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Abstract - *Delivery lead-time has become a factor of competitiveness for companies and an important criterion of purchase for the customers today.Thus, in order to increase their profit, companies must not focus only on price but also need to quote the right delivery lead time to their customers.Some authors to find a way in quoting the right delivery lead-time while considering an M/M/1 system. In M/M/1, all customersare accepted.This can lead to longer lead times in the queue. Firms can react by quoting longer lead timesin order to cope with this situation. However, this leads to lower demand and revenue. Starting from this observation, we investigate in this paper whether a customer rejection policy can be more beneficial for the firm than an all-customers' acceptance policy. Indeed, our idea is based on the fact that rejecting some customers might help to quote shorter lead time for the accepted customers, which might lead to higher demand and profit. We model this rejection policy based on an M/M/1/K system. We analytically determine the optimal firm's policy (optimal price and quoted lead time) in case of M/M/1/1 system. Then, we compare the optimal firm's profit under M/M/1/1 with the optimal profit obtained by M/M/1. Two situations are considered: a system without holding and penalty costs and a system where these costs are included.*

Keywords:*Lead-time quotation, Pricing, M/M/1/K, M/M/1.*

1. Introduction

The delivery lead-time, which represents the elapsed time between the placement of the order by the customer and the receipt of this order, has became a factor of competitiveness for companies and an important purchase criterion for manycustomers.Geary and Zonnenberg (2000) reported that top performers among 110 organizations conducted initiatives not only to reduce costs and maintain reliability, but also to improve delivery speed and flexibility. Baker et al.(2001) found that less than 10% of end consumers and less than 30% of corporate customers base their purchasing decisions on price only; for a substantial majority of purchasers both price and delivery lead time are crucial factors that determine their purchase decisions. Thus, in order to increase their profit, companies must not focus only on price but also need to quote the right delivery lead time to their customers. A short quoted lead time can lead to higher demand but canalso result in late delivery, which affects the firm's reputation for on-time delivery and deters future customers (Slotnick, 2014). In addition, companies risk to lose markets if they are not capable of respecting the quoted delivery lead-times (Kapuscinski and Tayur, 2007). A long quoted lead time can reduce the risk of late delivery but leads to lower demand. This raises the following relevant question: What is the best lead time that must be quoted by a company when customers are not only sensitive to price but also to lead time? Some authors tried to answer this question while considering an M/M/1 system (Palaka at al., 1998, So and Song, 1998, and Pekgün et al., 2008). As stated by Gross et al. (2008); Kleinrock (1975); Thomopoulos (2012), one of the characteristics of M/M/1 is the infinite system capacity. Thus, the M/M/1 accepts all customers, which can lead to longer lead times in the queue. In order to cope with this situation, firms can react by quoting longer lead times in order to maintain the desired service level. However, this leads to lower demand and revenue.Starting from this observation, we investigate in this paper whether a customer rejection policy can be more beneficial for the firm than an all-customers acceptance policy. Indeed, our idea is based on the fact that rejecting some customers might help to quote shorter lead time for the accepted customers, which might lead to higher demand and profit. We model this rejection policy based on an M/M/1/K system. We analytically determine the optimal firm's policy (optimal price and quoted lead time) in case of M/M/1/1 system. Then, we compare the optimalfirm's profit under M/M/1/1 with the optimal profit obtained by M/M/1. Two situations are considered: a system without holding and penalty costs and a system where these costs are included.

The rest of this paper isorganized as follows. A literature review on M/M/1 systems with leadtime-dependent demand is presented in the next section. Then, we develop in section 3 the formulation of the M/M/1/K system with price-and leadtime-sensitive demand.In Section 4, we analytically solve the M/M/1/K system for K=1 without holding and penalty cost and compare the results to those obtained with M/M/1. We dedicate section 5 to the case with holding and penalty cost.We finally conclude and give future work directions.

2. Literature Review

In the past 2 decades, a considerable number of researchers in economics and operations management have studied: price-, rebate-, space-, quality, and advertising-dependent demand. Huang et al.(2013) suggested that there may exist further research opportunities for using lead time–dependent demand as they had found only few publications belonging to lead time–dependent categories.

A highlighted by Huang et al. (2013), the M/M/1 model is widely used in the literature to incorporate a price- and lead-time sensitive demand in a make-to-order system In what follows, we review such models.

Palaka et al. (1998)studied thelead-time setting, pricing decisions, and capacity utilization of a profit maximizing firm that faces a linear price- and leadtime-sensitive demand. Costs related to congestion (holding cost) and late deliveries (penalty cost) are considered in Palaka et al.'s model. So and Song(1998) developed an analytical framework for a firm to understand the strong interrelationships among pricing, delivery time guarantees, demand, and the overall profitability of offering the services. The authors useda log linear function to model the demand as a function of price, delivery time, and delivery reliability level. Pekgün et al. (2008)studied centralization and decentralization of pricing and lead-time decisions of a Make-To-Order (MTO) firm, while using the same setting ofPalaka et al. (1998)for their decentralized model but without holding and penalty cost.
There are some other research that used the M/M/1 to model their MTO system in order to model the lead-time- and price-dependent demand(Ho and Zheng, 2004; Liu et al., 2007; Ray and Jewkes, 2004; Zhao et al., 2012).Ray and Jewkes(2004) conduct a research about costumer lead-time management with demand and price sensitive to lead-time. In their research, the price is sensitive to lead-time. Demand is modeled as linear function, so is the price sensitive lead-time. Ho and Zheng(2004), they model the demand in MNL model. They conduct research in single firm and competitive multi-firm.Ho and Zheng(2004) discuss about the competitive market that the demand sensitive to lead-time. They use game theory approach to show how a firms should react to the market and another firm's lead-time strategy. The demand is modeled as costumer utility (satisfactory). Liu et al.(2007) conduct research about the decentralized supply chain in single firm. They mainly use Stackelberg game with supplier and retailer to quote the lead time and price. Zhao et al. (2012) discuss about the lead time and pricing decision for two types of costumer: lead-time sensitive or price sensitive. They use two policies which are uniform or differentiated model. Zhao et al. (2012) model the demand in the willingness-to-pay model for single firm problem.

3. Proposed Model (The M/M/1/K)

As in Palaka et al.(1998), we consider a make-to-order firm where the capacity is assumed to be constants while price, quoted lead-time and demand are decision variables. Customers are served in first-come, first-served basis (FCFS). The arrival processes are assumed to be Poisson process. The processing time of customersin the system is assumed to be exponentially distributed. Contrary to the assumptions of M/M/1 model where all customers are accepted (as in Palaka et al., 1998 and Pekgun et al., 2008),we reject clients when there is already K clients in the systemThus, we model the system as an M/M/1/K model.

Similarly to Liu et al. (2007); Palaka et al. (1998); and Pekgün et al., (2008), the demand is assumed to be a linear decreasing function in price and quoted lead-time.

$$\Lambda(p,l) = a - b_1 p - b_2 l, \tag{1}$$

where:

p = price of the good/service set by the firm,
l = quoted lead-time,
$\Lambda(p,l)$ = expected demand for the good/service withprice p and quoted lead-time l,
a = market potential,
b_1 = price sensitivity of demand,
b_2 = lead-time sensitivity of demand,

Since the demand is downward sloping in both price and quoted lead-time, b_1 and b_2 are restricted to be non-negative

According to Palaka et al. (1998), this linear demand function is tractable and has several desirable properties. For instance, with such a linear demand, the price elasticity is increasing in both price and quoted lead-time. Customers would be more sensitive to long lead-times when they are paying more for the goods or service. Similarly, customers would be more sensitive to high prices when they also have longer waiting times.

In order to prevent the firms from quoting unrealistically short lead-times, we assume that the firm maintains a certain minimum service level. The service level is defined as the probability of meeting the quoted lead-time ($P(W \leq l) \geq s$).

Since we assume an M/M/l/K queueing system with mean service rate, μ, and mean arrival rate (or, demand), λ, throughput rate (effective demand), $\bar{\lambda}$,the expected number of customers in the system is given by L_s (see eq. (2)), and the actual lead-time (time in the system) is exponentially distributed with mean $L_s/\bar{\lambda}$ (see eq. (3)). The probability that the firm is able to meet the quoted lead-time l($P(W\leq l)$)and the probability that a job is late($P(W>l)$)is given in eq. (4). These equations are based on Gross et al.(2008).

$$L_s = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} \text{ with } \rho = \frac{\lambda}{\mu} \tag{2}$$

$$W = \frac{L_s}{\bar{\lambda}} \text{ with } \bar{\lambda} = \lambda(1-P_K) \text{ and } P_K = \frac{1-\rho}{1-\rho^{K+1}}\rho^K \text{ if } \rho \neq 1 \text{ or } P_K = \frac{1}{K+1} \text{ if } \rho = 1 \tag{3}$$

$$P(W \leq l) = 1 - \sum_{k=0}^{K-1}\left(\sum_{i=0}^{k}\frac{(\mu\cdot l)^i}{i!}e^{-\mu\cdot l}\right)\frac{P_k}{1-P_K} \text{ and } P(W > l) = \sum_{k=0}^{K-1}\left(\sum_{i=0}^{k}\frac{(\mu\cdot l)^i}{i!}e^{-\mu\cdot l}\right)\frac{P_k}{1-P_K} \tag{4}$$

The objective of the firm is to maximize its revenue, which includes the following three main parts:

(1) **Expected revenue (net of direct costs)**is represented by $\bar{\lambda}(p-m)$, where $m$is the unit direct variable cost.
(2) **Total Congestion costs**is expressed as the mean number of jobs in the system multiplied by the unit holding cost ($L_s \times F$).This cost typically represents the in-process inventory holding cost.
(3) **Total Lateness penalty cost** is expressed as (penalty per job per unit lateness) × (number of overdue clients) × (expected lateness given that a job is late). The number of overdue clients is equal to: (throughput rate) × (probability that a job is late). The penalty cost per job per unit lateness (denoted by c) reflects the direct compensation paid to customers for not meeting the quoted lead-time. Mathematically, this total Lateness penalty cost is given by $\left(c \times \bar{\lambda} \times P(W \geq l) \times W\right)$.

as the firm's optimization problem can be modeled as follows:

(P0) $$\underset{l,p,\lambda}{\text{Maximize}} \quad \bar{\lambda}(p-m) - (L_s \times F) - \left(c \times \bar{\lambda} \times P(W \geq l) \times W\right) \tag{5}$$

$$\text{Subject to } \lambda \leq a - b_1 p - b_2 l \tag{6}$$

$$1-\sum_{k=0}^{K-1}\left(\sum_{i=0}^{k}\frac{(\mu\cdot l)^i}{i!}e^{-\mu\cdot l}\right)\frac{P_k}{1-P_K}\geq s \tag{7}$$

$$\rho=\frac{\lambda}{\mu} \tag{8}$$

$$P_K=\frac{1-\rho}{1-\rho^{K+1}}\rho^K \text{ if } \rho\neq 1 \text{ and } P_K=\frac{1}{K+1} \text{ if } \rho=1 \tag{9}$$

$$\bar{\lambda}=\lambda(1-P_K) \tag{10}$$

$$\lambda, p, l, \bar{\lambda}\geq 0 \tag{11}$$

where,

Decision Variables			Parameters		
p	=	Price of the good/service set by the firm,	a	=	market potential)
l	=	Quoted lead-time,	b_1	=	Price sensitivity of demand,
λ	=	Mean arrival rate (demand),	b_2	=	Lead-time sensitivity of demand,
			μ	=	Mean service rate (Production capacity),
			m	=	Unit direct variable cost,
			s	=	Service level defined by company,
			P_K	=	Probability of rejectedcustomer,
			K	=	System capacity.

In this formulation, constraint (7) imposes that the mean demand (λ) does not exceed the demand obtained with price (p) and quoted lead-time (l). Constraint (8) expresses the service level constraint. Constraint (10) calculates the probability ofrejecting customer. Constraint (11) is the number of customers that are served and exit the system. Constraint (12) is the non-negativity constraint.

Solving the general problem analytically seemsdifficult. Hence, in the section 4 we start by modeling the system with K = 1 and without penalty and holding costs. Then in the section 5, we will investigate whether the new model can be better or not by comparing with existing M/M/1 models. In the section 6, we will add the penalty cost and holding cost in modeling the system with K = 1 and do the comparison of the new model (with penalty and holding cost) with the existing M/M/1 in section 7.

4. M/M/1/1 Without Penalty Cost and Holding Cost

In this section, we model the system with K = 1. Since the unit holding cost and the penalty cost are removed, the objective function will be only to maximize the expected revenue. The Probability $(W>l)=e^{-\mu l}$ (proven in Appendix A). Hence, service level constraint can be written as $1-e^{-\mu l}\geq s$. The formulation of this problem becomes:

(P1) $$\underset{\lambda,l,p,\bar{\lambda}}{\text{Maximize}}\ \bar{\lambda}(p-m) \tag{12}$$

$$\text{Subject to } \lambda\leq a-b_1p-b_2l \tag{13}$$

$$1-e^{-\mu l}\geq s \tag{14}$$

$$\rho=\frac{\lambda}{\mu} \tag{15}$$

$$P_1=\frac{\rho}{1+\rho} \tag{16}$$

$$\bar{\lambda}=\lambda(1-P_1) \tag{17}$$

$$\lambda, p, l, \bar{\lambda} \geq 0 \tag{18}$$

Eq. (15) can be rewritten as $\mu l \geq \ln(1/(1-s))$. Then, by integrating the equality constraint (eq. (16-19)) into the objective function, we can simplify the formulation as:

$$\text{(P1')} \underset{\lambda,l,p}{\text{Maximize}} \left(\frac{\lambda\mu}{\mu+\lambda}\right)\times(p-m) \tag{19}$$

$$\text{Subject to } \lambda \leq a - b_1 p - b_2 l \tag{20}$$

$$\mu l \geq \ln(1/(1-s)) \tag{21}$$

$$\lambda, p, l \geq 0 \tag{22}$$

Using the new formulation (P1') we will solve it in single variable optimization. We start from the demand constraint., the demand constraint (eq. (21)) is binding at optimality in our new problem (see proof in appendix B), thus:

$$\lambda = a - b_1 p - b_2 l \Leftrightarrow p = \frac{a - b_2 l - \lambda}{b_1} \tag{23}$$

the service level constraint (eq.(22)) is also binding at optimality in our new problem (see proof in appendix C). Hence, $l = \ln(1/(1-s))/\mu$. We denote $\ln(1/(1-s))$ by z, and get:

$$l = \frac{z}{\mu} \tag{24}$$

Substitute $l = z/\mu$ into eq. (25), we obtain

$$p = \frac{a\mu - b_2 z - \lambda\mu}{\mu b_1} \tag{25}$$

Substituting eq. (26) into the objective function, we get a new formulation with single variable (λ) as:

$$\text{(P1'')} \underset{\lambda}{\text{Maximize}} \left(\frac{\lambda a\mu - \lambda b_2 z - \lambda^2\mu - \lambda m\mu b_1}{\mu b_1 + \lambda b_1}\right) \tag{26}$$

Proposition 1. There exists $\lambda \geq 0$ in problem P1" such as $f(\lambda) \geq 0$ iff $(a\mu - b_2 z)/\mu b_1 \geq m$ (proof in appendix D).

To find the solution (optimum point) from eq. (27), first we need to find stationary point. Hence we have to find the necessary condition where the first order condition is equal to 0.

$$\frac{d}{d\lambda}\left(\frac{\lambda a\mu - \lambda b_2 z - \lambda^2\mu - \lambda m\mu b_1}{\mu b_1 + \lambda b_1}\right) = 0 \Leftrightarrow \frac{\mu b_1\left(a\mu - b_2 z - m\mu b_1 - 2\lambda\mu - \lambda^2\right)}{(\mu b_1 + \lambda b_1)^2} = 0 \tag{27}$$

numerator of eq. (28)should be equal to zero.

$$\mu b_1\left(a\mu - b_2 z - m\mu b_1 - 2\lambda\mu - \lambda^2\right) = 0 \Leftrightarrow a\mu - b_2 z - m\mu b_1 - 2\lambda\mu - \lambda^2 = 0 \tag{28}$$

The discriminant (Δ) of eq. (29) should be greater or equal to 0 to have roots.

$$\Delta \quad = 4\mu^2 + 4a\mu - 4b_2 z - 4m\mu b_1$$

It is proven in appendix E that ($\Delta \geq 0$). Hence eq. (29) has two real roots which are:

$$\lambda_1 = -\mu - \sqrt{\mu^2 + a\mu - b_2 z - m\mu b_1} \text{ and } \lambda_2 = -\mu + \sqrt{\mu^2 + a\mu - b_2 z - m\mu b_1} \tag{29}$$

The $\lambda_1$is negative value. Hence, there is only one feasible stationary point λ_2.Under proposition 1, λ_2 is positive(see appendix F).In appendix G, it is proven that objective function is concave in $\lambda, l, p \geq 0$.Hence, the λ_2 is also the optimum point that can be obtained as in proposition 2.

Proposition 2. For problem P1:

1. The optimum demand is $\lambda^* = -\mu + \sqrt{\mu^2 + a\mu - b_2 z - m\mu b_1}$ with $z = \ln(1/(1-s))$,

2. The optimum lead-time is $l^* = \ln(1/(1-s))/\mu$,
3. The optimum price is $p^* = (a - b_2 l^* - \lambda^*)/b_1$, and
4. The optimum profit = $(\lambda^* \mu/(\mu + \lambda^*)) \times (p^* - m)$.

5. Comparison with Existing M/M/1 Without Penalty Cost and Holding Cost

In this section, we compare our modeling (M/M/1/1) with the existing M/M/1 taken from Pekgün et al. (2008)as they don't consider holding and penalty costs. We use a base case with parameters: b_2 = 6, b_1 = 4; μ = 10; s = 0.95; m = 5, We vary the market potential (a) and other parameters.For each pair of value, for example (a, b_2), it gives us a relative value of the optimal profit obtained from the M/M/1 and M/M/1/1. This relative value followseq.(30):

$$\frac{\text{Profit}^{\text{M/M/1/1}} - \text{Profit}^{\text{M/M/1}}}{\text{Profit}^{\text{M/M/1}}} \times 100\% \tag{30}$$

First, we compare the result based on the variation of market potential (a) and lead-time sensitivity (b_2). A positive value means that the approach with rejections (M/M/1/1 model) is better than the approach without rejections (M/M/1 model).As seen in table 1, the M/M/1 can be better than M/M/1 in particularly when the market potential is small and lead-time sensitivity is high.

Table 1- Comparison based on a and b_2

b_2	M/M/1 vs M/M/1/1					
20	-	40.87%	17.94%	8.29%	3.42%	0.66%
19	-	37.10%	15.27%	6.12%	1.57%	-0.99%
18	-	33.39%	12.61%	3.96%	-0.30%	-2.65%
17	-	29.73%	9.96%	1.79%	-2.18%	-4.33%
16	-	26.13%	7.31%	-0.40%	-4.07%	-6.03%
15	-	22.57%	4.66%	-2.59%	-5.99%	-7.74%
14	-	19.05%	2.01%	-4.80%	-7.92%	-9.48%
13	-	15.57%	-0.65%	-7.03%	-9.87%	-11.24%
12	-	12.11%	-3.32%	-9.29%	-11.86%	-13.04%
11	-	8.68%	-6.02%	-11.57%	-13.88%	-14.87%
10	-	5.26%	-8.74%	-13.90%	-15.94%	-16.74%
9	-	1.85%	-11.49%	-16.28%	-18.06%	-18.67%
8	-	-1.56%	-14.30%	-18.71%	-20.23%	-20.65%
7	-	-4.98%	-17.17%	-21.22%	-22.48%	-22.71%
6	-	-8.43%	-20.13%	-23.83%	-24.83%	-24.86%
5	-	-11.92%	-23.20%	-26.56%	-27.30%	-27.14%
	20	30	40	50	60	70
	a					

Note:

-	Problem is infeasible for both M/M/1 and M/M/1/1

Second, we compare based on the variation of market potential (a) and price sensitivity (b_1).When the costumer become more sensitive to price, the M/M/1 will give us worse performance than M/M/1/1. This happen because as we increase the price sensitivity, we will decrease the demand. In table 2, it can be seen that the M/M/1 is better policy when the price sensitivity is low.

Table 2 - Comparison based on a and b_1

b_1	M/M/1 vs M/M/1/1					
14	-	-	-	-	-	-
13	-	-	-	-	-	4.54%
12	-	-	-	-	-	-8.43%
11	-	-	-	-	4.54%	-15.87%
10	-	-	-	-	-8.43%	-20.13%
9	-	-	-	4.54%	-15.87%	-22.51%
8	-	-	-	-8.43%	-20.13%	-23.83%
7	-	-	4.54%	-15.87%	-22.51%	-24.51%
6	-	-	-8.43%	-20.13%	-23.83%	-24.83%
5	-	4.54%	-15.87%	-22.51%	-24.51%	-24.91%
4	-	-8.43%	-20.13%	-23.83%	-24.83%	-24.86%
3	4.54%	-15.87%	-22.51%	-24.51%	-24.91%	-24.72%
2	-8.43%	-20.13%	-23.83%	-24.83%	-24.86%	-24.53%
1	-15.87%	-22.51%	-24.51%	-24.91%	-24.72%	-24.30%
	20	30	40	50	60	70
	a					

Third, we compare based on the variation of market potential (a) and service level (s) (see Table 3). To maintain the service level, the firms can quote any lead-time in M/M/1 as there is no penalty in the overdue clients. However, if the service level is set to be close to 1, it will cause the firms quoting a very long lead-time. This will inflict their profit in the small market level as the demand will decrease if the lead-time ridiculously long.

Table 3- Comparison based on a and s

s	M/M/1 vs M/M/1/1					
0.999	-	18,48%	1,57%	-5,17%	-8,24%	-9,77%
0.995	-	7,35%	-7,07%	-12,47%	-14,68%	-15,59%
0.99	-	2.61%	-10.87%	-15.74%	-17.58%	-18.23%
0.98	-	-2.12%	-14.77%	-19.12%	-20.59%	-20.98%
0.97	-	-4.90%	-17.10%	-21.16%	-22.43%	-22.66%
0.96	-	-6.89%	-18.79%	-22.65%	-23.76%	-23.89%
0.95	-	-8.43%	-20.13%	-23.83%	-24.83%	-24.86%
0.94	-	-9.70%	-21.23%	-24.81%	-25.71%	-25.68%
	20	30	40	50	60	70
	a					

Fourth, we compare based on the variation of market potential (a) and production capacity (μ) (see Table 4).If the company has a small production capacity, it will inflict the service time. The firms will take longer time to serve the clients if they have small production capacity. This cause the lead-time became longer. Hence, the policy to reject some of the clients is better in this situation.

Table 4-Comparison based on a and μ

μ	M/M/1 vs M/M/1/1					
10	-	-8.43%	-20.13%	-23.83%	-24.83%	-24.86%
9	-	-5.72%	-17.35%	-20.83%	-21.77%	-21.82%

μ	M/M/1 vs M/M/1/1					
8	-	-1.70%	-13.51%	-16.94%	-17.94%	-18.09%
7	-	4.32%	-8.15%	-11.82%	-13.05%	-13.42%
6	-	13.62%	-0.53%	-4.90%	-6.63%	-7.40%
5	-	28.77%	10.79%	4.86%	2.15%	0.67%
4	-	56.01%	28.86%	19.59%	14.97%	12.21%
3	-	116.15%	61.96%	44.62%	35.87%	30.51%
2	-	359.61%	144.13%	98.95%	78.23%	66.03%
1	-	-	962.02%	359.37%	240.29%	186.72%
	20	30	40	50	60	70
	a					

It is shown that the use of the client rejection policy can be better in some cases even when the penalty and holding cost are removed. Therefore, we expect that when penalty and holding cost are added, the rejection policy will even perform better. We investigate this problem in the next section

6. M/M/1/1 With Penalty Cost and Holding Cost

With the addition of penalty cost and holding cost, the objective will be based on three parts: expected revenue, total congestion costs, and total lateness penalty costs. The formulation of these three parts have been presented earlier.The service level constraint inthis case is similar to the previous case: $1-e^{-\mu l} \geq s$ as the Probability $(W > l) = e^{-\mu l}$ (proven in Appendix A). Thus the formulation of this problem is:

(P2) $$\underset{\lambda, p, l, \bar{\lambda}}{Maximize} \quad \bar{\lambda}(p-m) - \frac{F\lambda}{\mu+\lambda} - \frac{c\bar{\lambda}}{\mu} e^{-\mu l} \qquad (31)$$

Subject to $$\lambda \leq a - b_1 p - b_2 l \qquad (32)$$

$$1 - e^{-\mu l} \geq s \qquad (33)$$

$$\rho = \frac{\lambda}{\mu} \qquad (34)$$

$$P_1 = \frac{\rho}{1+\rho} \qquad (35)$$

$$\bar{\lambda} = \lambda(1 - P_1) \qquad (36)$$

$$\bar{\lambda}, \lambda, l, p \geq 0 \qquad (37)$$

Integrating the equality constraints (eq. (34 – 37)) to the objective function and rewrite $1-e^{-\mu l} \geq s$ to $\mu l \geq \ln(1/(1-s))$, we get a new problem as:

(P2') $$\underset{\lambda, l, p}{\text{Maximize}} \quad \frac{\lambda\left(p\mu - m\mu - F - ce^{-\mu l}\right)}{\mu+\lambda} \qquad (38)$$

Subject to $$\lambda \leq a - b_1 p - b_2 l \qquad (39)$$

$$\mu l \geq \ln(1/(1-s)) \qquad (40)$$

$$\lambda, p, l \geq 0 \qquad (41)$$

The demand constraint (eq. (39)) is absolutely binding at optimality in our case (see demonstration in appendix H). Thusby removing the price (p), the formulation became:

(P2") $$\underset{\lambda, l}{Maximize} \quad \frac{\lambda\left[\left(\mu(a - b_2 l - \lambda)/b_1\right) - m\mu - F - ce^{-\mu l}\right]}{\mu+\lambda} \qquad (42)$$

Subject to $\mu l \geq \ln(1/(1-s))$ (43)

$l, p \geq 0$ (44)

There is also a feasibility condition of this problem as explained in proposition 3.

Proposition 3. The problem P2" is feasible (λ,l ≥ 0 and profit ≥ 0) iff $\frac{a-b_2 l}{b_1} \geq m$ and $a\mu - \mu b_2 l - \mu m b_1 - F b_1 - c b_1 e^{-\mu l} \geq 0$ (proof in appendix J).

Unlike the first case where the penalty and holding costs are not considered, the service level constraint is not necessarily binding in this case, which complicates the solving approach. Indeed, for large values of c, the actual service level has to be very high (close to 1) to avoid a high penalty cost. This indicates that the actual service level can be greater than the imposed service level (s).The detailed proof is given in appendix I. We now present the main steps to get the optimal solution given in proposition 4.

To solve the problem, we apply the Lagrangian multiplier method. The stationary points of problem (P2") must satisfy:

$$\frac{\partial L}{\partial \lambda} = 0, \frac{\partial L}{\partial l} = 0, \gamma \frac{\partial L}{\partial \gamma} = 0, \gamma \geq 0 \text{ and } \mu l \geq \ln\left(\frac{1}{1-s}\right)$$

where,

$$L(\lambda, l, \gamma) = \frac{\lambda\left[\left(\mu(a - b_2 l - \lambda)/b_1\right) - m\mu - F - ce^{-\mu l}\right]}{\mu + \lambda} + \gamma\left\{\mu l - \ln\left(\frac{1}{1-s}\right)\right\} \quad (45)$$

As we have already explained, we have two situations:

(1) the service level constraint (39) is non-binding: $s < s_c$,

(2) the service level constraint (39) is binding: $s \geq s_c$,

where the critical value for the service level (s_c) equals to $1 - b_2/b_1 c$. With this critical service level (s_c), the lead-time can be found based on the two mutually exclusive cases of the service level:

$$1 - e^{-\mu l} = Max\{s_c, s\} \Leftrightarrow l^* = \frac{1}{\mu}\ln\left(\frac{1}{1 - Max\{s_c, s\}}\right)$$

$$\Leftrightarrow l^* = \frac{1}{\mu}\ln x \text{ where, } x = Max\{1/(1-s), b_1 c/b_2\}$$

Next, to find the demand (λ), we derive eq. (46) based λ, we can find the demand of this problem.

$$\frac{\partial L}{\partial \lambda} = 0 \quad \Leftrightarrow \quad \frac{\mu\left(a\mu - \mu b_2 l - 2\mu\lambda - \mu m b_1 - F b_1 - b_1 c e^{-\mu l} - \lambda^2\right)}{b_1(\mu + \lambda)^2} = 0 \quad (46)$$

Numerator of eq. (47) should be equal to zero.

$$\mu\left(a\mu - \mu b_2 l - 2\mu\lambda - \mu m b_1 - F b_1 - b_1 c e^{-\mu l} - \lambda^2\right) = 0 \quad (47)$$

The discriminant (Δ) of eq. (54) should be greater than 0 to have real roots.

$$\Delta = 4\mu^2 + 4a\mu - 4\mu b_2 l - 4\mu m b_1 - 4F b_1 - 4 b_1 c e^{-\mu l} \quad (48)$$

In appendix K, it is proven that the discriminant (Δ) is bigger than zero. Hence eq. (50) has two roots which are:

$$\lambda_1 = -\mu - \sqrt{\mu^2 + a\mu - \mu b_2 l - \mu b_1 m - F b_1 - b_1 c e^{-\mu l}} \quad \text{and}$$

$$\lambda_2 = -\mu + \sqrt{\mu^2 + a\mu - \mu b_2 l - \mu b_1 m - F b_1 - b_1 c e^{-\mu l}} \quad (49)$$

$\lambda_1$is negative. Under proposition 3 λ_2 hasa value greater than zero(see proof in appendix L). It is proven in appendix M that the objective is concave function in $\lambda, l, p \geq 0$. Thus, the lead-time (l^*) and demand (λ_2) provide the optimal solution.. As a summary, the optimum point of this problem can be found based on the proposition 4.

Proposition 4. For problem P2:

1. The optimum lead-time $l^* = \ln x/\mu$ with $x = Max\{1/(1-s), b_1 c/b_2\}$.
2. The optimum demand can be found by using equation $\lambda^* = -\mu + \sqrt{\mu^2 + a\mu - \mu b_2 l^* - \mu b_1 m - Fb_1 - b_1 c e^{-\mu l^*}}$,
3. the optimum price is $p^* = \left(a - b_2 l^* - \lambda^*\right)/b_1$.
4. The optimum profit $= \lambda^*\left(p^*\mu - m\mu - F - ce^{-\mu l^*}\right)/\left(\mu + \lambda^*\right)$.

Next based on the result found in this section, we will investigate this rejection policy with penalty and holding cost by comparing with the existing M/M/1 model.

7. Comparison with Existing M/M/1 With Penalty Cost and Holding Cost

In this section, we compare our model (M/M/1/1) with the existing M/M/1 taken from Palaka et al.(1998).We vary the market potential (*a*) and other parameters.For each pair of value, for example (*a*, b_2), we calculate the relative gain obtained by using M/M/1/1 instead of M/M/1. This relative value follows equation (26). We use the same base case as in the previous comparison with additions of $F = 2$ and $c = 10$.

First, we compare the result based on the variation of market potential (*a*) and lead-time sensitivity (b_2). As expected, there are more cases where M/M/1/1 is better than M/M/1 (see Table 5). In the M/M/1 the holding cost can be very high because all clients are accepted. Hence, as seen in table 5, the M/M/1/1 is better when dealing with customers that are very sensitive to lead-time with low the market potential.

Table 5- Comparison based on a and b_2

b_2	M/M/1 vs M/M/1/1					
20	-	53.96%	26.95%	15.50%	9.58%	6.11%
19	-	49.95%	24.23%	13.33%	7.74%	4.49%
18	-	46.02%	21.53%	11.17%	5.90%	2.86%
17	-	42.16%	18.84%	9.02%	4.05%	1.22%
16	-	38.37%	16.17%	6.86%	2.19%	-0.43%
15	-	34.64%	13.51%	4.69%	0.33%	-2.09%
14	-	30.96%	10.85%	2.52%	-1.54%	-3.76%
13	-	27.34%	8.20%	0.34%	-3.43%	-5.45%
12	-	23.77%	5.56%	-1.85%	-5.34%	-7.16%
11	-	20.24%	2.91%	-4.05%	-7.26%	-8.89%
10	-	16.74%	0.25%	-6.27%	-9.21%	-10.65%
9	-	13.28%	-2.42%	-8.52%	-11.19%	-12.43%
8	-	9.84%	-5.10%	-10.80%	-13.19%	-14.25%
7	-	6.42%	-7.81%	-13.11%	-15.24%	-16.10%
6	-	3.01%	-10.56%	-15.47%	-17.33%	-18.01%
5	-	-0.40%	-13.35%	-17.88%	-19.49%	-19.97%
	20	30	40	50	60	70
	a					

Second, we compare based on the variation of market potential (*a*) and price sensitivity (*b_1*) (see Table 6). In M/M/1, the lead-time can become very long because we accept all clients. Not to mention the penalty cost for overdue clients, the profit will be worse with high value of price sensitivity of clients. Hence, for clients that are very sensitive to price, the M/M/1/1 could be better policy.

Table 6- Comparison based on a and b_1

b_1	M/M/1 vs M/M/1/1					
13	-	-	-	-	-	-
12	-	-	-	-	-	26.18%
11	-	-	-	-	42.97%	11.99%
10	-	-	-	-	20.26%	2.93%
9	-	-	-	35.50%	7.00%	-3.27%
8	-	-	-	14.45%	-1.51%	-7.76%
7	-	-	28.29%	2.01%	-7.34%	-11.16%
6	-	-	8.70%	-6.00%	-11.56%	-13.87%
5	-	21.30%	-2.99%	-11.49%	-14.77%	-16.10%
4	-	3.01%	-10.56%	-15.47%	-17.33%	-18.01%
3	14.50%	-8.05%	-15.75%	-18.51%	-19.47%	-19.70%
2	-2.69%	-15.24%	-19.53%	-20.96%	-21.32%	-21.23%
1	-13.22%	-20.20%	-22.45%	-23.04%	-22.98%	-22.67%
	20	30	40	50	60	70
	a					

Third, we compare based on the variation of market potential (*a*) and service level (*s*) (see Table 7).If the firms set the service level close to 1, it will cause very long lead-time in M/M/1 model. The long lead-time will cause high holding cost which affect the demand and ruin the profit. Thus, rejecting some clients could be an alternative to keep the high profit. The M/M/1/1could be the solution in this type of case.

Table 7- Comparison based on a and s

s	M/M/1 vs M/M/1/1					
0.999	-	27,99%	8,68%	0,74%	-3,09%	-5,15%
0.995	-	16,78%	0,28%	-6,25%	-9,19%	-10,63%
0.99	-	12.21%	-3.25%	-9.22%	-11.81%	-12.99%
0.98	-	7.90%	-6.63%	-12.10%	-14.35%	-15.29%
0.97	-	5.58%	-8.48%	-13.68%	-15.75%	-16.56%
0.96	-	4.07%	-9.69%	-14.72%	-16.67%	-17.41%
0.95	-	3.01%	-10.56%	-15.47%	-17.33%	-18.01%
0.94	-	2.22%	-11.20%	-16.02%	-17.83%	-18.46%
	20	30	40	50	60	70
	a					

Fourth, we compare based on the variation of market potential (*a*) and production rate (μ) (see Table 8).The production capacity affects the service time of the firms. If the production capacity is small; the service time for each client will be very big (service time = $1/\mu$). This long service will cause clients to wait. This will have an impact on the high holding cost. Hence, for a firms with small production capacity, it is better to reject some costumer.

Table 8- Comparison based on a and μ

μ	M/M/1 vs M/M/1/1					
12	-	-3,15%	-16,06%	-21,16%	-23,10%	-23,74%

μ	M/M/1 vs M/M/1/1					
11	-	-0,55%	-13,70%	-18,65%	-20,51%	-21,14%
10	-	3,01%	-10,56%	-15,47%	-17,33%	-18,01%
9	-	7,95%	-6,36%	-11,41%	-13,40%	-14,21%
8	-	14,97%	-0,67%	-6,15%	-8,45%	-9,51%
7	-	25,30%	7,18%	0,79%	-2,08%	-3,57%
6	-	41,28%	18,43%	10,30%	6,40%	4,19%
5	-	68,27%	35,52%	24,01%	18,26%	14,82%
4	-	121,78%	64,17%	45,51%	36,16%	30,47%
3	-	277,90%	122,19%	84,63%	66,97%	56,50%
2	-	-	312,26%	183,40%	136,67%	111,73%
1	-	-	-	1496,14%	550,83%	363,51%
	20	30	40	50	60	70
	a					

Fifth, we compare based on the variation of market potential (*a*) and holding cost (*F*) (see Table 9). The holding cost affect the total profits. In M/M/1, there is a possibility that a client has a very long lead-time. This cause the company to stock their goods in a long time. This long holding period will cause the expensive holding cost. Thus, it decreases the profit. This condition explains that reject some clients could be a better policy.

Table 9- Comparison based on a and F

F	M/M/1 vs M/M/1/1					
11	-	46,09%	21,57%	11,21%	5,93%	2,89%
10	-	40,94%	17,99%	8,33%	3,46%	0,70%
9	-	35,92%	14,43%	5,44%	0,98%	-1,51%
8	-	31,00%	10,88%	2,55%	-1,52%	-3,74%
7	-	26,18%	7,35%	-0,36%	-4,05%	-6,00%
6	-	21,44%	3,81%	-3,30%	-6,60%	-8,30%
5	-	16,76%	0,27%	-6,26%	-9,20%	-10,63%
4	-	12,14%	-3,30%	-9,27%	-11,84%	-13,02%
3	-	7,56%	-6,90%	-12,33%	-14,55%	-15,48%
2	-	3,01%	-10,56%	-15,47%	-17,33%	-18,01%
1	-	-1,55%	-14,29%	-18,70%	-20,22%	-20,64%
0	-	-6,13%	-18,14%	-22,07%	-23,25%	-23,41%
	20	30	40	50	60	70
	a					

Sixth, we compare based on the variation of market potential (*a*) and penalty cost (*c*) (see Table 10). Because we set the service level to be high (95%), it also means that there are only 5% of overdue clients. In high market potential and all client's acceptance policy, the total profit is higher than the total penalty cost. Thus, it makes the penalty cost insignificant. However, it can be seen that there is a decrease in the superiority of M/M/1 to M/M/1/1 in function of the penalty cost. It can be concluded that in big market potential, there is a limit of increasing the penalty cost where the M/M/1 is superior. In small market potential, the 5% of overdue clients is significant. The penalty cost is affecting that the profit. Hence, reject some client is better as there isn't any penalty in rejecting clients.

Table 10- Comparison based on a and c

c	M/M/1 vs M/M/1/1					
10	-	3,01%	-10,56%	-15,47%	-17,33%	-18,01%
9	-	2,78%	-10,74%	-15,63%	-17,48%	-18,14%
8	-	2,55%	-10,93%	-15,79%	-17,62%	-18,27%
7	-	2,32%	-11,11%	-15,95%	-17,76%	-18,40%
6	-	2,09%	-11,30%	-16,11%	-17,90%	-18,53%
5	-	1,87%	-11,48%	-16,27%	-18,05%	-18,66%
4	-	1,64%	-11,67%	-16,43%	-18,19%	-18,79%
3	-	1,41%	-11,85%	-16,59%	-18,33%	-18,92%
2	-	1,18%	-12,04%	-16,75%	-18,48%	-19,05%
1	-	0,96%	-12,23%	-16,91%	-18,62%	-19,18%
0	-	0,73%	-12,41%	-17,07%	-18,76%	-19,31%
	20	30	40	50	60	70
	a					

8. Conclusion

In this paper, we provide the general model of M/M/1/K for case with and without holding and penalty cost. We solve both case analytically for K=1. We compare our M/M/1/1 model with the existing M/M/1 model taken from Pekgün et al.(2008) and Palaka et al.(1998). In the case where the penalty and holding cost aren't considered, the M/M/1/K is better when the customers are lead-time sensitive, the market potential is small, and the firm's production capacity (mean service rate) is small. In the case where the penalty and holding cost are considered, the M/M/1/K is better when the customers are lead-time sensitive, the market potential is small, the firm's production capacity (mean service rate) is small and the holding cost is high. We currently working in an extension of this research which is K > 1. Another possible extension is modeling the system in M/D/1.

9. Reference

Baker, W., Marn, M., Zawada, C., 2001. Price smarter on the Net. Harv. Bus. Rev. 79, 122–127.

Geary, S., Zonnenberg, J.P., 2000. What It Means To Be Best In Class. Supply Chain Manag. Rev. 4, 43–48.

Gross, D., Shortle, J.F., Thompson, J.M., Harris, C., 2008. Fundamentals of queueing theory, Fourth Edi. ed. John Wiley & Sons, Inc., Hoboken, New Jersey.

Ho, T.H., Zheng, Y.-S., 2004. Setting Customer Expectation in Service Delivery: An Integrated Marketing-Operations Perspective. Manage. Sci. 50, 479–488. doi:10.1287/mnsc.1040.0170

Huang, J., Leng, M., Parlar, M., 2013. Demand Functions in Decision Modeling : A Comprehensive Survey and Research Directions. Decis. Sci. 44, 557–609. doi:10.1111/deci.12021

Kapuscinski, R., Tayur, S., 2007. Reliable Due-Date Setting in a Capacitated MTO System with Two Customer Classes. Oper. Res. 55, 56–74. doi:10.1287/opre.1060.0339

Kleinrock, L., 1975. QUEUEING SYSTEMS, VOLUME 1: THEORY, VOLUME 1: ed, John Wiley & Sons, Inc. John Wiley & Sons, Inc., New York. doi:10.1002/net.3230060210

Liu, L., Parlar, M., Zhu, S.X., 2007. Pricing and Lead Time Decisions in Decentralized Supply Chains. Manage. Sci. 53, 713–725. doi:10.1287/mnsc.1060.0653

Palaka, K., Erlebacher, S., Kropp, D.H., 1998. Lead-time setting, capacity utilization, and pricing decisions under lead-time dependent demand. IIE Trans. 30, 151–163. doi:10.1080/07408179808966447

Pekgün, P., Griffin, P.M.P., Keskinocak, P., 2008. Coordination of marketing and production for price and leadtime decisions. IIE Trans. 40, 12–30. doi:10.1080/07408170701245346
Ray, S., Jewkes, E.M., 2004. Customer lead time management when both demand and price are lead time sensitive. Eur. J. Oper. Res. 153, 769–781. doi:10.1016/S0377-2217(02)00655-0
Slotnick, S.A., 2014. Lead-time quotation when customers are sensitive to reputation. Int. J. Prod. Res. 52, 713–726. doi:10.1080/00207543.2013.828176
So, K.C., Song, J.-S., 1998. Price, delivery time guarantees and capacity selection. Eur. J. Oper. Res. 111, 28–49. doi:10.1016/S0377-2217(97)00314-7
Sztrik, J., 2011. Basic Queueing Theory. University of Debrecen: Faculty of Informatics, Debrecen.
Thomopoulos, N.T., 2012. Fundamentals of Queuing Systems: Statistical Methods for Analyzing Queuing Models. Springer Science & Business Media.
Zhao, X., Stecke, K.E., Prasad, A., 2012. Lead Time and Price Quotation Mode Selection: Uniform or Differentiated? Prod. Oper. Manag. 21, 177–193. doi:10.1111/j.1937-5956.2011.01248.x

Appendix

A. The probability *(W>l)*)

In Palaka et al. (1998), So and Song(1998), and Pekgün et al. (2008), Service level constrain is defined as the probability that the firm is able to meet the quoted lead-time (P(W $\leq l$) $\geq$ s). And Sztrik(2011) formulate P(W $\leq l$) for the M/M/1/K as

$$P(W \leq l) = 1 - \sum_{k=0}^{K-1}\left(\sum_{i=0}^{k}\frac{(\mu \cdot l)^i}{i!}e^{-\mu \cdot l}\right)\frac{P_k}{1-P_K}$$

For the problem of the M/M/1/1, the K equal to 1 ($K=1$). Hence P(W $\leq l$) for the M/M/1/1

$$1 - \sum_{k=0}^{K-1}\left(\sum_{i=0}^{k}\frac{(\mu \cdot l)^i}{i!}e^{-\mu \cdot l}\right)\frac{P_k}{1-P_K} = 1 - \sum_{k=0}^{0}\left(\sum_{i=0}^{k}\frac{(\mu \cdot l)^i}{i!}e^{-\mu \cdot l}\right)\frac{P_k}{1-P_K}$$
$$= 1-\left(e^{-\mu l}\right)\frac{P_0}{1-P_1}$$

with,

$$P_0 = \frac{1-\rho}{1-\rho^{K+1}} = \frac{1}{1+\rho} \text{ and } P_K = P_1 = \frac{1-\rho}{1-\rho^{K+1}}\rho^K = \frac{\rho}{1+\rho}$$

Hence,

$$1-\left(e^{-\mu l}\right)\frac{P_0}{1-P_1} \quad = 1-e^{-\mu l}\left(\left(\frac{1}{(1+\rho)}\right)\times\left(\frac{(1+\rho)}{1}\right)\right)$$
$$= 1-e^{-\mu l}$$

Therefore, the service level constrain for the M/M/1/1 is $1-e^{-\mu l} \geq s$.

B. Proof demand constraint is binding

As in Palaka et al.(1998) and Pekgün et al.(2008), suppose that the optimal solution is given by price, p^*, quoted lead-time l^*, and demand rateλ^*, and that $\lambda^* < \Lambda(p^*, l^*.)$. Since the revenues are non-decreasing in *p*, one could increase the price to *p′* (while holding the demand rate and quoted lead-time constant) until $\lambda^* = \Lambda(p',1')$. This change will increase revenueswithout increasing the queueing costs, direct variablecosts, and lateness penalties. Therefore, (p^*, l^*, and λ^*) cannot be an optimal solution.To give a better explanation, we demonstrate it as:

$$\underset{\lambda,l,p,\bar{\lambda}}{\text{Maximize}} \quad \bar{\lambda}(p-m)$$

$$\text{Subject to} \quad \lambda \le a - b_1 p - b_2 l$$

$$1-\sum_{k=0}^{K-1}\left(\sum_{i=0}^{k}\frac{(\mu\cdot l)^i}{i!}e^{-\mu\cdot l}\right)\frac{P_k}{1-P_K} \ge s$$

Assume: $\lambda^* \le a - b_1 p^* - b_2 l^*$, we will have $F^* = \bar{\lambda}^*\left(p^* - m\right)$. If we increase the price to *p′*, then we will have $F' = \bar{\lambda}^*\left(p' - m\right)$ which is bigger than the F^*. Hence, there is a contradiction which is mean that the constraint is binding.

C. Proof Service level constraint is binding

The service constraint must be binding at optimality as in Palaka et al.(1998) and Pekgün et al.(2008). For the M/M/1/1, applying the Lagrangian multiplier method, we see that a stationary point to problem (P1) must satisfy:

$$\frac{\partial L}{\partial \lambda} = 0, \frac{\partial L}{\partial l} = 0, \gamma\frac{\partial L}{\partial \gamma} = 0, \gamma \ge 0 \text{ and } \mu l \ge \ln\left(\frac{1}{1-s}\right)$$

where,

$$L(\lambda,l,\gamma) = \left(\frac{\lambda\mu}{\mu+\lambda}\right)\times(p-m) + \gamma\left\{\mu l - \ln\left(\frac{1}{1-s}\right)\right\}$$

with $\lambda = a - b_1 p - b_2 l$, hence $p = \dfrac{a - b_2 l - \lambda}{b_1}$

$$L(\lambda,l,\gamma) = \left(\frac{\lambda\mu}{\mu+\lambda}\right)\times\left(\frac{a - b_2 l - \lambda - m b_1}{b_1}\right) + \gamma\left\{\mu l - \ln\left(\frac{1}{1-s}\right)\right\}$$

$$\frac{\partial L}{\partial l} = 0 \quad \Leftrightarrow \quad \gamma\mu - \frac{\mu\lambda b_2}{b_1(\mu+\lambda)} = 0$$

$$\gamma = \frac{\lambda b_2}{b_1(\mu+\lambda)}$$

$$\gamma\frac{\partial L}{\partial \gamma} = 0 \quad \Leftrightarrow \quad \gamma\left\{\mu l - \ln\left(\frac{1}{1-s}\right)\right\} = 0$$

We know that $\gamma = \lambda b_2 / \left(b_1(\mu+\lambda)\right)$ with $\lambda, \mu, b_1, b_2 \ge 0$. Hence, $\gamma \ne 0$. Thus, $\mu l - \ln\left(1/(1-s)\right) = 0$ which means service level constraint is binding.

D. Feasibility Condition and Non-Negative Objective Function

Note that: $\lambda = a - b_1 p - b_2 l$ from appendix B and $\lambda \ge 0$, hence:

$$a - b_1 p - b_2 l \ge 0 \Leftrightarrow \quad a - b_2 l \ge b_1 p$$

$$\frac{(a - b_2 l)}{b_1} \ge p$$

Note that: $1-e^{-(\mu)l}=s$ from appendix C

$$1-e^{-(\mu)l}=s \Leftrightarrow \mu l=\ln\left(1/(1-s)\right)$$
$$\Leftrightarrow l=\frac{\ln\left(1/(1-s)\right)}{\mu}$$
$$\Leftrightarrow l=\frac{z}{\mu}$$

Equation above implies that

$$\frac{(a-b_2 l)}{b_1}\ge p \Leftrightarrow \frac{a\mu-b_2 z}{\mu b_1}\ge p \Leftrightarrow \text{The feasibility condition}$$

Note that $p\ge m$ to have the positive objective function. It implies that:

$$\frac{a\mu-b_2 z}{\mu b_1}\ge m$$

E. Proof that Discriminant ≥ 0

$$\Delta=4\mu^2+4a\mu-4b_2 z-4m\mu b_1\ge 0$$
$$\Leftrightarrow \mu^2+a\mu-b_2 z-m\mu b_1\ge 0 \text{ with } z=\mu l$$
$$\Leftrightarrow \mu^2+a\mu-b_2\mu l-m\mu b_1\ge 0$$
$$\Leftrightarrow \mu+a-b_2 l-mb_1\ge 0$$

The $a-b_2 l-mb_1$ is equivalent to λ with $p=m$ and λ is non-negative. Hence, it is proven that $\mu+a-b_2 l-mb_1\ge 0$. Thus, the $\Delta\ge 0$

F. The $\lambda_2 \ge 0$ corresponds to the feasibility condition.

Suppose $\lambda_2\ge 0$

$$\Leftrightarrow -\mu+\sqrt{\mu^2+a\mu-b_2 z-m\mu b_1}\ge 0$$
$$\Leftrightarrow \sqrt{\mu^2+a\mu-b_2 z-m\mu b_1}\ge \mu$$
$$\Leftrightarrow \mu^2+a\mu-b_2 z-m\mu b_1\ge \mu^2$$
$$\Leftrightarrow a\mu-b_2 z-m\mu b_1\ge 0$$
$$\Leftrightarrow a\mu-b_2 z\ge m\mu b_1$$
$$\Leftrightarrow \frac{a\mu-b_2 z}{\mu b_1}\ge m$$

This result corresponds to the feasibility condition & non-negative objective function

G. Proof that the objective function is concave in M/M/1 without penalty and holding cost

This part of the appendix demonstrates that the objective function for the M/M/1/1 without penalty and holding cost is concave. The objective function of this problem is:

$$\underset{\lambda,l,p,\bar{\lambda}}{\text{Maximize}}\ \bar{\lambda}(p-m) \Leftrightarrow \underset{\lambda}{\text{Maximize}}\left(\frac{\lambda a\mu-\lambda b_2 z-\lambda^2\mu-\lambda m\mu b_1}{\mu b_1+\lambda b_1}\right)$$

Suppose that the $\lambda_1 = 0$, then the objective become:

$$\begin{aligned}\overline{\lambda}(p-m) &= \frac{\lambda a\mu - \lambda b_2 z - \lambda^2\mu - \lambda m\mu b_1}{\mu b_1 + \lambda b_1} \\ &= \frac{0}{\mu b_1} \\ &= 0\end{aligned}$$

If the $\lambda_2 = +\infty$, thenthe objective function is:

$$\begin{aligned}\overline{\lambda}(p-m) &= \frac{\lambda a\mu - \lambda b_2 z - \lambda^2\mu - \lambda m\mu b_1}{\mu b_1 + \lambda b_1} \\ &= \frac{-\infty}{+\infty} \\ &= -\infty\end{aligned}$$

It can be summarized as shown in table below

			The objective with λ^*
The objective with	$\lambda_1=0$	$F(\lambda_1) = 0$	$F(\lambda^*) \geq F(\lambda_1)$
	$\lambda_2= +\infty$	$F(\lambda_2) = -\infty$	$F(\lambda^*) \geq F(\lambda_2)$

This prove that the objective with $\lambda_1 = 0$ and $\lambda_2 = +\infty$ compared to λ^* is lower. Hence, the objective function is concave in $\lambda > 0$.

H. Proof demand constraint is binding (M/M/1/1 with holding and penalty cost)

As in Palaka et al.(1998) and Pekgün et al.(2008), suppose that the optimal solution is given by price, p^*, quoted lead-time l^*, and demand rate λ^*, and that $\lambda^* < \Lambda(p^*, 1^*.)$. Since the revenues are non-decreasing in p, one could increase the price to p' (while holding the demand rate and quoted lead-time constant) until $\lambda^* = \Lambda(p', 1')$. This change will increase revenues without increasing the queueing costs, direct variable costs, and lateness penalties. Therefore, (p^*, l^*, and λ^*) cannot be an optimal solution. To give a better explanation, we demonstrate it as:

$$\underset{\lambda, l, p, \overline{\lambda}}{\text{Maximize}} \quad \overline{\lambda}(p-m) - \frac{F\lambda}{\mu+\lambda} - \frac{c\overline{\lambda}}{\mu} e^{-\mu l}$$

$$\text{Subject to} \quad \lambda \leq a - b_1 p - b_2 l$$

$$1 - \sum_{k=0}^{K-1}\left(\sum_{i=0}^{k} \frac{(\mu \cdot l)^i}{i!} e^{-\mu \cdot l}\right)\frac{P_k}{1-P_K} \geq s$$

Assume: $\lambda^* \leq a - b_1 p^* - b_2 l^*$, we will have $F^* = \overline{\lambda}^*(p^* - m)$. If we increase the price to p', then we will have $F' = \overline{\lambda}^*(p' - m)$ which is bigger than the F^*. Hence, there is a contradiction which is mean that the constraint is binding.

I. Proof Service level constraint (M/M/1/1 with holding and penalty cost)

As in Palaka et al.(1998), applying the Lagrangian multiplier method, we see that a stationary point to problem (P2) must satisfy:

$$\frac{\partial L}{\partial \lambda}=0,\frac{\partial L}{\partial l}=0,\gamma\frac{\partial L}{\partial \gamma}=0,\gamma\geq 0 \text{ and } \mu l\geq \ln\left(\frac{1}{1-s}\right)$$

where,

$$L(\lambda,l,\gamma)=\frac{\lambda\left(p\mu-m\mu-F-ce^{-\mu l}\right)}{\mu+\lambda}+\gamma\left\{\mu l-\ln\left(\frac{1}{1-s}\right)\right\}$$

with $\lambda=a-b_1p-b_2l$, hence $p=\dfrac{a-b_2l-\lambda}{b_1}$

$$L(\lambda,l,\gamma)=\frac{\lambda\left[\left(\mu(a-b_2l-\lambda)/b_1\right)-m\mu-F-ce^{-\mu l}\right]}{\mu+\lambda}+\gamma\left\{\mu l-\ln\left(\frac{1}{1-s}\right)\right\}$$

$$\frac{\partial L}{\partial l}=0 \quad\Leftrightarrow\quad \gamma\mu-\frac{\lambda\left[(\mu b_2/b_1)-\mu ce^{-\mu l}\right]}{\mu+\lambda}=0$$

$$\gamma\mu(\mu+\lambda)-\mu(\lambda b_2/b_1)+\mu\lambda ce^{-\mu l}=0$$

$$\frac{\gamma(\mu+\lambda)}{\lambda c}-\frac{b_2}{cb_1}+e^{-\mu l}=0$$

$$1-\frac{b_2}{cb_1}+\frac{\gamma(\mu+\lambda)}{\lambda c}=1-e^{-\mu l}$$

Substitute $1-\dfrac{b_2}{cb_1}=s_c$, we have:

$$s_c+\frac{\gamma(\mu+\lambda)}{\lambda c}=1-e^{-\mu l} \tag{I1}$$

$$\gamma\frac{\partial L}{\partial \gamma}=0 \quad\Leftrightarrow\quad \gamma\left\{\mu l-\ln\left(\frac{1}{1-s}\right)\right\}=0$$

From $\gamma(\partial L/\partial\gamma)=0$, we know that there are two situations ($\gamma=0$ or $\mu l-\ln(1/(1-s))=0$). Suppose that $\gamma=0$, it implies that $\mu l-\ln(1/(1-s))>0$ which also imply that the service level constraint is non-binding. Since $\gamma=0$, equation (I1) implies that $s_c=1-e^{-\mu l}$. In addition, if the service level is non-binding, then $1-e^{-\mu l}>s$. It imply that $s_c>s$.

Next, suppose $\gamma\neq 0$, thus it implies $\mu l-\ln(1/(1-s))=0$ which also means that the service level constraint is binding. Hence $1-e^{-\mu l}=s$. Combining with the eq. (I1), it implies that: $s_c+\dfrac{\gamma(\mu+\lambda)}{\lambda c}=s$. This imply that $s_c\leq s$.

In the non-binding situation, the service level will be $s_c=1-e^{-\mu l}$ with $s_c>s$. And in the binding situation, the service level will be $1-e^{-\mu l}=s$ with $s_c\leq s$. Hence combining the both case, the service level will be: $1-e^{-\mu l}=Max\{s,s_c\}$

J. Feasibility Condition and Non-Negative Objective Function (M/M/1/1 with holding and penalty cost)

Note that: $\lambda=a-b_1p-b_2l$ from appendix B and $\lambda\geq 0$, hence:

$$a - b_1 p - b_2 l \geq 0 \Leftrightarrow a - b_2 l \geq b_1 p \Leftrightarrow \frac{(a - b_2 l)}{b_1} \geq p \qquad \text{(Condition J1)}$$

As $p \geq m$ and condition 1 imply that the feasibility condition of this problem is:
$\frac{(a - b_2 l)}{b_1} \geq m$, with $l^* = \frac{1}{\mu} \ln x$ and $x = Max\{1/(1-s), b_1 c/b_2\}$ as proven in section 6.

Then, to have the non-negative objective function which is correspond to have a feasibility condition:

$$\bar{\lambda}(p - m) - \frac{F\lambda}{\mu + \lambda} - \frac{c\bar{\lambda}}{\mu} e^{-\mu l} \geq 0 \Leftrightarrow \frac{\lambda\left(p\mu - m\mu - F - ce^{-\mu l}\right)}{\mu + \lambda} \geq 0$$

It is obvious that $\mu + \lambda \geq 0$ since the μ and λ is always bigger than zero. Hence, the numerator should be:

$$\lambda\left(p\mu - m\mu - F - ce^{-\mu l}\right) \geq 0$$

$$p\mu - m\mu - F - ce^{-\mu l} \geq 0$$

$$p \geq m + F/\mu + ce^{-\mu l}/\mu \qquad \text{(Condition J2)}$$

Combine this two p condition (J1 and J2), we have:

$\frac{(a - b_2 l)}{b_1} \geq p$ and $p \geq m + F/\mu + ce^{-\mu l}/\mu$

$$\Leftrightarrow \frac{(a - b_2 l)}{b_1} \geq m + F/\mu + ce^{-\mu l}/\mu$$

$$\Leftrightarrow \frac{(a - b_2 l)}{b_1} - m - F/\mu - ce^{-\mu l}/\mu \geq 0$$

$$\Leftrightarrow \mu(a - b_2 l) - \mu m b_1 - F b_1 - c b_1 e^{-\mu l} \geq 0$$

$$\Leftrightarrow a\mu - \mu b_2 l - \mu m b_1 - F b_1 - c b_1 e^{-\mu l} \geq 0$$

That equation should be bigger equal to zero to have a non-negative objective function.

K. Proof that Discriminant ≥ 0 (M/M/1/1 with holding and penalty cost)

$$\Delta = (-2\mu)^2 - 4(-1)\left(a\mu - \mu b_2 l - \mu m b_1 - F b_1 - b_1 c e^{-\mu l}\right) \geq 0$$

$$\Leftrightarrow 4\mu^2 + 4a\mu - 4\mu b_2 l - 4\mu m b_1 - 4F b_1 - 4b_1 c e^{-\mu l} \geq 0$$

$$\Leftrightarrow \mu^2 + a\mu - \mu b_2 l - \mu m b_1 - F b_1 - b_1 c e^{-\mu l} \geq 0$$

The $a\mu - \mu b_2 l - \mu m b_1 - F b_1 - b_1 c e^{-\mu l}$ corresponds to the non-negativity condition of appendix J. This imply that to have a non-negative objective functions, we have to have $a\mu - \mu b_2 l - \mu m b_1 - F b_1 - b_1 c e^{-\mu l} \geq 0$. With the $\mu \geq 0$, we can say that $\mu^2 + a\mu - \mu b_2 l - \mu m b_1 - F b_1 - b_1 c e^{-\mu l} \geq 0$ and $\Delta \geq 0$ are proven.

L. The $\lambda_2 \geq 0$ correspond to the feasibility condition (M/M/1/1 with holding and penalty cost)

$$\lambda_2 \geq 0 \quad \Leftrightarrow \quad -\mu + \sqrt{\mu^2 + a\mu - \mu b_2 l - \mu b_1 m - F b_1 - b_1 c e^{-\mu l}} \geq 0$$

$$\sqrt{\mu^2 + a\mu - \mu b_2 l - \mu b_1 m - F b_1 - b_1 c e^{-\mu l}} \geq \mu$$

$$\mu^2 + a\mu - \mu b_2 l - \mu b_1 m - F b_1 - b_1 c e^{-\mu l} \geq \mu^2$$

$$a\mu - \mu b_2 l - \mu b_1 m - Fb_1 - b_1 ce^{-\mu l} \geq 0$$

This final equation corresponds to the non-negativity of the objective function in appendix J.

M. Proof that the objective function is concave in M/M/1 with penalty and holding cost

This part of the appendix demonstrates that the objective function for the M/M/1/1 with penalty and holding cost is concave. The objective function of this problem is:

$$\underset{\lambda,p,l,\bar{\lambda}}{Maximize} \quad \bar{\lambda}(p-m) - \frac{F\lambda}{\mu+\lambda} - \frac{c\bar{\lambda}}{\mu}e^{-\mu l}$$

$$\Leftrightarrow \underset{\lambda,l}{Maximize} \quad \frac{\lambda\left[\left(\mu\left(a-b_2 l-\lambda\right)/b_1\right)-m\mu-F-ce^{-\mu l}\right]}{\mu+\lambda}$$

Suppose that the $\lambda_1 = 0$ and $l_1 = 0$, then the objective become:

$$\begin{aligned}\bar{\lambda}(p-m) - \frac{F\lambda}{\mu+\lambda} - \frac{c\bar{\lambda}}{\mu}e^{-\mu l} &= \frac{\lambda\left[\left(\mu\left(a-b_2 l-\lambda\right)/b_1\right)-m\mu-F-ce^{-\mu l}\right]}{\mu+\lambda}\\ &= \frac{0\left[\left(a\mu/b_1\right)-m\mu-F-c\right]}{\mu}\\ &= \frac{0}{\mu b_1}\\ &= 0\end{aligned}$$

Suppose that the $\lambda_2 = +\infty$ and $l_1 = 0$, then the objective become:

$$\begin{aligned}\bar{\lambda}(p-m) - \frac{F\lambda}{\mu+\lambda} - \frac{c\bar{\lambda}}{\mu}e^{-\mu l} &= \frac{\lambda\left[\left(\mu\left(a-b_2 l-\lambda\right)/b_1\right)-m\mu-F-ce^{-\mu l}\right]}{\mu+\lambda}\\ &= \frac{\infty\left[\left(-\infty/b_1\right)-m\mu-F-c\right]}{\infty}\\ &= \frac{-\infty}{\infty}\\ &= -\infty\end{aligned}$$

Suppose that the $\lambda_1 = 0$ and $l_1 = +\infty$, then the objective become:

$$\begin{aligned}\bar{\lambda}(p-m) - \frac{F\lambda}{\mu+\lambda} - \frac{c\bar{\lambda}}{\mu}e^{-\mu l} &= \frac{\lambda\left[\left(\mu\left(a-b_2 l-\lambda\right)/b_1\right)-m\mu-F-ce^{-\mu l}\right]}{\mu+\lambda}\\ &= \frac{0\left[\left(-\infty/b_1\right)-m\mu-F-\infty\right]}{\mu}\\ &= \frac{0}{\mu}\\ &= 0\end{aligned}$$

Suppose that the $\lambda_2 = +\infty$ and $l_2 = +\infty$, then the objective become:

$$\begin{aligned}\bar{\lambda}(p-m) - \frac{F\lambda}{\mu+\lambda} - \frac{c\bar{\lambda}}{\mu}e^{-\mu l} &= \frac{\lambda\left[\left(\mu\left(a-b_2 l-\lambda\right)/b_1\right)-m\mu-F-ce^{-\mu l}\right]}{\mu+\lambda}\\ &= \frac{\infty\left[\mu\left(a-\infty-\infty/b_1\right)-m\mu-F-\infty\right]}{\infty}\\ &= \frac{-\infty}{\infty}\end{aligned}$$

$$= -\infty$$

It can be summarized as shown in table below

			The objective with λ^*, l^*
The objective with	$\lambda_1 = 0, l_1 = 0$	$F(\lambda_1, l_1) = 0$	$F(\lambda^*, l^*) \geq F(\lambda_1, l_1)$
	$\lambda_2 = +\infty, l_1 = 0$	$F(\lambda_2, l_1) = -\infty$	$F(\lambda^*, l^*) \geq F(\lambda_2, l_1)$
	$\lambda_1 = 0, l_2 = +\infty$	$F(\lambda_1, l_2) = 0$	$F(\lambda^*, l^*) \geq F(\lambda_1, l_2)$
	$\lambda_2 = +\infty, l_2 = +\infty$	$F(\lambda_2, l_2) = -\infty$	$F(\lambda^*, l^*) \geq F(\lambda_2, l_2)$

This prove that the other compared to λ^* is lower. Hence, the objective function is concave.