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Simple Tracking Output Feedback $H_\infty$ Control for Switched Linear Systems: Lateral Vehicle Control Application

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Abstract: In this paper, the problem of the switched $H_\infty$ tracking output feedback control problem is studied. The control design problem is addressed in the context of discrete-time switched linear systems. Then, the design of continuous-time case becomes trivial. Linear Matrix Inequality (LMI) and Linear Matrix Equality (LME) representations are used to express all sufficient conditions to solve the control problem. Some transformations leading to sufficient conditions for the control problem are also used. All conditions are established for any switching using a switched Lyapunov function and a common Lyapunov function. The effectiveness of the proposed control approach is shown through a steering vehicle control implementation. Interesting simulation results are obtained using real data acquired by an instrumented car.

Keywords: Switched linear systems, $H_\infty$ norm, poly-quadratic stability, steering control.

1. INTRODUCTION

Many real complex systems have several operating modes, each of them corresponds to a local dynamical behavior. In many cases, all operating modes can be described by nonlinear mathematical models. Unfortunately, the use of such models becomes a very hard task. One solution used for this problem, is the multi-model representation. In practice, this approach is used to design gain-scheduled methods (Stilwell and Rugh [1999], Apkarian and Gahinet [1995]), linear parameter varying control (Apkarian et al. [1997, 1995]), fuzzy systems (Castillo and Melin [2008], Mendel [2004], Sugeno and Kang [1988]). In same reasoning, the switched systems have been developed (Branicky [1998], Liberzon and Morse [1999], Daafouz et al. [2002], Sun and Ge [2005], Lin and Antsaklis [2007], Koenig et al. [2008], Lin and Antsaklis [2009], Koenig and Marx [2009]). Such systems are defined by a finite number of subsystems and switching rules. In particular, the switched linear systems are obtained by linearization of nonlinear systems in the vicinity of some operating modes (or operating points). The design of switched systems remains an attractive problem and widely addressed for linear systems by using common Lyapunov function (Liberzon and Morse [1999], Sun and Ge [2005], Lin and Antsaklis [2007, 2009]) and switched Lyapunov functions (Branicky [1998], Daafouz et al. [2002], Du et al. [2007]).

The vehicle dynamics control problem is used to show the effectiveness of the proposed controllers. The lateral control seems to be a good example for the switching control application. Many vehicle controllers have been reported (see, Cerone et al. [2009], Plochl and Edelmann [2009] and the references therein) and some of them assume that the vehicle model is exactly known, and such...
an assumption is generally not satisfied. Consequently, the switched systems can be successfully used.

This paper is organized as follows. Section 2 describes the Linear bicycle vehicle model and the problem statement. Section 3 presents the design method of switched control. The simulation results using the real data are given in Section 4. Section 5 summarizes conclusions and perspectives.

2. SWITCHED CONTROL: PROBLEM STATEMENT

For switched control design, let us recall the single track vehicle model commonly used for lateral control.

2.1 Single track vehicle model

The lateral model used here is composed of lateral and yaw motions which are described by:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Ff(t) \\
y(t) &= Cx(t)
\end{align*}
\]

where

\[
A = \begin{bmatrix}
-2C_f + 2C_r & -V_x - 2C_f L_f - 2C_r L_f / mV_x & 0 \\
-2C_f L_f - 2C_r L_f / mV_x & -2C_f L_f^2 + 2C_r L_f^2 / mV_x & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
C_f & 2L_f C_f / l_z & 0
\end{bmatrix}^T
\]

\[
x = [V_y \; \dot{\psi} \; \psi]^T, \quad C = [1 \; 0 \; V_x]
\]

\[
F = [-g \; 0 \; 0]^T, \quad x(t) \in \mathbb{R}^n \text{ is the state vector, } u(t) = \delta(t) \in \mathbb{R}^m \text{ is the control input, } y(t) \in \mathbb{R}^p \text{ is the output, and } f(t) = \phi_x(t) \in \mathbb{R}^n \text{ is the disturbance input that satisfies } f \in L_2[0, \infty). \quad A, B, C \text{ and } F \text{ are system matrices with appropriate size.}
\]

Let us recall that to obtain the model (1), the following linear tire force model is used:

\[
F_{sf} = C_f \left( \delta - \frac{V_y + \dot{\psi} L_f}{V_x} \right), \quad F_{yr} = -C_r \left( \frac{V_y - \dot{\psi} L_r}{V_x} \right)
\]

2.2 Problem formulation

In model (1), several parameter variations can be listed: cornering stiffnesses $C_f$ and $C_r$, longitudinal speed $V_x$. In (1), all parameters are considered constant. Unfortunately, many driving actions act on these parameters. The characteristics of Figure 1 highlight the braking maneuver in which $C_f$ and $C_r$ have two different modes. We can observe that the lateral tire forces operate in linear region with small sideslip angle (less than 2 deg). According to the characteristic of Figure 1 and model (1), the following switching rules can be considered:

1- First rule on $C_f$ and $C_r$:

\[
\begin{align*}
C_{f,r}^1 &= C_{f,r}^1 \quad \text{if } \beta < \beta^* \\
C_{f,r}^2 &= C_{f,r}^2 \quad \text{if } \beta > \beta^*
\end{align*}
\]

2- Second rule on $V_x$:

\[
\begin{align*}
V_x &= V_{x_1} \quad \text{if } V_x \in [V_{x_1} - \Delta, V_{x_1} + \Delta] \\
V_x &= V_{x_2} \quad \text{if } V_x \in [V_{x_2} - \Delta, V_{x_2} + \Delta] \\
&\vdots \\
V_x &= V_{x_M} \quad \text{if } V_x \in [V_{x_{M-1}} - \Delta, V_{x_M} - \Delta]
\end{align*}
\]

with $\beta^*$ is the switching threshold on the sideslip angle and $\Delta = \frac{V_{x_2} - V_{x_1}}{2}$. However, the cornering stiffnesses coefficients ($C_f$ and $C_r$) are measured by expensive sensors (around 100 K€). For this, a switched controller can then be designed with unmeasurable premise variable. Moreover, the system must be robust against the premise variables (Kiss et al. [2011], Ichalal et al. [2010]). In our case, the premise variables are $\beta$ and $V_x$ which can be estimated online (see Villagra et al. [2009]). Since, the LTI model (1) can be viewed as a switched one:

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{M} \alpha_i(t) [A_i x(t) + B_i u(t) + F_i \omega(t)] \\
y(t) &= \sum_{i=1}^{M} \alpha_i(t) C_i x(t)
\end{align*}
\]

The function $\alpha_i(t)$ is the switching signal

\[
\alpha_i : \mathbb{R}^+ \rightarrow \{0, 1\} \quad \sum_{i=1}^{M} \alpha_i(t) = 1, \quad t \in \mathbb{R}^+
\]

Our aim is to design switched controllers, such that the output $y(t)$ of the closed-loop system tracks any given reference output $y_r(t)$ of the following reference model:

\[
\begin{align*}
\dot{x}_r(t) &= \sum_{i=1}^{M} \alpha_i(t) [A_i x_r(t) + F_i \omega(t)] \\
y_r(t) &= \sum_{i=1}^{M} \alpha_i(t) C_i x_r(t)
\end{align*}
\]

where, $y_r(t) \in \mathbb{R}^p$ has the same dimension as $y(t)$. $x_r(t) \in \mathbb{R}^n$ and $\omega(t) \in \mathbb{R}^m$ are respectively the reference state and the bounded reference input. $A_r$, $C_r$ and $F_r$ are appropriately dimensioned with $A_r$ Hurwitz. The control design procedure assumes that both $x(t)$ and $x_r(t)$ are measurable outputs. For our purpose, we define the following tracking output error:

\[
\begin{align*}
\tilde{y}(t) &= y(t) - y_r(t)
\end{align*}
\]

Therefore, the following augmented system is obtained:

\[
\begin{align*}
\dot{\xi}(t) &= \sum_{i=1}^{M} \alpha_i(t) [A i \zeta(t) + B i u(t) + F i \omega(t)] \\
\tilde{y}(t) &= \sum_{i=1}^{M} \alpha_i(t) C_i \zeta(t)
\end{align*}
\]
where $A_{ai} = \begin{bmatrix} A_i & 0 \\ 0 & A_{ri} \end{bmatrix}$, $B_{ai} = \begin{bmatrix} B_i \\ 0 \end{bmatrix}$, $F_{ai} = \begin{bmatrix} F_i & 0 \\ 0 & F_{ri} \end{bmatrix}$, $C_{ai} = \begin{bmatrix} C_i - C_{ri} \end{bmatrix}$, $\xi(t) = \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix}$, $\omega(t) = \begin{bmatrix} f(t) \\ r(t) \end{bmatrix}$.

The discrete-time system corresponding to (8) using the first order Euler approximation at frequency 200Hz is

$$
\begin{align*}
\xi_{k+1} &= \sum_{i=1}^{M} \alpha_i(k) \left[ \bar{A}_{ai} \xi_k + \bar{B}_{ai} u_k + F_{ai} \omega_k \right] \\
\bar{y}_k &= \sum_{i=1}^{M} \alpha_i(k) C_{ai} \xi_k
\end{align*}
$$

and the functional switching signal $\alpha_i(k)$ is

$$
\alpha_i : \mathbb{Z}^+ \rightarrow \{0, 1\}, \quad \sum_{i=1}^{M} \alpha_i(k) = 1, \quad k \in \mathbb{Z}^+
$$

In the sequel, the switched control problem for discrete-time system (9) is given.

**Problem:** Consider the following $H_{\infty}$ discrete-time tracking output feedback controller for system (9)

$$
u_k = -\sum_{i=1}^{M} \alpha_i(k) \bar{K}_i \bar{y}_k
$$

where the gains $\bar{K}_i \in \mathbb{R}^{(n+n_r)}$ are computed such that:

S1. the closed loop system $\xi_{k+1} = \sum_{i=1}^{M} \alpha_i(k)(\bar{A}_{ai} - \bar{B}_{ai} \bar{K}_i \bar{C}_{ai}) \xi_k$ is asymptotically stable when $\omega(k) = 0$;

S2. the transfer function $H_{\infty}(s) = \sum_{i=1}^{M} C_{ai}(sI - (\bar{A}_{ai} - \bar{B}_{ai} \bar{K}_i \bar{C}_{ai}))^{-1} F_{ai}$, from $\omega(k)$ to $\bar{y}(k)$ satisfies the $H_{\infty}$ norm $\|H_{\infty}(s)\|_{\infty} < \gamma$ for positive scalar $\gamma$.

To deal with the above problem, let us recall the following Schur complement commonly used in literature:

**Lemma 1.** Let $X = X^T > 0$, $N = N^T > 0$ and $W$ be given matrices. By Schur complement, the following statements are equivalent:

$$X - WN^{-1}W^T > 0 \Leftrightarrow \begin{bmatrix} X & W \\ W^T & N \end{bmatrix} > 0 \quad (12)$$

**Assumption 1.** We assume that for $i \in \{1, \ldots, M\}$, the pair $(\bar{A}_{ai}, \bar{B}_{ai})$ is stabilizable.

3. **DESIGN OF DISCRETE-TIME SWITCHED $H_{\infty}$ TRACKING OUTPUT FEEDBACK CONTROL**

The main objective of this section is to find some sufficient conditions and compute the gains of (11) for system (9), in order to stabilize the following closed-loop system

$$
\begin{align*}
\xi_{k+1} &= \sum_{i=1}^{M} \alpha_i(k) \left[ \bar{A}_{ai} - \bar{B}_{ai} \bar{K}_i \bar{C}_{ai} \right] \xi_k + \bar{F}_{ai} \omega_k \\
\bar{y}_k &= \sum_{i=1}^{M} \alpha_i(k) \bar{C}_{ai} \xi_k
\end{align*}
$$

and satisfy specification S2. For this, we establish the following theorem.

**Theorem 1.** Under assumption 1, if there exist a constant $\gamma > 0$, matrices $X_{ai} > 0$, $X_{aij} > 0$, $M_{ai}$ and $N_{ai}$ such that the following LMI and LME are satisfied:

$$
\begin{bmatrix}
-X_{ai} & 0 & X_{ai} \bar{A}_{ai} - C_{ai}^T N_{ai}^T B_{ai} & X_{ai} \bar{C}_{ai}^T \\
\ast & -\gamma^2 I & \ast & \ast \\
\ast & \ast & -X_{aij} & 0 \\
\ast & \ast & \ast & -I
\end{bmatrix} < 0 \quad (14)
$$

for $(i, j) \in \{1, \ldots, M\}^2$, then, the stabilizing gains of (11) are given by $\bar{K}_i = N_i M_{ai}^{-1}$.

**Proof.** To give sufficient conditions for the existence of (11) such that the closed-loop system (13) satisfies specifications S1 and S2, the following inequality should be verified:

$$V_k + \bar{y}_k^T \bar{y}_k - \gamma^2 \omega_k^T \omega_k < 0 \quad (16)$$

where $V_k = \sum_{i=1}^{M} \alpha_i(k) \xi_k^T P_{ai} \bar{y}_k$ is the switched Lyapunov functions and $P_i$ are positive definite matrices. Computing the difference $V_{k+1} - V_k$ along (13), the relation (16) becomes

$$
\begin{align*}
\sum_{j=1}^{M} \alpha_j(k+1) \xi_k^T P_{aj} \xi_k + \sum_{i=1}^{M} \alpha_i(k) \xi_k^T P_{ai} \xi_k \\
+ \bar{y}_k^T \bar{y}_k - \gamma^2 \omega_k^T \omega_k < 0
\end{align*}
$$

To consider all switches, the following particular cases are considered

$$
\begin{align*}
\alpha_i(k) = 1 \quad \text{and} \quad \alpha_i \neq 0, \\
\alpha_i(k+1) = 1 \quad \text{and} \quad \alpha_i \neq 0.
\end{align*}
$$

Using (18), (17) becomes

$$
\begin{align*}
\xi_k^T \left[ (\bar{A}_{ai} - \bar{B}_{ai} \bar{K}_i) \bar{C}_{ai} \right] + C_{ai} P_{ai} \bar{C}_{ai} - P_{ai} \xi_k + \omega_k^T \left[ F_{ai}^T P_{ai} F_{ai} - \gamma^2 I \right]
\end{align*}
$$

which can be rewritten as

$$
\begin{align*}
\begin{bmatrix}
\xi_k^T \\
\omega_k^T
\end{bmatrix} \times
\begin{bmatrix}
\Pi_{i,j} - P_{ai} \\
\bar{F}_{aij} \bar{P}_{aij} \bar{A}_{aij} - \bar{B}_{aij} \bar{K}_{aij} \bar{C}_{aij}
\end{bmatrix} - \gamma^2 I + \bar{P}_{aij} F_{aij} < 0
\end{align*}
$$

where $\Pi_{i,j} = (\bar{A}_{ai} - \bar{B}_{ai} \bar{K}_i \bar{C}_{ai})^T P_{aj} (\bar{A}_{ai} - \bar{B}_{ai} \bar{K}_i \bar{C}_{ai}) + C_{aij} \bar{C}_{aij}$. Then, (19) is negative for $[\xi_k \omega_k^T] \neq 0$ if

$$
\begin{align*}
\begin{bmatrix}
\Pi_{i,j} - P_{ai} \\
\bar{F}_{aij} \bar{P}_{aij} (\bar{A}_{aij} - \bar{B}_{aij} \bar{K}_{aij} \bar{C}_{aij}) - \gamma^2 I + \bar{P}_{aij} F_{aij}
\end{bmatrix} < 0
\end{align*}
$$

By Schur complement (12), (20) becomes

$$
\begin{bmatrix}
-X_{ai} & 0 & X_{ai} \bar{A}_{ai} - C_{ai}^T N_{ai}^T B_{ai} & X_{ai} \bar{C}_{ai}^T \\
\ast & -\gamma^2 I & \ast & \ast \\
\ast & \ast & -X_{aij} & 0 \\
\ast & \ast & \ast & -I
\end{bmatrix} < 0 \quad (21)
$$

Now, pre- and post-multiplying (21) by $Z_{ai} = \text{diag}(X_{ai} = P_{ai}^{-1}, I, I, I) = P_{ai}$, (21) becomes

$$
\begin{align*}
\begin{bmatrix}
-X_{ai} & 0 & X_{ai} \bar{A}_{ai} - B_{ai} \bar{K}_{ai} \bar{C}_{ai} & X_{ai} \bar{C}_{ai} \\
\ast & -\gamma^2 I & \ast & \ast \\
\ast & \ast & -X_{aij} & 0 \\
\ast & \ast & \ast & -I
\end{bmatrix} < 0
\end{align*}
$$

(22)
Now, to avoid the nonlinearities $B_ai,C_ai,X_ai$ and $X_ai \hat{C}_i^T K_i^T B_i^T$, let us consider the following matrices transformation describing a linear matrix equality:

$$M_ai \hat{C}_ai = \hat{C}_ai X_ai$$  \hspace{1cm} (23)

and substituting $K_ai = N_ai M_ai^{-1}$ into (22), (14) is obtained.

Remark 1. The design of such a controller for continuous-time system (8) becomes trivial and can easily be deduced from the above one. For more details about this problem, some elements are given in Appendix A.

4. SIMULATION RESULTS USING REAL DATA

The simulations are conducted using a full Non Linear Four Wheels Vehicle Model (NLFWVM) (Menhour et al. [2013]) of a Peugeot 406. For our simulations, we suppose that the reference model describing the lateral displacement of the vehicle is given by

$$\begin{cases}
\dot{x}_i(t) = -x_i(t) + r(t) \\
y_i(t) = x_i(t)
\end{cases}$$  \hspace{1cm} (24)

where $r(t) = V_i x_i(t) \sin(\psi_i(t)) + V_i y_i(t) \cos(\psi_i(t)) + a_y(t)$. For our simulation tests, the reference input $r(t)$ is constructed from the data acquired during an experimental test. These tests are conducted on a race track with a professional driver under good conditions (see grey curves of Figures 3, 4, 6 and 5). For our simulations, the reference output $y_i$ is computed using measured lateral acceleration $a_y$, yaw angle $\psi_i$, longitudinal $V_i$ and lateral $V_i$ speeds. Figure 2 shows the road bank angle $f(t) = \omega(t)$ used as unknown input for our simulations and its spectral domain is located in a low frequency range. For all trials, the car’s acquisition device operates at frequency $200 \text{ Hz}$. For our simulations, switching rule (3) is used and two values $V_{s1} = 60 \text{ km/h}$ and $V_{s2} = 90 \text{ km/h}$ are chosen (see also the upper part of Figure 3). Consequently, two local models are obtained $M = 2$ and $(i,j) \in \{1, 2\}$. Then, the stability of two controllers (11) and (A) are guaranteed by the resolution of 4 LMIs/LMEs of theorem 1 and of 2 LMIs/LMEs of theorem 2 to obtain two Lyapunov matrices ($X_{ai}$ and $X_{ai}$) and a common Lyapunov matrix $X$ respectively. The LMIs/LMEs of theorems 1 and 2 are solved using YALMIP software (Löfberg [2004]). The obtained gains of discrete-time controller (11) are:

$K_{a1} = 0.05644$ and $K_{a2} = 0.04887$

and the $H_\infty$ performances are obtained for $\gamma^* = 2.6$.

Figures 4, 5, 6 and 7 show that the closed-loop simulation results obtained with two controllers and NLFWVM are similar to the measured ones. The main dynamical variables plotted are the derivative of lateral deviation (Figure 6) and trajectories (Figure 5).

Figure 7 shows also the performances of two controllers in terms of tracking output errors $\tilde{y}$ which are less than 0.1 m/s. These tracking output errors are quite small, but, the controller (11) have less than the controller (A), this may be due to the conservatism of Lyapunov function approach used to design (A).

The steering angles input performances of the two controllers are depicted on Figure 4. The computed steering angles are similar to the measured ones.

Figures 8 and 9 show the unknown input $f(t) = \phi(t)$ attenuation properties of continuous-time subsystems $((A_i - B_i K_i C_i), F_i, C_i)$ and discrete-time subsystems $((A_{ai} - B_{ai} K_{ai} C_{ai}), F_{ai}, C_{ai})$ from the unknown input to controlled output variable. Moreover, by computing, $||\omega||_2 = ||f||_2 + ||r||_2$ and $||\tilde{y}||_2 = ||y - y_r||_2$, which gives for continuous-time

$$\frac{||\tilde{y}||_2}{||\omega||_2} = 0.0021 < \gamma^* = 2.6$$

and for discrete-time

$$\frac{||\tilde{y}||_2}{||\omega||_2} = 0.0010 < \gamma^* = 2.6$$

this shows the effectiveness of the proposed controllers.
Fig. 5. Trajectories: Reference and controlled model

Fig. 6. Tracking output: reference $y_r$ and model $y$

Fig. 7. Tracking output errors $\hat{y}$

Fig. 8. Singular values of system $((A_{ai} - B_{ai}K_{ai}C_{ai}), F_{ai}, C_{ai})$ for $V_x = 60$ and $90 \text{ km/h}$, between $f$ to $\hat{y}$

5. CONCLUSIONS AND FUTURE WORK

Two switched $H_\infty$ tracking output feedback controllers are proposed. Such controllers are developed for continuous-
time and discrete-time switched linear systems. Based on a common Lyapunov function and switched Lyapunov functions methods, sufficient conditions for the existence of two controllers are established. All these conditions are expressed in terms of LMIs/LMEs constraints. The design of such controllers gains is reduced to solve a set of LMIs/LMEs using YALMIP software. The effectiveness of the proposed controllers is shown through the steering vehicle control implementation. The simulation tests are conducted using real data.

The steering vehicle control example assumes the lateral speed to be a measurable state. However, such a variable is measured by an expensive sensor. For this problem, it will be interesting to design a robust $H_\infty$ unknown input observer for system (1) with yaw motion as measurable output.

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Appendix A. SWITCHED $H_\infty$ TRACKING OUTPUT FEEDBACK CONTROL: CONTINUOUS-TIME CASE

**Problem**: Consider the following switched controller for continuous-time system (8)

\[
 u(t) = -\sum_{i=1}^{M} a_i(t) K_{ai} \hat{y}(t) \tag{A.1}
\]

where the gains $K_{ai} \in \mathbb{R}^{p \times (n+n_v)}$ are computed such that the specifications S1 and S2 in continuous-time are ensured.

For the above design problem, the following theorem is trivial and can easily be established using common Lyapunov function.

**Theorem 2.** Suppose that for $i \in \{1, \ldots, M\}$, the pair $(A_{ai}, B_{ai})$ is stabilizable. If there exist a positive scalar $\gamma > 0$, matrices $X > 0$, $M_{ai}$ and $N_{ai}$ such that the following LMI and LME are satisfied:

\[
 \begin{bmatrix}
 \Phi_{ai} & F_{ai} & X \tilde{C}_{ai}^T \\
 a_i(t) K_{ai} & 0 & 0 \\
 0 & -\gamma^2 I & -I \\
 \end{bmatrix} < 0 \tag{A.2}
\]

\[
 C_{ai} X = M_{ai} C_{ai} \tag{A.3}
\]

for $i \in \{1, \ldots, M\}$, then the stabilizing gains of (A) are given by $K_{ai} = N_{ai} M_{ai}^{-1}$ with $F_{ai} = X A_{ai}^T + A_{ai} X - C_{ai}^T N_{ai}^T B_{ai} - B_{ai} N_{ai} C_{ai}$.

**Remark 2.** The proof of the above theorem can be deduced using the same reasoning of the discrete-time case.

**Notation 1.** $(.)^T$ stands for the transpose matrix, $(.) > 0$ ($\geq 0$) denotes a symmetric positive definite matrix (semidefinite), we use an asterisk $(*)$ to represent a term that is induced by symmetry, $\text{diag}(.)$ stands a diagonal block matrix. $V_z$: longitudinal speed [km/h] and $V_y$: lateral speed [km/h], $\psi$: yaw rate [rad/s], $\psi$: yaw angle [rad], \( \delta \): wheel steer angle [rad], $\phi_z$: road bank angle [rad], $C_{f,r}$: front and rear cornering stiffness [N/rad], $\beta$: tire sideslip angle [rad], $L_f, r$: distances from the CoG to the front and rear axles [m], $I_z$: yaw moment of inertia [Kg.m²], $m$: vehicle mass [kg], $g$: acceleration due to gravity [m/s²], CoG: Center of Gravity.