continuation outside the disk of convergence is not allowed, but the case $|\alpha| = R_f$ is. In this case the series $\sum_{n=0}^{\infty} a_n\alpha^n$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, can be divergent or converge but not absolutely (for instance $\sum_{n=0}^{\infty}(-1/4)^n\binom{2n}{n} = 1/\sqrt{2} \in \mathbf{G}_0$). Absolute convergence was crucial to us to prove that $\mathbf{G}$ is a ring, a property we shall prove below for $\mathbf{G}_0$ and $\mathbf{G}_\infty$. The problem is to perform correctly the multiplication on $\mathbf{G}_s$. It is well-known that the Cauchy product $\sum_{n=0}^{\infty}\sum_{j+k=n} a_j b_k$ might not be equal $(\sum_{n=0}^{\infty} a_n) \cdot (\sum_{n=0}^{\infty} b_n)$ when both series converge only conditionally. For instance,

$$\left(\sum_{n=0}^{\infty}(-1)^n\frac{\binom{2n}{n}}{4^n}\right)^2 \neq \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\sum_{j+k=n}\binom{2j}{j}\binom{2k}{k}$$

because $\sum_{j+k=n}\binom{2j}{j}\binom{2k}{k} = 4^n$ for all $n \geq 0$, so that the series on the right-hand side is divergent. However, $(\sum_{n=0}^{\infty} a_n) \cdot (\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty}\sum_{j+k=n} a_j b_k$ when both series converge and at least one converges absolutely (Mertens' theorem). Unfortunately, our proof of [13, Theorem 1] offers no control on the p-adic valuations of the denominators of the Taylor coefficients of f, so that given $\xi \in \mathbf{G}_s$ we cannot assert that the function $f \in \mathcal{G}$ we construct there (such that $f(1) = \xi$ and R_f is arbitrarily large) belongs to $\mathcal{G}_s$. Our first result shows we can solve this problem but there is a cost: we are no longer able to prove that the coefficients are in $\mathbb{Q}(i)$ and that $R_f > 1$.

Theorem 1. *For any $\xi \in \mathbf{G}_s$ there exists $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{G}_s$ such that $R_f \geq 1$ and $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series equal to ξ.*

More precisely, using transfer results from analysis of singularities we construct such an f with the property that $a_n = \mathcal{O}(n^{-1-\varepsilon})$ for some $\varepsilon > 0$ that depends on ξ. In general, our method does not yield a function f with $R_f > 1$. To try to obtain f as in Theorem 1, one may apply Euler's acceleration method $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} a_k$, where the right-hand side is sometimes much more rapidly convergent than the left-hand side. But this process is not strong enough to systematically increase the radius of convergence in our setting. For instance, with $a_n = \frac{1}{(n+1)^s}$, $s \geq 2$, the radius of convergence of the G-function

$$\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)^s}$$

is still equal to 1. ([1]) There might be more efficient acceleration transforms (preserving G-functions) but we don't know of any. We also explain in Section 4 why our approach in [13], based on analytic continuation, is apparently inoperent here.

As explained above, Theorem 1 implies directly the following result.

Corollary 1. *The $\overline{\mathbb{Q}}$-vector spaces $\mathbf{G}_s$ make up a total filtration of the $\overline{\mathbb{Q}}$-algebra $\mathbf{G}_\infty$, i.e.*

$$\mathbf{G}_s \subset \mathbf{G}_{s+1} \quad \text{and} \quad \mathbf{G}_s \cdot \mathbf{G}_t \subset \mathbf{G}_{s+t} \quad \text{for any } s, t \in \mathbb{N}.$$

In particular, $\mathbf{G}_0$ and $\mathbf{G}_\infty$ are $\overline{\mathbb{Q}}$-subalgebras of $\mathbf{G}$ and, conjecturally, $\mathbf{G}_\infty = \mathbf{G}$.

The main tool in [13] is analytic continuation; we proved that connection constants of G-functions belong to $\mathbf{G}$. It is not clear to us whether this result can be adapted to $\mathbf{G}_s$ (see §4); however if we restrict to the border of the disk of convergence we have the following result.

Theorem 2. *Let $f \in \mathcal{G}_s$, and $\rho \in \overline{\mathbb{Q}}$ be a singularity of f with $|\rho| = R_f$. Then there exist $C \in \mathbf{G}_s \setminus \{0\}$, $t \in \mathbb{Q}$, $k \in \mathbb{N}$, with either $t \notin \mathbb{N}$ or $k \geq 1$, and a polynomial P of degree $< t$ with coefficients in $\mathbf{G}_s$, such that as $z \to \rho$ with $|z| \leq R_f$:*

$$f(z) = P(\rho - z) + C(\rho - z)^t (\log(\rho - z))^k (1 + o(1)).$$

[1] Simply observe that $1 \geq \frac{1}{2^n} \sum_{k=0}^{n} \frac{\binom{n}{k}}{(k+1)^s} \geq \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n (n/2+1)^s} \gg \frac{1}{n^{s+1/2}}$.

Since the coefficients of P are the $(-1)^j f^{(j)}(\rho)/j!$ with $0 \le j < t$, they obviously belong to $\mathbf{G}_s$; the main point here is that the "next" coefficient C (i.e., the "first non-holomorphic" one) is also in $\mathbf{G}_s$. Applying Theorem 2 to all singularities of f of modulus R_f yields the following corollary (using transfer results).

Corollary 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{G}_s$. Then there exist $t \in \mathbb{Q}$, $j \in \mathbb{N}$, a finite non-empty set S of algebraic numbers ζ such that $|\zeta| = 1$, and non-zero elements $C_\zeta \in \mathbf{G}_s$ for any $\zeta \in S$, such that, as $n \to \infty$,*

$$a_n = \frac{(\log(n))^j}{\Gamma(-t)\, R_f^n\, n^{t+1}} \Big(\sum_{\zeta \in S} C_\zeta \zeta^{-n} + o(1) \Big)$$

where $\Gamma(-t)$ should be understood as 1 if $t \in \mathbb{N}$.

With respect to [13, Eq. (6.2)], the new point in this corollary is that $C_\zeta \in \mathbf{G}_s$.

Finally, we apply our results to a refinement of the notion of G-approximations of complex numbers, introduced in [13].

Theorem 3. *Let $\mathbb{K}$ be a subfield of $\overline{\mathbb{Q}}$, $\xi \in \mathbb{C}^\star$ and $s, t \in \mathbb{N} \cup \{\infty\}$. Then the following assertions are equivalent:*

- (i) *There exist $f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{G}_s$ and $g = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{G}_t$ with coefficients $a_n, b_n \in \mathbb{K}$, such that $R_f \ge 1$, $R_g \ge 1$, and $\xi = f(1)/g(1)$.*
- (ii) *There exist two sequences $(u_n)_{n\ge0}$ and $(v_n)_{n\ge0}$ in $\mathbb{K}^{\mathbb{N}}$ such that $\sum_{n=0}^{\infty} u_n z^n \in \mathcal{G}_s$, $\sum_{n=0}^{\infty} v_n z^n \in \mathcal{G}_t$, and $\lim_{n\to\infty} \frac{u_n}{v_n} = \xi$.*

With $\mathbb{K} = \overline{\mathbb{Q}}$, assertion (i) means that $\xi \in \frac{\mathbf{G}_s}{\mathbf{G}_t}$, that is $\xi = x/y$ with $x \in \mathbf{G}_s$ and $y \in \mathbf{G}_t \setminus \{0\}$. Theorem 3 refines [13, Theorem 3], which is essentially the same statement with $\mathbb{K} = \mathbb{Q}$ and $\mathbf{G}$ instead of $\mathbf{G}_s$ and $\mathbf{G}_t$.

Considering $a_n = \sum_{k=1}^{n} \frac{1}{k^3}$ and $b_n = 1$, we see that $\zeta(3) \in \frac{\mathbf{G}_3}{\mathbf{G}_0}$. This can also be seen from Apéry's celebrated construction, i.e. with

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2, \; a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2 \Big(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{n}} \Big),$$

though it is a non-trivial task (see [11, 5]) to prove that $\sum_{n=0}^{\infty} a_n z^n \in \mathcal{G}_3$ and that $a_n/b_n \to \zeta(3)$. Of course, by multiplying these two specific sequences a_n and b_n by $\frac{1}{(n+1)^s}$, $s \ge 0$, we see that $\zeta(3) \in \frac{\mathbf{G}_{3+s}}{\mathbf{G}_s}$ as well.

As J. Cresson pointed out to us, Theorem 3 shows that any real number $\xi = x/y$ with $x, y \in \mathbf{G}$ and $y \ne 0$ is elementary (in the sense of [22, Definition 9]). Therefore Yoshinaga's example of a non-elementary real number ξ_0 (constructed in [22, Proposition 17]) satisfies $\xi_0 \notin \mathrm{Frac}\,(\mathbf{G})$. Recall that conjecturally $\mathbf{G} = \mathcal{P}[1/\pi]$ where (as above) $\mathcal{P}$ is the ring of periods, and that Yoshinaga has proved that periods are elementary (so that $\xi_0 \notin \mathcal{P}$).

To conclude this introduction, we would like to point out an important difference between the present results and those of [13]. Any given G-function is annihilated by a G-operator of which, at any algebraic point, all solutions can be written in terms of G-functions. This enabled us to use analytic continuation in a crucial way in [13]. On the opposite, when globally bounded functions come into the play, the situation seems different. We suggest the following conjecture.

Conjecture 1. *Consider a differential operator $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ and some $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$. Let us assume that the differential equation $Ly(z) = 0$ has a local basis of solutions at $z = \alpha$ consisting of functions of the form $f(z-\alpha)$ if $\alpha \in \overline{\mathbb{Q}}$, or $f(1/z)$ if $\alpha = \infty$, where each $f(z)$ is a G-function in $\mathcal{G}_0$. Then $Ly(z) = 0$ has a $\mathbb{C}$-basis of solutions made of algebraic functions over $\overline{\mathbb{Q}}(z)$.*

In some sense, this conjecture is analogous to a result proved in [18] for E-functions, namely that if an E-operator admits a $\mathbb{C}$-basis of holomorphic solutions at $z = 0$, then such a basis can be made of functions of the form $\sum_{j=1}^{\mu} P_j(z) e^{\alpha_j z}$, where $P_j(z) \in \overline{\mathbb{Q}}(z)$ and $\beta_j \in \overline{\mathbb{Q}}$. Such exponential polynomials are the simplest examples of E-functions, while algebraic functions over $\overline{\mathbb{Q}}(z)$ are the simplest G-functions. If, in Conjecture 1, we remove the assumption that the G-functions $f(z)$ are in $\mathcal{G}_0$, then the solutions are not always algebraic functions; we do not know if a classification of the solutions is still possible, even under the further assumption that L is a G-operator ([2]).

As pointed out to us by B. Adamczewski, G. Christol and E. Delaygue, Conjecture 1 follows from Grothendieck's conjecture on p-curvatures (see §4 for details). In any case, it prevents us from using analytic continuation, and it explains why all results in the present paper concern only the (closed) disk of convergence.

The structure of this paper is as follows. We gather examples in §2, and prove in §3 the results stated in this introduction; the main tool is transfer results from analysis of singularities. At last, we discuss in §4 the problem of analytic continuation.

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2 Examples

In this section, we present various examples of numbers in $\mathbf{G}_s$. From the well-known formulas or definitions

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}, \qquad \log(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n},$$

[2] A G-operator is a differential operator in $\overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ minimal, for the degree in $\frac{d}{dz}$, among those which annihilate some G-function; an E-operator is a differential operator in $\overline{\mathbb{Q}}[z, \frac{d}{dz}]$ minimal, for the degree in z, among those which annihilate some E-function.

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\qquad \frac{\pi^2}{12}-\frac{\log(2)^2}{2}=\sum_{n=1}^{\infty}\frac{1}{n^2 2^n},$$
$$\zeta(s_1,s_2,\ldots,s_n)=\sum_{n_1>n_2>\cdots n_k\geq 1}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}$$

we deduce that $\pi,\log(2)\in\mathbf{G}_1$, $\frac{\pi^2}{12}-\frac{\log(2)^2}{2}\in\mathbf{G}_2$, $\zeta(s)\in\mathbf{G}_s$ for $s\geq 2$, and $\zeta(s_1,s_2,\ldots,s_n)\in\mathbf{G}_{s_1+\cdots+s_n}$ for $s_1\geq 2$, $s_j\geq 1$. On the other hand, π is a unit of the ring $\mathbf{G}$ of G-values and more precisely we oberve that $1/\pi$ is in $\mathbf{G}_0$, as Ramanujan's formula shows:

$$\frac{1}{\pi}=\sum_{n=0}\frac{(42n+5)\binom{2n}{n}^3}{2^{12n+4}}.$$

However we conjecture that $\pi\notin\mathbf{G}_0$.

Some hypergeometric functions ${}_{p+1}F_p$ with rational parameters are globally bounded, though it is not easy to prove this directly in a specific instance; Christol has obtained a characterization of this property in [7] (see also [10] for a discussion and a refinement). In particular, any hypergeometric series with only 1's as lower parameters is globally bounded. Its values at algebraic points inside the unit disk are thus in $\mathbf{G}_0$.

We proved in [13, Proposition 1] that the Beta values $B(x,y):=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ are units of $\mathbf{G}$ for any $x,y\in\mathbb{Q}$ for which $B(x,y)$ is defined and not zero; observe that $B(\frac{1}{2},\frac{1}{2})=\pi$.

Proposition 1. *For any $x,y\in\mathbb{Q}$ for which $B(x,y)$ is defined and not zero, we have $\frac{1}{B(x,y)}\in\mathbf{G}_0$ and $B(x,y)\in\pi\mathbf{G}_0\subset\mathbf{G}_1$.*

Moreover, for any integers $a\geq 1$ and $b\geq 2$ we have

$$\Gamma(a/b)^b\in\pi^{b-1}\mathbf{G}_0\subset\mathbf{G}_{b-1}\quad \textit{and}\quad \frac{1}{\Gamma(a/b)^b}\in\frac{1}{\pi}\mathbf{G}_0\subset\mathbf{G}_0.$$

Proof. We start from Gauss' famous hypergeometric summation formula [20, p. 28, Eq. (1.7.6)]

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

which is valid for $\mathrm{Re}(c)>\mathrm{Re}(a+b)$, the series being absolutely convergent. When $a,b,c\in\mathbb{Q}$, $c\notin\mathbb{Z}_{\leq 0}$, the series is obviously a G-value and furthermore if $c=1$ then the value of the series is in $\mathbf{G}_0$. This last statement follows from Christol's above-mentioned characterization, and also directly from the well-known fact that there exists a positive integer A such that

$$A^n\frac{(a)_n}{n!}\in\mathbb{Z}\text{ for any }n\in\mathbb{N}.$$

The minimal value of A depends on the p-adic valuations of $a=u/v$ for all primes p, but $A=v^2$ is suitable. Applying Gauss' identity with a and b changed to $-x$ and $-y$

respectively, we see that for any non-negative $x, y \in \mathbb{Q}$ such that $x + y > 0$,

$$\frac{1}{B(x,y)} = \frac{xy}{x+y} \frac{\Gamma(1+x+y)}{\Gamma(1+x)\Gamma(1+y)} = \frac{xy}{x+y} \sum_{n=0}^{\infty} \frac{(-x)_n(-y)_n}{n!^2} \in \mathbf{G}_0.$$

We can extend this inclusion to any other well-defined value of $\frac{1}{B(x,y)}$, $x, y \in \mathbb{Q}$, by means of the functional equation $\Gamma(s+1) = s\Gamma(s)$. As in [13, Proposition 1], we have that $B(x,y) = \frac{\beta\pi}{B(1-x,1-y)}$ for some algebraic number β, which proves that $B(x,y) \in \pi\mathbf{G}_0$.

Furthermore, let $a \geq 1$ and $b \geq 2$. Then

$$(a-1)! \prod_{j=1}^{b-1} B\big(\frac{a}{b}, \frac{ja}{b}\big) = \Gamma\big(\frac{a}{b}\big)^b$$

so that $\Gamma(\frac{a}{b})^b \in \pi^{b-1}\mathbf{G}_0 \subset \mathbf{G}_{b-1}$. On the other hand, it is also straightforward to prove that, for any integers $a \geq 1$, $b \geq 2$,

$$\prod_{j=1}^{b-2} B\big(\frac{a}{b}, \frac{ja}{b}\big) = \frac{\Gamma\big(\frac{a}{b}\big)^{b-1}}{\Gamma\big(a - \frac{a}{b}\big)} = \beta \frac{\Gamma\big(\frac{a}{b}\big)^b}{\pi}$$

for some non-zero algebraic number β; this implies $\frac{\pi}{\Gamma(\frac{a}{b})^b} \in \mathbf{G}_0$. □

When a G-value is proved to be in a certain $\mathbf{G}_s$ with $s \geq 1$, one may wonder if it is in fact in $\mathbf{G}_t$ for some $t < s$. This is in general a very difficult question to answer. We have searched carefully among the large amount of alternative representations of these numbers in the literature, and we conjecture that the above mentioned results for π, $\frac{\pi^2}{12} - \frac{\log(2)^2}{2}$, $\zeta(s)$, $\zeta(s_1, s_2, \dots, s_n)$ are best possible. Furthermore, we proved in [13, Lemma 8] that $\log(\alpha) \in \mathbf{G}_1$ for any non-zero algebraic number α and any branch of the logarithm; we conjecture that in fact $\log(\alpha) \in \mathbf{G}_1 \setminus \mathbf{G}_0$ (in particular, $\pi \notin \mathbf{G}_0$).

Given $\xi \in \mathbf{G}$, it seems natural to consider the integer $\delta(\xi)$ defined as the least $s \geq 0$ such that $\xi \in \mathbf{G}_s$; such an s always exists (conjecturally) since $\mathbf{G}_\infty = \cup_s \mathbf{G}_s$. For instance we have $\delta(1) = \delta(1/\pi) = 0$, $\delta(\pi) \leq 1$, and conjecturally $\delta(\pi) = 1$. Since $\mathbf{G}_s \cdot \mathbf{G}_t \subset \mathbf{G}_{s+t}$ we have $\delta(\xi\xi') \leq \delta(\xi) + \delta(\xi')$ for any $\xi, \xi' \in \mathbf{G}$. We believe that this inequality is *not always* an equality: for instance $\pi \cdot \frac{1}{\pi} = 1$, and conjecturally $\delta(1) < \delta(\pi) + \delta(1/\pi)$. Another possible example is given by $\xi = \Gamma(\frac{a}{b})^b$ and $\xi' = \Gamma(\frac{b-a}{b})^b$ with coprime integers $b > a > 0$ such that $b \geq 3$: we have $\xi\xi' \in \pi^b\overline{\mathbb{Q}}^\star \subset \mathbf{G}_b$ so that $\delta(\xi\xi') \leq b$, whereas we believe that $\delta(\xi) = \delta(\xi') = b-1$. We refer to [21] for a different notion of degree of periods, with similar properties.

3 Proofs of the main results

3.1 Notations and transfer theorems

Let $s \in \mathbb{N}$, $f \in \mathcal{G}_s$, and ρ be a singularity of f such that $|\rho| = R_f$. Using the André-Chudnovski-Katz theorem (see [4, p. 719] for a discussion, and [13, Theorem 6] for a statement) and considering the first non-holomorphic term in a local expansion of f at ρ, we obtain $t(\rho,f)\in\mathbb{Q}$, $k(\rho,f)\in\mathbb{N}$, $C_{\rho,f}\in\mathbb{C}^\star$, and a polynomial $P_{\rho,f}\in\mathbb{C}[X]$ of degree $< t(\rho,f)$, such that

$$f(z) = P_{\rho,f}(\rho-z) + C_{\rho,f}\log(\rho-z)^{k(\rho,f)}(\rho-z)^{t(\rho,f)}(1+o(1)) \tag{3.1}$$

as $z\to\rho$ with $|z|\leq R_f$. Moreover the function $\log(\rho-z)^{k(\rho,f)}(\rho-z)^{t(\rho,f)}$ is not holomorphic at $z=\rho$, so that either $t(\rho,f)\notin\mathbb{N}$ or $k(\rho,f)\geq 1$. We remark that $C_{\rho,f}\in\mathbf{G}$ (see [13]); this weaker version of Theorem 1 will be re-proved below using a strategy different from the one of [13].

Transfer results (see [15], Chapter VI, §2) enable one to deduce an asymptotic estimate of a_n from the local behaviour of $f(z)=\sum_{n=0}^\infty a_n z^n$ at all $\rho\in\mathcal{S}$, where $\mathcal{S}$ is the finite non-empty set of singularities of f of modulus R_f. Indeed, let $t(f)=\min_{\rho\in\mathcal{S}} t(\rho,f)$, and let $k(f)$ denote the maximal value of $k(\rho,f)$ among the $\rho\in\mathcal{S}$ such that $t(\rho,f)=t(f)$. Then letting $\mathcal{S}'$ denote the set of all $\rho\in\mathcal{S}$ such that $t(\rho,f)=t(f)$ and $k(\rho,f)=k(f)$, we have:

$$a_n = \frac{\log(n)^{k(f)}}{\Gamma(-t(f))n^{t(f)+1}}\Big(\sum_{\rho\in\mathcal{S}'} D_\rho\rho^{-n}+o(1)\Big) \quad \text{if } t(f)\notin\mathbb{N}, \tag{3.2}$$

$$a_n = \frac{\log(n)^{k(f)-1}}{n^{t(f)+1}}\Big(\sum_{\rho\in\mathcal{S}'} D_\rho\rho^{-n}+o(1)\Big) \quad \text{if } t(f)\in\mathbb{N}, \tag{3.3}$$

where

$$D_\rho = \begin{cases} (-1)^{k(f)}C_{\rho,f}\rho^{t(f)} & \text{if } t(f)\notin\mathbb{N},\\[2ex] (-1)^{k(f)+t(f)}t(f)!k(f)C_{\rho,f}\rho^{t(f)} & \text{if } t(f)\in\mathbb{N}.\end{cases}$$

Of course we have $k(f)\geq 1$ if $t(f)\in\mathbb{N}$.

3.2 Proof of Theorem 1

Let $\xi\in\mathbf{G}_s$. There exists $f=\sum_{n=0}^\infty a_n z^n\in\mathcal{G}_s$ such that $R_f\geq 1$ and $f(1)=\xi$. If $R_f>1$ then the series $\sum a_n$ converges absolutely and is equal to ξ; therefore we may assume that $R_f=1$. We denote by $\mathcal{S}$ the set of singularities of f on the unit circle, and let

$$g(z) = f(z)\prod_{\rho\in\mathcal{S}\setminus\{1\}}(\rho-z)^{1-t(\rho,f)}.$$

Since $t(\rho,f)\in\mathbb{Q}$ for any ρ, the product is an algebraic function holomorphic at 0, and therefore a globally bounded G-function; this yields $g\in\mathcal{G}_s$. Let $\widetilde{\mathcal{S}}$ denote the set of singularities of g on the unit circle. Then $\widetilde{\mathcal{S}}\subset\mathcal{S}$ and $R_g\geq 1$; it may happen that $\widetilde{\mathcal{S}}=\emptyset$ so that $R_g>1$. Moreover for any $\rho\in\widetilde{\mathcal{S}}\setminus\{1\}$ we have $t(\rho,g)\geq 1$. If $1\in\mathcal{S}$ then $1\in\widetilde{\mathcal{S}}$ and $t(1,g)=t(1,f)>0$ since $f(z)$ has a finite limit ξ as $z\to 1$, $|z|<1$. Letting $\varepsilon=\min(1,t(1,f))$ in this case, and $\varepsilon=1$ if $1\notin\mathcal{S}$, we obtain in both cases that

$t(\rho,g) \geq \varepsilon > 0$ for any $\rho \in \widetilde{\mathcal{S}}$. If $\widetilde{\mathcal{S}} \neq \emptyset$ then (3.2) and (3.3) yield $b_n = \mathcal{O}(n^{-1-\varepsilon}(\log n)^{k(g)})$ where $g(z) = \sum_{n=0}^{\infty} b_n z^n$; this estimate holds also if $\widetilde{\mathcal{S}} = \emptyset$ since $R_g > 1$ in this case. Therefore $\sum b_n$ is an absolutely convergent series, equal to $g(1)$. Multiplying all coefficients b_n with the fixed algebraic number $f(1)/g(1) = \prod_{\rho\in\mathcal{S}\setminus\{1\}}(\rho-1)^{t(\rho,f)-1}$ concludes the proof of Theorem 1.

3.3 Proof of Theorem 2

Let $f \in \mathcal{G}_s$ and ρ be a singularity of f of modulus R_f. Eq. (3.1) shows that the coefficients of $P_{\rho,f}$ are the $(-1)^j f^{(j)}(\rho)/j!$ with $0 \leq j < t(\rho,f)$. Since $f^{(j)} \in \mathcal{G}_s$ we deduce that $P_{\rho,f} \in \mathbf{G}_s[X]$. Let us prove now that $C_{\rho,f} \in \mathbf{G}_s$. Letting $j = \lceil t(\rho,f) \rceil$ we have $\deg P_{\rho,f} \leq j-1$. If $P_{\rho,f} \neq 0$ then $C_{\rho,f}/C_{\rho,f^{(j)}} \in \mathbb{Q}^\star$ and $P_{\rho,f^{(j)}} = 0$ so that we may restrict to the case where $P_{\rho,f} = 0$. Then we argue by induction on $k(\rho,f)$. If $k(\rho,f) = 0$ then $C_{\rho,f}$ is the value at $z = \rho$ of the function $f(z)(\rho-z)^{-t(\rho,f)} \in \mathcal{G}_s$ so that $C_{\rho,f} \in \mathbf{G}_s$. Now assume that the result holds for any function with a smaller value of $k(\rho,f)$, and let $g(z) = f(z)(\rho-z)^{-t(\rho,f)} \in \mathcal{G}_s$. Then we have

$$g'(z) = -k(\rho,f)C_{\rho,f}\frac{\log(\rho-z)^{k(\rho,f)-1}}{\rho-z}(1+o(1))$$

with the derivative $g' \in \mathcal{G}_s$ as well, so that $k(\rho,f)C_{\rho,f} \in \mathbf{G}_s$. Since $k(\rho,f)$ is a positive integer, this concludes the induction and the proof of Theorem 2.

3.4 Proof of Theorem 3

The proof is the same as the corresponding theorem in [13]. We simply have to use the additionnal informations provided by Corollary 2 on the coefficients C_ζ. Since there is no subtility, we leave the details to the reader.

4 Conclusion and comments on Conjecture 1

In this section, we explain the difficulties to use here the method of [13] to express any $\xi \in \mathbf{G}$ as $f(1)$ where f has Taylor coefficients in $\mathbb{Q}(i)$ and $R_f > 1$ can be chosen arbitrarily.

To prove Theorem 1 in [13], we used analytic continuation and in particular made a large use of the André-Chudnovski-Katz theorem on local solutions of G-operators at algebraic points. This was even the first step to prove that $\mathbf{G}$ is ring. In the present situation, a possible analogue property, $\mathcal{P}$ say, would be that, given a function $f \in \mathcal{G}_0$ with minimal G-operator L, there exists at any algebraic point ρ a local basis of solutions of L made up of functions involving only series in $\mathcal{G}_0$. However Property $\mathcal{P}$ is false in general, as the following example shows. Let us consider the hypergeometric function

$$f(z) = {}_2F_1\left[\begin{matrix}a, b\\ 1\end{matrix}; z\right] = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!^2}z^n \qquad (4.1)$$

with $a, b \in \mathbb{Q}$, $a + b = 1$ and $|z| < 1$. As mentioned in Section 2, we have $f \in \mathcal{G}_0$. It is a solution of a hypergeometric differential equation of order 2, which admits at $z = 1$ the basis of solutions $f(1-z)$ and $g(1-z) + \log(1-z)f(1-z)$ with

$$g(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \Big(\frac{1}{k+a} + \frac{1}{k+b} - \frac{2}{k+1}\Big) \frac{(a)_n (b)_n}{n!^2} z^n.$$

Then we have

$$f(z) = -\frac{1}{\Gamma(a)\Gamma(b)} \big(g(1-z) + \log(1-z)f(1-z)\big)$$

for $|1-z| < 1$ and $|\arg(1-z)| < \pi$. Clearly, the series $g(z)$ is in $\mathcal{G}_1$ but not in $\mathcal{G}_0$. This observation is not surprising because a second solution at $z = 0$ is $g(z) + \log(z)f(z)$, again involving G-functions not all in $\mathcal{G}_0$.

In fact, Conjecture 1 stated in the Introduction implies that Property $\mathcal{P}$ holds only for a very small class of G-functions, namely algebraic functions over $\overline{\mathbb{Q}}(z)$. In fact, that conjecture deals with the more general situation where a basis of solutions of a differential operator $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ at $z = 0$ is made of G-functions in $\mathcal{G}_0$. (The seemingly more general case for $z = \alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$ can be immediately reduced to $z = 0$ by a suitable change of variable.) We mentioned in the Introduction that Conjecture 1 follows from Grothendieck's conjecture for p-curvatures (stated in [19, p. 2] for instance). The following argument is due to B. Adamczewski, G. Christol and E. Delaygue. For any prime ideal $\mathfrak{p}$ over a sufficiently large prime number p, this local basis provides by reduction of L mod $\mathfrak{p}$ a local basis of solutions consisting in formal power series (see [2, Lemma 4.4]). Therefore the $\mathfrak{p}$-curvature of L is zero (see [6, §3]), and Grothendieck's conjecture provides a basis of solutions algebraic over $\overline{\mathbb{Q}}(z)$.

Since we are not able to use analytic continuation, we did not succeed in proving any result concerning $f(\xi)$ when $f \in \mathcal{G}_s$ and $|\xi| > R_f$. It is not even clear to us whether one should expect that $f(\xi) \in \mathbf{G}_s$.

However we have seen in several examples that when ρ is a singularity of $f \in \mathcal{G}_s$ with $|\rho| = R_f$, the local expansion

$$f(z) = \sum_{n=-N}^{\infty} \sum_{k=0}^{K} c_{n,k} (\rho - z)^{n/d} \log(\rho - z)^k$$

as $z \to \rho$ has coefficients $c_{n,k} \in \mathbf{G}_s$. Arguing as in [13, §4] this means that all connection constants of f with respect to a local basis at ρ belong to $\mathbf{G}_s$; it would be a refinement upon Theorem 2. In this respect we remark that the connection constants of f with respect to local bases at algebraic points z with $|z| < R_f$ are simply the derivatives $f^{(j)}(z)$ of f at z, so that they belong to $\mathbf{G}_s$.

To conclude, let us mention that algebraic numbers can be written as $f(1)$ with $f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{G}_0$ having coefficients a_n in $\mathbb{Q}(i)$: this follows from [13, Lemma 7]. However we do not know if all elements of $\mathbf{G}_0$ have the same property.

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