2L convex polyominoes: discrete tomographical aspects

Khalil Tawbe, Laurent Vuillon

2L-CONVEX POLYOMINOES: DISCRETE TOMOGRAPHICAL ASPECTS

K. TAWBE, L. VUILLON

Abstract. This paper uses the theoretical material developed in a previous article by the authors in order to reconstruct a subclass of 2L-convex polyominoes. The main idea is to control the shape of these polyominoes by combining 4 types of geometries. Some modifications are made in the reconstruction algorithm of Chrobak and Dürr for HV-convex polyominoes in order to impose these geometries.

1. Introduction

The present paper uses the theoretical material developed in a previous article by the authors [14] in order to reconstruct a sub-class of 2L-convex polyominoes. Indeed, 2L-convex polyominoes are the first difficult class of polyominoes in terms of tomographical reconstruction in the hierarchy of kL-polyominoes and in this article we design an algorithm of reconstruction for a sub-class of 2L-convex which is the first step in the whole comprehension of the hierarchy of kL-polyominoes.

One main problem in discrete tomography consists on the reconstruction of discrete objects according to their horizontal and vertical projection vectors. In order to restrain the number of solutions, we could add convexity constraints to these discrete objects. There are many notions of discrete convexity of polyominoes (namely HV-convex [3], Q-convex [4], L-convex polyominoes [7]) and each one leads to interesting studies. One natural notion of convexity on the discrete plane is the class of HV-convex polyominoes that is polyominoes with consecutive cells in rows and columns. Following the work of Del Lungo, Nivat, Barcucci, and Pinzani [3] we are able using discrete tomography to reconstruct polyominoes that are HV-convex according to their horizontal and vertical projections. In addition to that, for an HV-convex polyomino P every pairs of cells of P can be reached using a path included in P with only two kinds of unit steps (such a path is called monotone). A polyomino is called kL-convex if for every two cells we find a monotone path with at most k changes of direction. Obviously a kL-convex polyomino is an HV-convex polyomino. Thus, the set of kL-convex polyominoes for $k \in \mathbb{N}$ forms a hierarchy of HV-convex polyominoes according to the number of changes of direction of monotone paths. This

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notion of L-convex polyominoes has been considered by several points of view. In [5] combinatorial aspects of L-convex polyominoes are analyzed, giving the enumeration according to the semi-perimeter and the area. In [6] it is given an algorithm that reconstructs an L-convex polyomino from the set of its maximal L-polyominoes. Similarly in [7] it is given another way to reconstruct an L-convex polyomino from the size of some special paths, called bordered L-paths.

In fact $2L$-convex polymoninoes are more geometrically complex and there was no result for their direct reconstruction. We could notice that Duchi, Rinaldi, and Schaeffer are able to enumerate this class in an interesting and technical article [9]. But the enumeration technique gives no idea for the tomographical reconstruction.

The first subclass that creates the link with $2L$-convex polyominoes is the class of HV-centered polyominoes. In [14], it is showed that if P is an HV-centered polyomino then P is $2L$-convex. Note that the tomographical properties of this subclass have been studied in [8] and its reconstruction algorithm is well known.

The main contribution of this paper is an $O(m^3n^3)$-time algorithm for reconstructing a subclass of $2L$-convex polyominoes using the geometrical properties studied in [14], and the algorithm of Chrobak and Dürr [8]. In particular, we add well chosen clauses to the original construction of Chrobak and Dürr in order to control the $2L$-convexity using 2SAT satisfaction problem.

This paper is divided into 5 sections. After basics on polyominoes, section 3 talks about the geometrical properties of a subclass of $2L$-convex polyominoes [14]. In section 4, the algorithm of Chrobak and Dürr for the reconstruction of the HV-convex polyominoes is given [8]. Section 5 describes the reconstruction of different subclasses of $2L$-convex polyominoes starting by the classes γ and $\Im_{2L}^{0,0}$ and ending by the other classes using an horizontal reflexion called S_H.

2. Definition and notation

A planar discrete set is a finite subset of the integer lattice $\mathbb{N}^2$ defined up to translation. A discrete set can be represented either by a set of cells, i.e. unitary squares of the cartesian plane, or by a binary matrix, where the 1's determine the cells of the set (see Fig.1). A polyomino P is a finite connected

set of adjacent cells (in the sequel, we use a 4-neighborhood, that is two cells are adjacent if they are chairing a segment), defined up to translation, in the cartesian plane. A polyomino is said to be *column-convex* (resp. *row-convex*) if every column (resp. row) is connected (see [2, 13]). Finally, a polyomino is said to be *convex* (or HV-convex) if it is both column and row-convex (see Fig. 2).

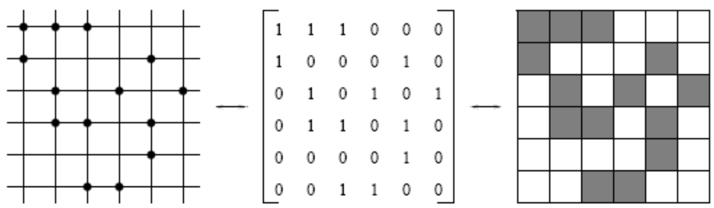

FIGURE 1. A finite set of $\mathbb{N} \times \mathbb{N}$, and its representation in terms of a binary matrix and a set of cells.

To each discrete set S, represented as a $m \times n$ binary matrix, we associate two integer vectors $H = (h_1, ..., h_m)$ and $V = (v_1, ..., v_n)$ such that, for each $1 \leq i \leq m$, $1 \leq j \leq n$, h_i and v_j are the number of cells of S (elements 1 of the matrix) which lie on row i and column j, respectively. The vectors H and V are called the *horizontal* and *vertical* projections of S, respectively (see Fig.3). Moreover if S has H and V as horizontal and vertical projections, respectively, then we say that S satisfies (H, V). Using the usual matrix notations, the element (i, j) denotes the entry in row i and column j.

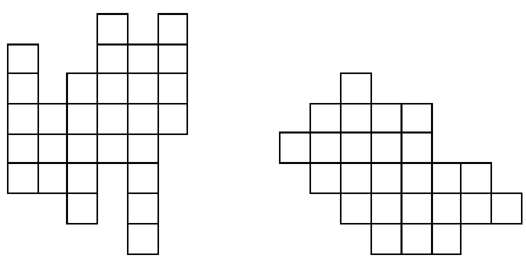

FIGURE 2. Column convex and HV-convex polyomino.

For any two cells A and B in a polyomino, a *path* $\prod_{AB}$, from A to B, is a sequence $(i_1, j_1), (i_2, j_2), ..., (i_r, j_r)$ of adjacent disjoint cells belonging in P, with $A = (i_1, j_1)$, and $B = (i_r, j_r)$. For each $1 \leq k \leq r - 1$, we say that the two consecutive cells $(i_k, j_k), (i_{k+1}, j_{k+1})$ form:

- an *east* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k + 1$;
- a *north* step if $i_{k+1} = i_k - 1$ and $j_{k+1} = j_k$;
- a *west* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k - 1$;
- a *south* step if $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k$.

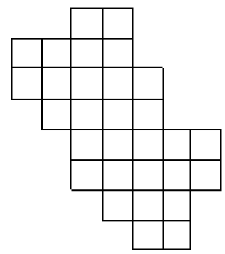

FIGURE 3. A polyomino P with $H = (2, 4, 5, 4, 5, 5, 3, 2)$ and $V = (2, 3, 6, 7, 6, 4, 2)$.

Finally, we define a path to be *monotone* if it is entirely made of only two of the four types of steps defined above.

Proposition 1 (Castiglione,Restivo [6])**.** *A polyomino P is HV-convex if and only if every pair of cells is connected by a monotone path.*

Let us consider a polyomino P. A path in P has a change of direction in the cell (i_k, j_k), for $2 \leq k \leq r-1$, if

$$i_k \neq i_{k-1} \Longleftrightarrow j_{k+1} \neq j_k.$$

Definition 1. *We call kL-convex an HV-convex polyomino such that every pair of its cells can be connected by a monotone path with at most k changes of direction respectively.*

In [6], it is proposed a hierarchy on convex polyominoes based on the number of changes of direction in the paths connecting any two cells of a polyomino.
For $k = 1$, we have the first level of hierarchy, i.e. the class of $1L$-convex polyominoes, also denoted L-convex polyominoes for the typical shape of each path having at most one single change of direction. In the present studies we focus our attention to the next level of the hierarchy, i.e. the class of $2L$-convex polyominoes, whose tomographical properties turn to be more interesting and substantially harder to be investigated than those of L-convex polyominoes (see Fig.4).

3. $2L$-convex polyominoes

Let (H, V) be two projection vectors and let P be an HV-convex polyomino, that satisfies (H, V). By a classical argument P is contained in a rectangle R (called minimal bounding box) where in this box no projection gives a zero. Let $[\min(S), \max(S)]$ ($[\min(E), \max(E)]$, $[\min(N), \max(N)]$, $[\min(W), \max(W)]$) be the intersection of P's boundary on the lower (right, upper, left) side of R (see [3]). By abuse of notation, we call $\min(S)$ [resp. $\min(E)$, $\min(N)$, $\min(W)$] the cell at the position $(m, \min(S))$ [resp.

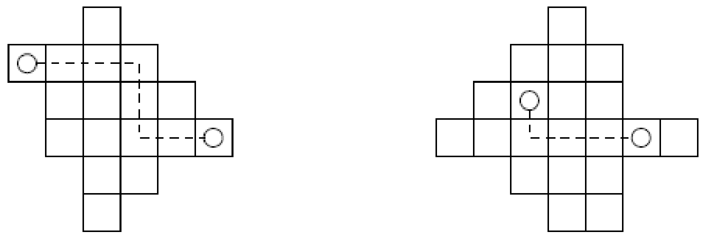

FIGURE 4. The convex polyomino on the left is 2L-convex, while the one on the right is L-convex. For each polyomino, two cells and a monotone path connecting them are shown.

$(\min(E), n)$, $(1, \min(N))$, $(\min(W), 1)$] and $\max(S)$ [resp. $\max(E)$, $\max(N)$, $\max(W)$] the cell at the position $(m, \max(S))$ [resp. $(\max(E), n)$, $(1, \max(N))$, $(\max(W), 1)$] (see Fig.5).

Definition 2. *The segment* $[min(S), max(S)]$ *is called the* S*-foot. Similarly, the segments* $[\min(E), \max(E)]$*,* $[\min(N), \max(N)]$ *and* $[\min(W), \max(W)]$ *are called* E*-foot,* N*-foot and* W*-foot.*

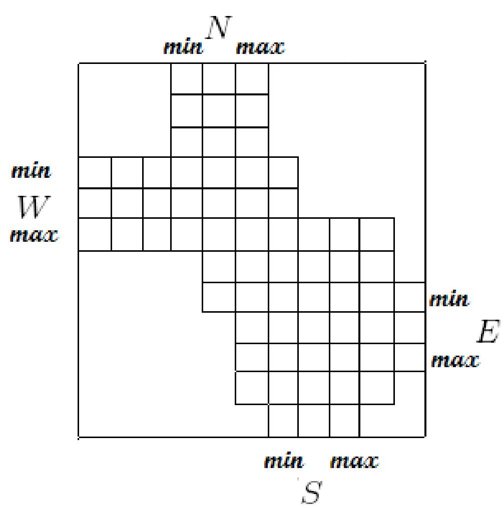

FIGURE 5. Min and max of the four feet in the rectangle R.

For a bounding rectangle R and for a given polyomino P, let us define the following sets:

- $WN = \{(i, j) \in P \mid i < \min(W) \text{ and } j < \min(N)\}$,
- $SE = \{(i, j) \in P \mid i > \max(E) \text{ and } j > \max(S)\}$,

- $NE = \{(i,j) \in P \mid i < \min(E) \textit{ and } j > \max(N)\}$,
- $WS = \{(i,j) \in P \mid i > \max(W) \textit{ and } j < \min(S)\}$.

Let $\mathcal{C}$ (resp. $\mathcal{C}_{2L}$) be the class of HV-convex polyominoes (resp. $2L$-convex polyominoes), thus we have the following classes of polyominoes regarding the position of the non-intersecting feet.

- $\Im^{0,0} = \{P \in \mathcal{C} | card(WN) = 0 \textit{ and } card(SE) = 0,$ $\max(W) < \min(E) \textit{ and } \max(N) < \min(S)\}$.
- $\Im^{0,0}_{2L} = \{P \in \mathcal{C}_{2L} | card(WN) = 0 \textit{ and } card(SE) = 0,$ $\max(W) < \min(E) \textit{ and } \max(N) < \min(S)\}$.
- $\Im'^{0,0} = \{P \in \mathcal{C} | card(NE) = 0 \textit{ and } card(SW) = 0,$ $\max(S) < \min(N) \textit{ and } \max(E) < \min(W)\}$.
- $\Im'^{0,0}_{2L} = \{P \in \mathcal{C}_{2L} | card(NE) = 0 \textit{ and } card(SW) = 0,$ $\max(S) < \min(N) \textit{ and } \max(E) < \min(W)\}$.
- $\gamma = \{P \in \mathcal{C} | \max(N) < \min(S) \textit{ and } \max(E) < \min(W)\}$.
- $\gamma' = \{P \in \mathcal{C} | \max(S) < \min(N) \textit{ and } \max(W) < \min(E)\}$.

Theorem 1 (Tawbe,Vuillon [14])**.** *Let P be an HV-convex polyomino in the class $\Im^{0,0}$. P is $2L$-convex if and only if there exists an L-path from:*

$$(1)\begin{cases} \max(N) \textit{ to } \max(E) \\ \textit{and} \\ \max(W) \textit{ to } \max(S) \end{cases}$$

or

$$(2)\begin{cases} \min(N) \textit{ to } \min(E) \\ \textit{and} \\ \min(W) \textit{ to } \min(S) \end{cases}$$

or

$$(3)\begin{cases} \min(N) \textit{ to } \min(E) \textit{ and } \max(W) \textit{ to } \max(S) \\ \textit{and} \\ 2L-path\ from\ \min(W) \textit{ to } \max(E) \end{cases}$$

or

$$(4)\begin{cases} \max(N) \textit{ to } \max(E) \textit{ and } \min(W) \textit{ to } \min(S) \\ \textit{and} \\ 2L-path\ from\ \min(N) \textit{ to } \max(S) \end{cases}$$

Corollary 1. *If P satisfies the conditions of Theorem 1, then P is in the class $\Im^{0,0}_{2L}$.*

The visualisation of the paths is shown below.

4. HV-Convex Polyominoes

Assume that H, V denote strictly positive row and column sum vectors. We also assume that $\sum_i h_i = \sum_j v_j$, since otherwise (H, V) do not have a realization.

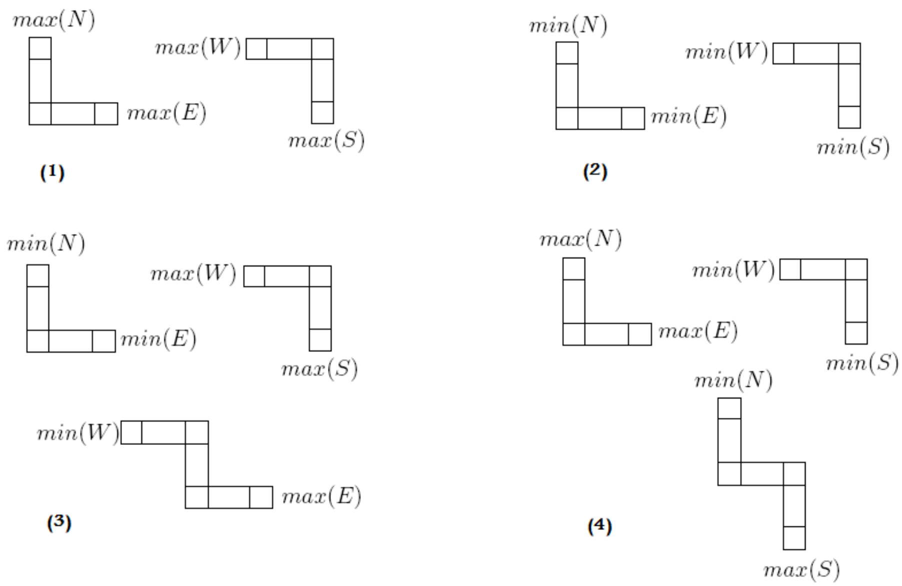

FIGURE 6. L-paths and 2L-paths on (1): first geometry, (2): second geometry, (3): third geometry, (4): fourth geometry.

The idea of Chrobak and Dürr [8] for the control of the HV-convexity is in fact to impose convexity on the four corner regions outside of the polyomino.

An object A is called an *upper-left corner region* if $(i+1, j) \in A$ or $(i, j+1) \in A$ implies $(i, j) \in A$. In an analogous fashion they can define other corner regions. Let $\overline{P}$ be the complement of P. The definition of HV-convex polyominoes directly implies the following lemma.

Lemma 1. *P is an HV-convex polyomino if and only if $\overline{P} = A \cup B \cup C \cup D$, where A, B, C, D are disjoint corner regions (upper-left, upper-right, lower-left and lower-right, respectively) such that (i) $(i-1, j-1) \in A$ implies (i, j) not in D, and (ii) $(i-1, j+1) \in B$ implies $(i, j) \notin C$.*

Given an HV-convex polyomino P and two row indices $1 \leq k, l \leq m$. P is *anchored* at (k, l) if $(k, 1), (l, n) \in P$. The idea of Chrobak and Dürr is, given (H, V), to reconstruct a 2SAT expression (a boolean expression in conjunctive normal form with at most two literals in each clause) $F_{k,l}(H, V)$ with the property that $F_{k,l}(H, V)$ is satisfiable iff there is an HV-convex polyomino realization P of (H, V) that is anchored at (k, l). $F_{k,l}(H, V)$ consists of several sets of clauses, each set expressing a certain property: "Corners" (Cor), "Disjointness" (Dis), "Connectivity" (Con), "Anchors" (Anc), "Lower bound on column sums" (LBC) and "Upper bound on row sums" (UBR).

$$Cor = \bigwedge_{i,j} \left\{ \begin{array}{llll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

The set of clauses Cor means that the corners are convex, that is for the corner A if the cell (i,j) belongs to A then cells $(i-1,j)$ and $(i,j-1)$ belong also to A. Similarly for corners B, C, and D.

$Dis = \bigwedge_{i,j} \{ \ X_{i,j} \Rightarrow \overline{Y}_{i,j} : \ for\ symbols\ X, Y \in \{A,B,C,D\}, X \neq Y \ \}$

The set of clauses Dis means that all four corners are pairwise disjoint, that is $X \cap Y = \emptyset$ for $X, Y \in \{A,B,C,D\}$.

$Con = \bigwedge_{i,j} \{ \ A_{i,j} \Rightarrow \overline{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \overline{C}_{i+1,j-1} \ \}$

The set of clauses Con means that if the cell (i,j) belongs to A then the cell $(i+1,j+1)$ does not belong to D, and similarly if the cell (i,j) belongs to B then the cell $(i+1,j-1)$ does not belong to C.

$Anc = \{ \ \overline{A}_{k,1} \wedge \overline{B}_{k,1} \wedge \overline{C}_{k,1} \wedge \overline{D}_{k,1} \wedge \overline{A}_{l,n} \wedge \overline{B}_{l,n} \wedge \overline{C}_{l,n} \wedge \overline{D}_{l,n} \ \}$

The set of clauses Anc means that we fix two cells on the west and east feet of the polyomino P, for $k, l = 1, ..., m$ the first one at the position $(k,1)$ and the second one at the position (l,n).

$$LBC = \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow \overline{C}_{i+v_j,j} & A_{i,j} \Rightarrow \overline{D}_{i+v_j,j} \\ B_{i,j} \Rightarrow \overline{C}_{i+v_j,j} & B_{i,j} \Rightarrow \overline{D}_{i+v_j,j} \end{array} \right\} \wedge \bigwedge_j \{ \ \overline{C}_{v_j,j} \quad \overline{D}_{v_j,j} \ \}$$

The set of clauses LBC implies that for each column j, we have that $\sum_i P_{i,j} \geq v_j$.

$$UBR = \bigwedge_j \left\{ \begin{array}{ll} \wedge_{i \leq min\{k,l\}} \overline{A}_{i,j} \Rightarrow B_{i,j+h_i} & \wedge_{k \leq i \leq l} \overline{C}_{i,j} \Rightarrow B_{i,j+h_i} \\ \wedge_{l \leq i \leq k} \overline{A}_{i,j} \Rightarrow D_{i,j+h_i} & \wedge_{max\{k,l\} \leq i} \overline{C}_{i,j} \Rightarrow D_{i,j+h_i} \end{array} \right\}$$

The set of clauses UBR implies that for each row i, we have that $\sum_j P_{i,j} \leq h_i$.

Define $F_{k,l}(H,V) = Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR$. All literals with indices outside the set $\{1,...,m\} \times \{1,...,n\}$ are assumed to have value 1.

Algorithm 1
Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$
W.l.o.g assume:$\forall i : h_i \in [1,n], \forall j : v_j \in [1,m], \sum_i h_i = \sum_j v_j \ and\ m \leq n.$
For $k, l = 1, ..., m$ **do begin**
If $F_{k,l}(H,V)$ is satisfiable,
then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**.
end
output "failure".

The following theorem allows to link the existence of HV-convex solution and the evaluation of $F_{k,l}(H,V)$. The crucial part of this algorithm comes from the constraints on the two sets of clauses LBC and UBR.

Theorem 2 (Chrobak,Dürr)**.** *$F_{k,l}(H,V)$ is satisfiable if and only if (H,V) have a realization P that is an HV-convex polyomino anchored at (k,l).*

Each formula $F_{k,l}(H,V)$ has size $O(mn)$ and can be computed in time $O(mn)$. Since 2SAT can be solved in linear time see [10, 1], Chrobak and Dürr give the following result.

Theorem 3 (Chrobak,Dürr)**.** *Algorithm 1 solves the reconstruction problem for HV-convex polyominoes in time $O(mn\min(m^2,n^2))$.*

5. Reconstruction of $2L$-convex polyominoes in γ and $\Im_{2L}^{0,0}$

The present section uses the theoretical material developed in the above sections in order to reconstruct $2L$-convex polyominoes in γ and $\Im_{2L}^{0,0}$. Some modifications are made to the reconstruction algorithm of Chrobak and Dürr for HV-convex polyominoes in order to impose our geometries. All the clauses that have been added and the modifications of the original algorithm are explained in the proofs of each subclass.

Finally, by defining an horizontal symmetry S_H, we show how to reconstruct P in the class γ' and $\Im_{2L}^{\prime 0,0}$.

5.1. **Clauses for the class γ.** In this section, we add the clause Pos and we modify the clause Anc of the original Chrobak and Dürr's algorithm in order to reconstruct if possible all polyominoes in the subclass γ.

$$Pos = \left\{ \; A_{(\max(E),1)} \wedge C_{(m,\max(N))} \; \right\}$$

$$Cor = \bigwedge_{i,j} \left\{ \begin{array}{llll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

$$Dis = \bigwedge_{i,j} \left\{ \; X_{i,j} \Rightarrow \overline{Y}_{i,j} : \; for\ symbols\ X,Y \in \{A,B,C,D\}, X \neq Y \; \right\}$$

$$Con = \bigwedge_{i,j} \left\{ \; A_{i,j} \Rightarrow \overline{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \overline{C}_{i+1,j-1} \; \right\}$$

$$Anc = \left\{ \begin{array}{l} \overline{A}_{\min(W),1} \wedge \overline{A}_{\min(E),n} \wedge \overline{B}_{\min(W),1} \wedge \overline{B}_{\min(E),n} \wedge \\ \overline{C}_{\min(W),1} \wedge \overline{C}_{\min(E),n} \wedge \overline{D}_{\min(W),1} \wedge \overline{D}_{\min(E),n} \wedge \\ \overline{A}_{1,\min(N)} \wedge \overline{A}_{m,\min(S)} \wedge \overline{B}_{1,\min(N)} \wedge \overline{B}_{m,\min(S)} \wedge \\ \overline{C}_{1,\min(N)} \wedge \overline{C}_{m,\min(S)} \wedge \overline{D}_{1,\min(N)} \wedge \overline{D}_{m,\min(S)} \wedge \\ \overline{A}_{\max(W),1} \wedge \overline{A}_{\max(E),n} \wedge \overline{B}_{\max(W),1} \wedge \overline{B}_{\max(E),n} \wedge \\ \overline{C}_{\max(W),1} \wedge \overline{C}_{\max(E),n} \wedge \overline{D}_{\max(W),1} \wedge \overline{D}_{\max(E),n} \wedge \\ \overline{A}_{1,\max(N)} \wedge \overline{A}_{m,\max(S)} \wedge \overline{B}_{1,\max(N)} \wedge \overline{B}_{m,\max(S)} \wedge \\ \overline{C}_{1,\max(N)} \wedge \overline{C}_{m,\max(S)} \wedge \overline{D}_{1,\max(N)} \wedge \overline{D}_{m,\max(S)} \end{array} \right\}$$

$$LBC = \bigwedge_i \left\{ \begin{array}{l} \wedge_{j<\min(N)} A_{i,j} \Rightarrow \overline{C}_{i+v_j,j} \\ \wedge_{\min(N)\leq j\leq \max(N)} C_{i+v_j,j} \Rightarrow \overline{A}_{i,j} \\ \wedge_{\max(N)<j<\min(S)} B_{i,j} \Rightarrow \overline{C}_{i+v_j,j} \\ \wedge_{\min(S)\leq j\leq \max(S)} \overline{B}_{i,j} \Rightarrow C_{i+v_j,j} \\ \wedge_{j>\max(S)} B_{i,j} \Rightarrow \overline{D}_{i+v_j,j} \end{array} \right\} \wedge \bigwedge_j \left\{ \; \overline{C}_{v_j,j} \quad \overline{D}_{v_j,j} \; \right\}$$

$$UBR = \bigwedge_j \left\{ \begin{array}{l} \wedge_{i<\min(E)} \overline{A}_{i,j} \Rightarrow B_{i,j+h_i} \\ \wedge_{\min(E)\leq i\leq \max(E)} \overline{B}_{i,j+h_i} \Rightarrow A_{i,j} \\ \wedge_{\max(E)<i<\min(W)} \overline{A}_{i,j} \Rightarrow D_{i,j+h_i} \\ \wedge_{\min(W)\leq i\leq \max(W)} \overline{A}_{i,j} \Rightarrow D_{i,j+h_i} \\ \wedge_{i>\max(W)} \overline{C}_{i,j} \Rightarrow D_{i,j+h_i} \end{array} \right\}$$

Define $\gamma(H,V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR$. All literals with indices outside the set $\{1,...,m\} \times \{1,...,n\}$ are assumed to have value 1.

Proposition 2. *If P is an HV-convex polyomino in γ, then P is a $2L$-convex polyomino.*

Proof. The proof is straightforward by using the L-paths between each pair of feet (see [14]). □

Algorithm 2
Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$
W.l.o.g assume:$\forall i : h_i \in [1,n], \forall j : v_j \in [1,m], \sum_i h_i = \sum_j v_j$.
For $\min(W), \min(E) = 1, ..., m$ and
$\min(S), \min(N) = 1, ..., n$ **do begin**
If $\gamma(H,V)$ is satisfiable,
then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**.
end
output "failure".

Proof of Algorithm 2. We make the following modifications of the original algorithm of Chrobak and Durr [8] in order to add the geometrical constraints of the class γ. The set Anc gives the feet of suitable size by fixing 8 cells outside the corners A, B, C, D. Thus these cells of the extremities of the feet are in the interior of the polyomino. The set Pos imposes the constraint of the relative positions of feet in the class γ. In particular the cell $A_{(\max(E),1)}$ implies that $\min(W) > \max(E)$ and the cell $C_{(m,\max(N))}$ implies that $\max(N) < \min(S)$ (see Fig.7) . Using the combination of the whole set of clauses, if $\gamma(H,V)$ is satisfiable then we are able to reconstruct an HV-convex with the constraints of the class γ. By Proposition 2 this HV-convex polyomino must be also $2L$-convex. □

5.2. **clauses for the class $\Im_{2L}^{0,0}$.** We code by a 2SAT formula the four geometries that characterize all $2L$-convex polyominoes in the class $\Im_{2L}^{0,0}$ in order to reconstruct them.

$$Pos = \left\{ \; C_{(\min(E),1)} \wedge C_{(m,\max(N))} \wedge A_{1,1} \wedge D_{m,n} \; \right\}$$
$$Cor = \bigwedge_{i,j} \left\{ \begin{array}{llll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$
$$Dis = \bigwedge_{i,j} \left\{ \; X_{i,j} \Rightarrow \overline{Y}_{i,j} : \; for \; symbols \; X, Y \in \{A,B,C,D\}, X \neq Y \; \right\}$$
$$Con = \bigwedge_{i,j} \left\{ \; A_{i,j} \Rightarrow \overline{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \overline{C}_{i+1,j-1} \; \right\}$$

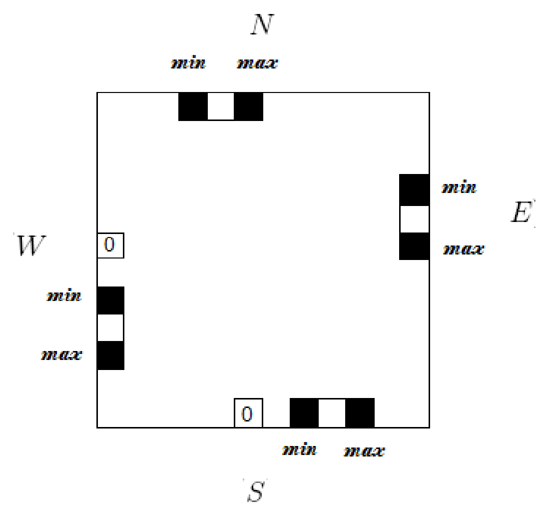

FIGURE 7. Relative position and anchors of the feet in the class γ

$$Anc = \left\{ \begin{array}{l} \overline{A}_{\min(W),1} \wedge \overline{A}_{\min(E),n} \wedge \overline{B}_{\min(W),1} \wedge \overline{B}_{\min(E),n} \wedge \\ \overline{C}_{\min(W),1} \wedge \overline{C}_{\min(E),n} \wedge \overline{D}_{\min(W),1} \wedge \overline{D}_{\min(E),n} \wedge \\ \overline{A}_{1,\min(N)} \wedge \overline{A}_{m,\min(S)} \wedge \overline{B}_{1,\min(N)} \wedge \overline{B}_{m,\min(S)} \wedge \\ \overline{C}_{1,\min(N)} \wedge \overline{C}_{m,\min(S)} \wedge \overline{D}_{1,\min(N)} \wedge \overline{D}_{m,\min(S)} \wedge \\ \overline{A}_{\max(W),1} \wedge \overline{A}_{\max(E),n} \wedge \overline{B}_{\max(W),1} \wedge \overline{B}_{\max(E),n} \wedge \\ \overline{C}_{\max(W),1} \wedge \overline{C}_{\max(E),n} \wedge \overline{D}_{\max(W),1} \wedge \overline{D}_{\max(E),n} \wedge \\ \overline{A}_{1,\max(N)} \wedge \overline{A}_{m,\max(S)} \wedge \overline{B}_{1,\max(N)} \wedge \overline{B}_{m,\max(S)} \wedge \\ \overline{C}_{1,\max(N)} \wedge \overline{C}_{m,\max(S)} \wedge \overline{D}_{1,\max(N)} \wedge \overline{D}_{m,\max(S)} \end{array} \right\}$$

$$LBC = \bigwedge_i \left\{ \begin{array}{l} \wedge_{j<\min(N)} A_{i,j} \Rightarrow \overline{C}_{i+v_j,j} \\ \wedge_{\min(N)\le j\le\max(N)} C_{i+v_j,j} \Rightarrow \overline{A}_{i,j} \\ \wedge_{\max(N)<j<\min(S)} B_{i,j} \Rightarrow \overline{C}_{i+v_j,j} \\ \wedge_{\min(S)\le j\le\max(S)} \overline{B}_{i,j} \Rightarrow C_{i+v_j,j} \\ \wedge_{j>\max(S)} B_{i,j} \Rightarrow \overline{D}_{i+v_j,j} \end{array} \right\} \wedge \bigwedge_j \left\{ \overline{C}_{v_j,j} \quad \overline{D}_{v_j,j} \right\}$$

$$UBR = \bigwedge_j \left\{ \begin{array}{l} \wedge_{i<\min(E)} \overline{A}_{i,j} \Rightarrow B_{i,j+h_i} \\ \wedge_{\min(E)\le i\le\max(E)} \overline{B}_{i,j+h_i} \Rightarrow A_{i,j} \\ \wedge_{\max(E)<i<\min(W)} \overline{A}_{i,j} \Rightarrow D_{i,j+h_i} \\ \wedge_{\min(W)\le i\le\max(W)} \overline{A}_{i,j} \Rightarrow D_{i,j+h_i} \\ \wedge_{i>\max(W)} \overline{C}_{i,j} \Rightarrow D_{i,j+h_i} \end{array} \right\}$$

$$REC = \left\{ A_{\min(W)-1,\min(N)-1} \wedge D_{\max(E)+1,\max(S)+1} \right\}$$

$$GEO1 = \left\{ \begin{array}{l} \overline{A}_{\max(W),\max(S)} \wedge \overline{B}_{\max(W),\max(S)} \\ \wedge \overline{C}_{\max(W),\max(S)} \wedge \overline{D}_{\max(W),\max(S)} \\ \wedge \overline{A}_{\max(E),\max(N)} \wedge \overline{B}_{\max(E),\max(N)} \\ \wedge \overline{C}_{\max(E),\max(N)} \wedge \overline{D}_{\max(E),\max(N)} \end{array} \right\}$$

$$GEO2 = \left\{ \begin{array}{l} \overline{A}_{\min(W),\min(S)} \wedge \overline{B}_{\min(W),\min(S)} \\ \wedge \overline{C}_{\min(W),\min(S)} \wedge \overline{D}_{\min(W),\min(S)} \\ \wedge \overline{A}_{\min(E),\min(N)} \wedge \overline{B}_{\min(E),\min(N)} \\ \wedge \overline{C}_{\min(E),\min(N)} \wedge \overline{D}_{\min(E),\min(N)} \end{array} \right\}$$

$$LGEO3 = \left\{ \begin{array}{l} \overline{A}_{\max(W),\max(S)} \wedge \overline{B}_{\max(W),\max(S)} \\ \wedge \overline{C}_{\max(W),\max(S)} \wedge \overline{D}_{\max(W),\max(S)} \\ \wedge \overline{A}_{\min(E),\min(N)} \wedge \overline{B}_{\min(E),\min(N)} \\ \wedge \overline{C}_{\min(E),\min(N)} \wedge \overline{D}_{\min(E),\min(N)} \end{array} \right\}$$

$$2LGEO3 = \left\{ \begin{array}{l} B_{\min(W),j} \Rightarrow \overline{C}_{\max(E),j-1} \wedge_{j>1} \\ \wedge \overline{X}_{\min(W),\max(N)+1} \\ \wedge \overline{X}_{\max(E),\min(S)-1}, \forall X \in \{A, B, C, D\} \end{array} \right\}$$

$$LGEO4 = \left\{ \begin{array}{l} \overline{A}_{\min(W),\min(S)} \wedge \overline{B}_{\min(W),\min(S)} \\ \wedge \overline{C}_{\min(W),\min(S)} \wedge \overline{D}_{\min(W),\min(S)} \\ \wedge \overline{A}_{\max(E),\max(N)} \wedge \overline{B}_{\max(E),\max(N)} \\ \wedge \overline{C}_{\max(E),\max(N)} \wedge \overline{D}_{\max(E),\max(N)} \end{array} \right\}$$

$$2LGEO4 = \left\{ \begin{array}{l} C_{i,\min(N)} \Rightarrow \overline{B}_{i-1,\max(S)} \wedge_{i>1} \\ \overline{X}_{\max(W)+1,\min(N)} \\ \wedge \overline{X}_{\min(E)-1,\max(S)}, \forall X \in \{A, B, C, D\} \end{array} \right\}$$

In order to reconstruct and to obtain the uniqueness of all $2L$-convex polyominoes in the class $\Im_{2L}^{0,0}$, we use all the combinations of the whole set of clauses that impose the union (or the sub-union) of the 4 geometries starting from all geometries and leading to each single one [14].

$\Im^{0,0}_{2L,geo1,geo2,geo3,geo4}(H, V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge REC \wedge GEO1 \wedge GEO2 \wedge LGEO3 \wedge LGEO4.$

$\Im^{0,0}_{2L,geo2,geo4}(H, V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge REC \wedge GEO2 \wedge LGEO4.$

$\Im^{0,0}_{2L,geo2,geo3}(H, V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge REC \wedge GEO2 \wedge LGEO3.$

$\Im^{0,0}_{2L,geo1,geo4}(H, V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge REC \wedge GEO1 \wedge LGEO4.$

$\Im^{0,0}_{2L,geo1,geo3}(H, V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge REC \wedge GEO1 \wedge LGEO3.$

$\Im^{0,0}_{2L,geo4}(H, V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge REC \wedge LGEO4 \wedge 2LGEO4.$

$\Im^{0,0}_{2L,geo3}(H, V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge REC \wedge LGEO3 \wedge 2LGEO3.$

$\Im^{0,0}_{2L,geo2}(H,V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge REC \wedge GEO2$.

$\Im^{0,0}_{2L,geo1}(H,V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge REC \wedge GEO1$.

Algorithm 3
Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$
W.l.o.g assume:$\forall i : h_i \in [1,n], \forall j : v_j \in [1,m], \sum_i h_i = \sum_j v_j$.
For $\min(W), \min(E) = 1, ..., m$ and
$\min(N), \min(S) = 1, ..., n$ **do begin**
If $\Im^{0,0}_{2L,geo1,geo2,geo3,geo4}(H,V)$ or $\Im^{0,0}_{2L,geo2,geo4}(H,V)$ or ... or $\Im^{0,0}_{2L,geo1}(H,V)$ is satisfiable,
then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**.
end
output "failure".

Proof of Algorithm 3. By Theorem 1 all 2L-convex polyominoes of the class $\Im^{0,0}_{2L}$ are given by combining the 4 geometries. Thus we combine all geometries using suitable set of clauses in order to try to reconstruct a polyomino in the class $\Im^{0,0}_{2L}$. We make the following modifications of the original algorithm of Chrobak and Durr [8] in order to add the geometrical constraints of the class $\Im^{0,0}_{2L}$. The set Pos imposes the constraint of the relative positions of feet in $\Im^{0,0}_{2L}$ (see Fig.8). The set $GEO1$ implies that we put a cell in the interior of the polyomino at the position $(\max(W), \max(S))$ (resp. $(\max(E), \max(N))$) and then by convexity an L-path between $\max(W)$ and $\max(S)$ (resp. $\max(N)$ and $\max(E)$). Thus we have exactly the definition of the first geometry. The set $GEO2$ (resp. $LGEO3$, $LGEO4$) gives the L-paths of the second (resp. third and fourth) geometry. The set $2LGEO3$ (resp. $2LGEO4$) controls the 2L-paths of the third (resp. fourth) geometry (see Fig.9).

In particular, $2LGEO3$ gives the 2L-path between $\min(W)$ and $\max(E)$ by using the clause $B_{\min(W),j} \Rightarrow \bar{C}_{\max(E),j-1}$. This clause says that if the cell $(\min(W), j)$ is in the corner B then the cell $(\max(E), j-1)$ is in the interior of the polyomino. By contraposition, we have the following clause $\wedge_j C_{\max(E),j-1} \implies \bar{B}_{\min(W),j}$, and this means that $(\max(E), j-1)$ is in the corner C while the cell $(\min(W), j)$ is in the interior of the polyomino. We would like to have the 2L-path between $\min(W)$ and $\max(E)$ thus we add two limit cases: $\bar{X}_{\min(W),\max(N)+1} \wedge \bar{X}_{\max(E),\min(S)-1}, \forall X \in \{A, B, C, D\}$ which impose that the cells $(\min(W), \max(N)+1)$ and $(\max(E), \min(S)-1)$ are in the interior of the polyomino (see Fig.10). Thus we have a 2L-path between $\min(W)$ and $\max(E)$. The same technique is applied for the clauses in $2LGEO4$. Remark that the clauses in $2LGEO3$ (resp. $2LGEO4$) is used only to determine the third (resp. the fourth) geometry because all other geometry combinations give L-paths between the feet and thus no reason to satisfy $2LGEO3$ and $2LGEO4$.

Using the conjunction of the whole set of clauses, if one of the $\Im^{0,0}_{2L,geo1,geo2,geo3,geo4}(H,V)$ or $\Im^{0,0}_{2L,geo2,geo4}(H,V)$ or ... or $\Im^{0,0}_{2L,geo1}(H,V)$ is satisfiable then we are able to reconstruct an HV-convex with the constraints of the class $\Im^{0,0}_{2L}$. $\square$

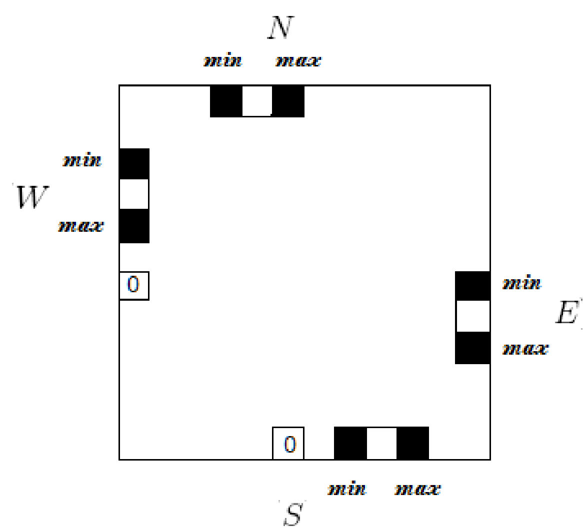

FIGURE 8. Relative position and anchors of the feet in the class $\Im^{0,0}_{2L}$

In order to compute the complexity of this algorthim, one can see that the possible positions of the four feet is $(n-h_m+1)(n-h_1+1)(m-v_1+1)(m-v_n+1) \leq n^2m^2$ (see [3]). And so by imposing the paths in the interior of the polyominoes using the algortihm of Chrobak and Dürr, we obtain the following result.

Theorem 4. *Algorithm 2, 3 solves the reconstruction problem for $2L$-convex polyominoes in γ or $\Im^{0,0}_{2L}$ in time $O(n^3m^3)$.*

5.3. **Reconstruction of the classes γ' and $\Im'^{0,0}_{2L}$ using the horizontal reflexion S_H.** Given two integer vectors $H = (h_1, ..., h_m)$ and $V = (v_1, ..., v_n)$. To reconstruct a polyomino P in the class γ' (resp. $\Im'^{0,0}_{2L}$), one can see that the horizontal reflexion $S_H : (i,j) \longrightarrow (m-i+1, j)$, $\forall i,j \in \{1,...,m\} \times \{1,...,n\}$ sends the projection vectors (H,V) to $(\tilde{H},V)$, where $\tilde{H} = (h_m, ..., h_1)$. Now from the two vectors of projections $(\tilde{H},V)$, one can reconstruct the polyomino P in the class γ (resp. $\Im^{0,0}_{2L}$) and then by the horizontal reflexion S_H, we reconstruct P in the class γ' (resp. $\Im'^{0,0}_{2L}$).

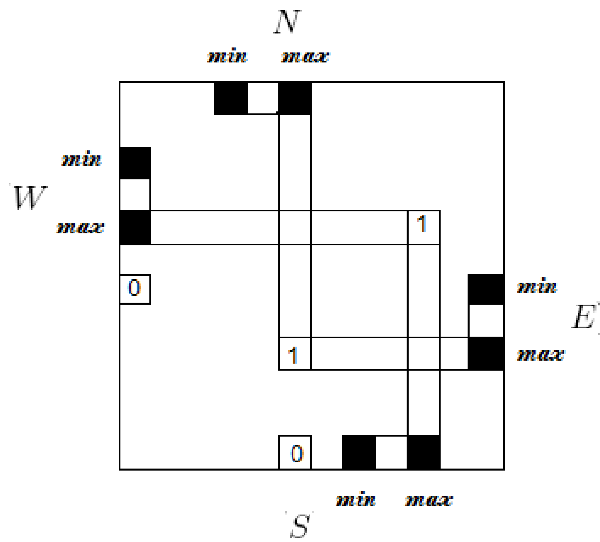

FIGURE 9. First geometry.

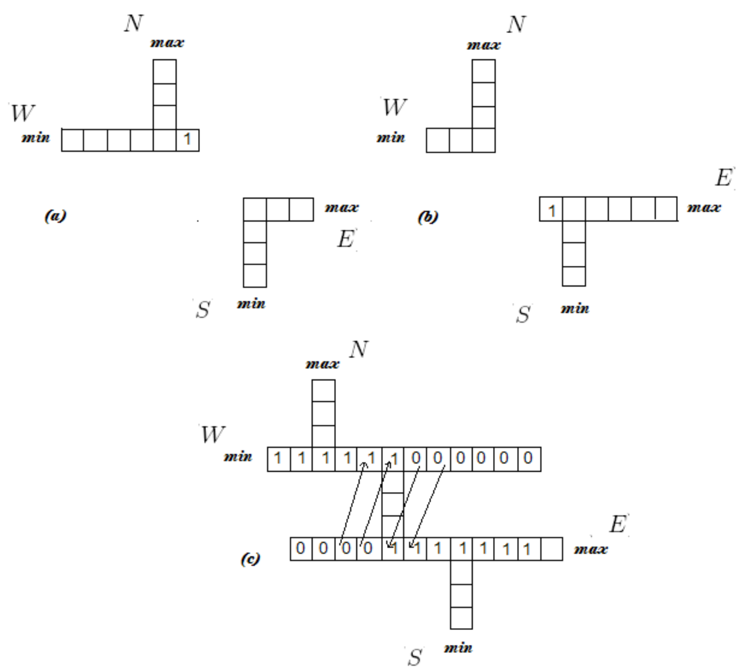

FIGURE 10. 2LGEO3: (a) and (b) are the limite cases for the third geometry in the class $\Im_{2L}^{0,0}$

N, S	W, E
$S \longrightarrow N$	$W \longrightarrow W$
$N \longrightarrow S$	$E \longrightarrow E$
	$\leq \Longleftrightarrow \geq$
$\min \longrightarrow \min$	$\min \longrightarrow \max$
$\max \longrightarrow \max$	$\max \longrightarrow \min$

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