On the shape of permutomino tiles

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Abstract

In this paper we explore the connections between two classes of polyominoes, namely the *permutominoes* and the *pseudo-square* polyominoes. A *permutomino* is a polyomino uniquely determined by a pair of permutations. Permutominoes, and in particular *convex permutominoes*, have been considered in various kinds of problems such as: enumeration, tomographical reconstruction, and algebraic characterization.

On the other hand, *pseudo-square* polyominoes are a class of polyominoes tiling the the plane by translation. The characterization of such objects has been given by Beauquier and Nivat, who proved that a polyomino tiles the plane by translation if and only if it is a pseudo-square or a pseudo-hexagon. In particular, a polyomino is pseudo-square if its boundary word may be factorized as $XY\widehat{X}\widehat{Y}$, where $\widehat{X}$ denotes the path X traveled in the opposite direction.

In this paper we relate the two concepts by considering the pseudo-square polyominoes which are also convex permutominoes. By using the Beauquier-Nivat characterization we provide some geometrical and combinatorial properties of such objects, and we show for any fixed X, each word Y such that $XY\widehat{X}\widehat{Y}$ is pseudo-square is prefix of an infinite word Y_∞ with period $4|X|_N|X|_E$. Also, we show that $XY\widehat{X}\widehat{Y}$ are centrosymmetric, i.e. they are fixed by rotation of angle π. The proof of this fact is based on the concept of *pseudoperiods*, a natural generalization of periods.

Keywords: Polyominoes, Permutominoes, Tilings, Palindromes, Pseudoperiods

1. Introduction

In the last years the problem of tiling planar surfaces has shown interesting mathematical aspects connected with computational theory, mathematical logic, discrete geometry, and also physics.

One possible explanation for so wide usage of tilings (also in different disciplines) is their capability to generate very complex configurations. As a confirmation confirmation of this assertion, let us just mention a classical result of Berger [5]: given a set of tiles, it is not decidable whether

there exists a tiling of the plane which involves all its elements. This result has been achieved by constructing an aperiodic set of tiles, and successively it has been strengthened by Gurevich and Koriakov [13] to the periodic case.

Further interesting results have been achieved by restricting the class of sets of tiles only to those having one single element. In particular Wijshoff and Van Leeuwen [17] considered the *exact polyominoes* (i.e. polyominoes which tile the plane by translation) and proved that the problem of recognizing them is decidable. The same problem was studied in [4], Beauquier and Nivat from a purely geometrical point of view and they found a characterization of all the exact polyominoes by using properties of the words which describe their boundaries. In particular they stated that the boundary word coding these polyominoes shows a pattern $XYZ\overline{X}\,\overline{Y}\,\overline{Z}$, called a *pseudo-hexagon*, where one of the variable may be empty in which case the pattern $XY\overline{X}\,\overline{Y}$ is called a *pseudo-square*. However, in their work, the authors were not concerned with the combinatorial properties of these structures.

Successively, in [2] the authors investigated some combinatorial and enumerative problems on exact polyominoes, focusing on the subclass of pseudo-square convex polyominoes. The results in [2] have been mostly achieved using the word representation for the boundary of the polyomino, and relying on several tools and properties arising from combinatorics on words. Later, following this research line, and using similar approaches, in [6] it was proved that if the tile is a parallelogram polyomino, then there are at most two distinct ways in which it can fill the plane by translation; in [8] the authors present a linear time algorithm for detecting pseudo-square polyominoes.

In this paper we introduce a new and challenging research line which aims at exploring the relations between exact polyominoes and *permutominoes*, a recently studied class of polyominoes arising in several mathematical and combinatorial problems and strictly connected with permutations.

Let us recall that a permutomino of size n is a polyomino determined by particular pairs (π_1, π_2) of permutations of length $n+1$ (see, for instance, Fig. 4). An equivalent definition for a permutomino is that, for each abscissa (ordinate) between 1 and n, there is exactly one vertical (horizontal) side in the boundary with that coordinate.

Permutominoes were introduced in [15], and then considered by F. Incitti while studying the problem of determining the $\widetilde{R}$-polynomials associated with a pair (π_1, π_2) of permutations [14]. In recent years, a particular class of permutominoes, namely convex permutominoes, and their associated permutations, have been widely studied [3, 9]. Some enumerative results known about convex permutominoes, which may be useful in this paper, are the following [9]:

i. the number of *parallelogram permutominoes* of size n is equal to the n-th *Catalan number*,

$$C_n = \frac{1}{n+1}\binom{2n}{n};$$

ii. the number of *convex permutominoes* of size n is:

$$2\,(n+3)\,4^{n-2}\;-\;\frac{n}{2}\binom{2n}{n}\qquad n \geq 1. \tag{1}$$

We also recall from [3] that permutations defining convex permutominoes are closely related with the so-called *square permutations* (or *convex permutations*), recently considered by several authors [16].

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In order to attain our objective, we start by considering pseudo-square polyominoes which are also convex permutominoes, i.e. *pseudo-square convex permutominoes* (*psc*-permutominoes for short). This problem raises interesting properties since it relates to combinatorics on words and classical problems on polyominoes. Moreover these objects are uniquely defined by permutations, thus they give us an effective way to tile the plane using permutations. Hence this problem shows strict relations with the vast combinatorics on pattern avoiding permutations [3].

Our first objective is to find a characterization of the boundary words of *psc*-permutominoes. More precisely, we are interested in the problem of establishing if, for a given word X, there is at least a word Y which is *compatible* with X, that is, $XY\widehat{X}\widehat{Y}$ represents a *psc*-permutomino.

Two further goals - which are strictly connected to the previous one - are to obtain the enumeration of *psc*-permutominoes according to the semi-perimeter, and to characterize permutations defining *psc*-permutominoes.

Using the Beauquier-Nivat characterization, and the results from [2], we show that for any given X there are at most two different words of minimal length Y and Y' compatible with X. These words are such that $|Y| = |Y'|$, and Y' can be obtained from Y by means of renaming of letters. Moreover, they individuate the *growing direction* of the permutomino, so we distinguish among the *upward* and the *downward* growing direction of a *psc*-permutomino and study these two cases separately.

In studying the *psc*-permutominoes with upward growing direction, we prove that X uniquely determines an infinite word Y_∞, and that every word Y which is up-compatible with X (i.e. $XY\widehat{X}\widehat{Y}$ is a *psc*-permutomino with upward growing direction) is a prefix of Y_∞. Moreover we show that Y_∞ has period $4\,|X|_E\,|X|_N$. The same results can be reformulated in the case of down-growing direction.

As an addition to the periodic nature of Y_∞, we show that *psc*-permutominoes are centrosymmetric, i.e. they are invariant under a rotation of angle π. The proof of this fact is based on the concept of *pseudoperiod*, a natural generalization of periods introduced in [1]. From a combinatorial point of view, it reveals a remarkable family of words who seem to share properties with *central words*.

Several challenging problems remain open and they are briefly presented and discussed in the final section. As a matter of fact, we point out that the same problems considered in this paper remain open if we go to consider the class of pseudo-hexagonal convex permutominoes, although some generalizations of our results seem to be possible.

We point out that this paper is an extended version of the abstract [7] presented at the 16th Conference on Discrete Geometry for Computer Imagery 6-8 April 2011 Nancy, France. Compared to that work, here we give a detailed study of the words X and Y leading to *psc*-permutominoes, we show that both X and Y must be palindromic words, which was only conjectured in [7], we provide new results, a new conjecture about the length of the word X of a *psc*-permutomino, and some examples concerning non convex pseudo square permutominoes, which deserve some some further inspection.

2. Two classes of polyominoes

In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square whose vertices have integer coordinates, and a *polyomino* is a finite connected union of cells having no cut point (see Fig. 1(a)). Polyominoes are defined up to translations, and we deal with polyominoes without "holes", i.e. polyominoes whose boundary is

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a single loop. A *column* (resp. *row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells whose centers lie on a vertical (resp. horizontal) line. A polyomino is said to be *column-convex* (resp. *row-convex*) when its intersection with any vertical (resp. horizontal) line is connected. A polyomino is convex if it is both column and row convex (see Fig. 1(b,c)).
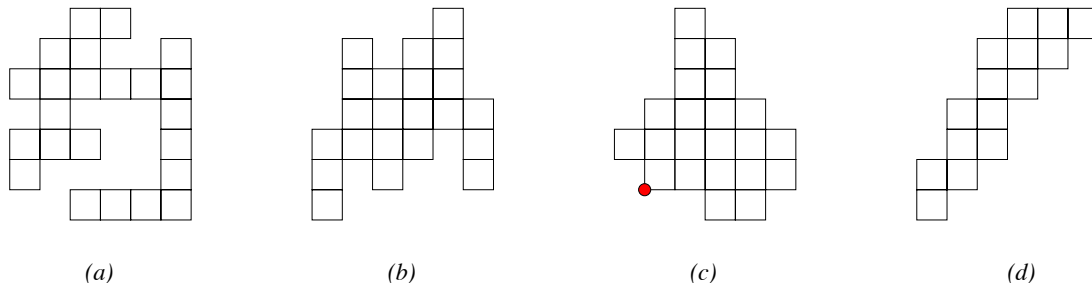

| (a) | (b) | (c) | (d) |

Figure 1: A polyomino (a), a column-convex polyomino (b), a convex polyomino (c).

A particular subclass of the class of convex polyominoes consists of the *parallelogram* polyominoes, defined by two lattice paths that use north (vertical) and east (horizontal) unitary steps, and intersect only at their origin and extremity. These paths are commonly called the *upper* and the *lower path* (see Fig. 1 (d)).

In what follows we will use to represent the boundary of a polyomino by a *boundary word* defined on the alphabet $\Sigma = \{N, S, E, W\}$, where N (resp. E, S, W) stands for the north (resp. east, south, west) unit step. The word representing a polyomino is obtained simply by following its boundary from a starting point in a clockwise orientation. For instance, the polyomino in Fig. 1(c), starting from the highlighted point, is represented by the word

$$X = NWNENENNNESESSSESESSWSWWNWW.$$

Moreover, if $X = x_1 x_2 \ldots x_r$ is a word, where $x_i \in \{N, E, S, W\}$, then the complement $\overline{X} = \overline{x}_r \ldots \overline{x}_2 \overline{x}_1$ is defined by $\overline{N} = S$, $\overline{S} = N$, $\overline{W} = E$, and $\overline{E} = W$. We define the length of a word X by $|X| = r$, and by $|X|_{x_i}$ the number of occurrences of the letter x_i in X. We observe that in any polyomino the length of the boundary word coincides with its perimeter.

2.1. Polyominoes that tile the plane

In [11], Beauquier and Nivat studied the class of exact polyominoes, i.e. polyominoes that tile the plane by translation. They found a characterization of all the exact polyominoes by using properties of the words which describe their boundaries. In particular they stated that the boundary words coding these polyominoes show a pattern $XYZ\widehat{X}\,\widehat{Y}\,\widehat{Z}$, called a *pseudo-hexagon*, where one of the variable may be empty in which case the pattern $XY\widehat{X}\widehat{Y}$ is called a *pseudo-square*.

In the rest of the paper, we will deal with pseudo-square convex polyominoes (briefly *psc-polyominoes*). As an example, the polyomino in Fig. 2 (a) is a *psc*-polyomino, and the decomposition

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of its boundary word is

$$
\begin{aligned}
X &= NNENEEN, \\
Y &= EE\overline{N}E = EESE, \\
\widehat{X} &= \overline{NEENENN} = SWWSWSS, \\
\widehat{Y} &= \overline{ENEE} = WNWW.
\end{aligned}
$$

In [2] the authors study the problem of enumerating *psc*-polyominoes according to the semi-perimeter, i.e. the length of the upper (lower) path. In order to obtain the enumeration, they determined some combinatorial properties about *psc*-polyominoes, some of which will be recalled here since they will be useful through the paper.
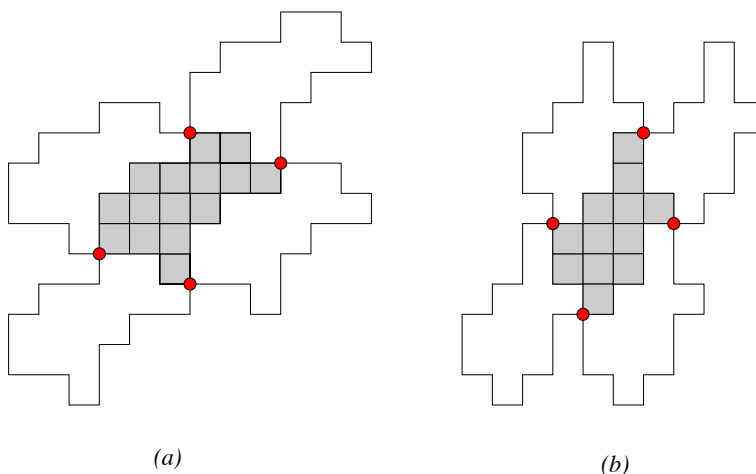

(a)   (b)

Figure 2: (a) a *psc*-polyomino (with highlighted cells) having only a decomposition of the first type, and the corresponding tiling of the plane; (b) a *psc*-polyomino having one decomposition of the second type, and the corresponding tile of the plane.

In order to simplify the enumeration, the whole class of *psc*-polyominoes can be partitioned into two sub-classes, according to the polyominoes' shape:

1. Pseudo-square parallelogram polyominoes (briefly, *psp*-polyominoes). A complete characterization of the elements of this class is stated by the following [2]:

   Proposition 1. *If $XY\widehat{X}\widehat{Y}$ is a decomposition of the boundary word of a pseudo-square parallelogram polyomino, then XY encodes its upper path, and YX its lower path. Moreover, the decomposition is unique, X starts and ends with N, and Y starts and ends with E.*

2. *Psc*-polyominoes which are not parallelogram. In this case, we can have at most two decompositions which are strictly related to each other:

(a) the first one is such that the path X of the decomposition starts from the lowest point in the leftmost column and ends in the leftmost point in the uppermost row (see Figure 3 (a)), then it has the form $N^+(E \vee N)^*N^+$. The starting point of such decomposition is denoted by A; we observe that also parallelogram polyominoes have this kind of decomposition;

(b) the second one is such that the path X starts from the uppermost point in the leftmost column and ends in the rightmost point in the uppermost row, (see Figure 3 (b), (c)). The starting point of such decomposition is called B.

We also observe that each psc-polyomino with a decomposition of the second type can be transformed – under an horizontal reflection – into another psc-polyomino whose decomposition is of the first type (note that the two polyominoes may coincide). So, without loss of generality we will only consider psc-polyominoes of with a decomposition of the first type, starting from A. Hence, from now on, the word X encoding the boundary of a generic psc-permutomino is assumed to have the form $N^+(E \vee N)^*N^+$.

According to the purposes of our paper, we are led to formulate the previous statements in a slightly different way.

Proposition 2. *Let $XY\widehat{X}\widehat{Y}$ be the decomposition of the boundary word of a psc-polyomino, and let us assume without loss of generality that the word X has the form $N^+(E \vee N)^*N^+$. The following two cases may occur:*

 - *Y is a north-east oriented monotone path (i.e. it uses only N and E steps), these are precisely the pseudo-square parallelogram polyominoes.*

 - *Y is a south-east oriented monotone path (i.e. it uses only S and E steps).*

According to the two cases we say that the polyomino has *upward (resp. downward) growing direction*. For instance, Figure 3 (a) depicts a psp-polyomino, having upward growing direction, while Figure 3 (b), (c) depict two psc-polyominoes having downward growing direction.

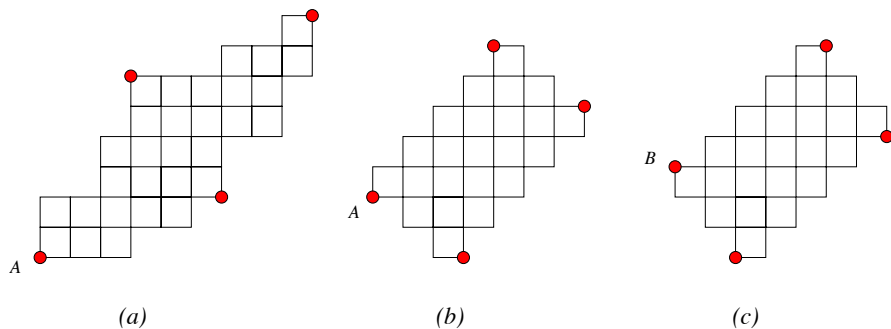(a)   (b)   (c)

Figure 3: (a) a psp-polyomino and its unique decomposition; in (b), (c) a non parallelogram psc-polyomino admitting decompositions of the first and of the second type; the starting points are A and B, respectively.

2.2. Polyominoes determined by permutations

Let P be a polyomino without holes, having n rows and n columns, $n \geq 1$, and we assume without loss of generality that the south-west corner of its minimal bounding rectangle is placed at $(1,1)$. Let $\mathcal{A} = \big(A_1, \ldots, A_{2(r+1)}\big)$ be the list of its vertices (i.e., corners of its boundary) ordered in a clockwise direction starting from the lowest leftmost vertex, with $A_i = (x_i, y_i)$. We say that P is a *permutomino* if $\mathcal{P}_1 = (A_1, A_3, \ldots, A_{2r+1})$ and $\mathcal{P}_2 = (A_2, A_4, \ldots, A_{2r+2})$ represent two permutations of length n. Obviously, if P is a permutomino, then $r = n$, see Fig. 4. As an immediate consequence we have that $m = n$, and so a permutomino has n rows and n columns (n is called its *size*), and the number of its vertices is $2(n+1)$. The two sets $\mathcal{P}_1$ and $\mathcal{P}_2$ can be regarded as two permutation matrices of $[n+1] = \{1, 2, \ldots, n+1\}$ having no common points; we indicate the permutations associated with them by $\pi_1(P)$ and $\pi_2(P)$, respectively (see Fig. 4). They are called the *permutations associated with P*.

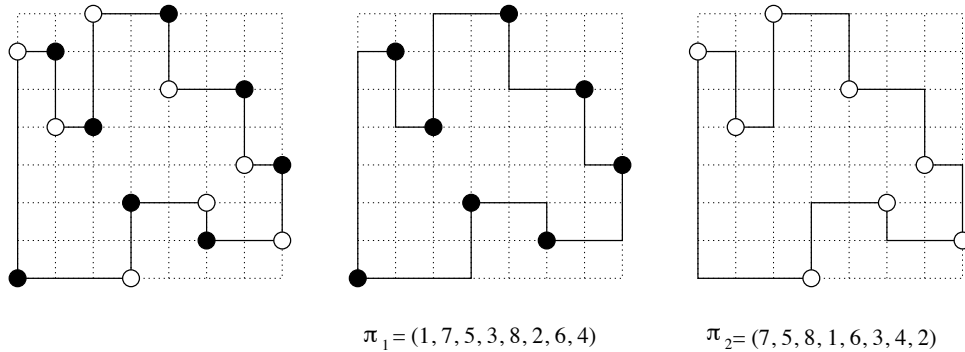$$\pi_1 = (1, 7, 5, 3, 8, 2, 6, 4) \qquad \pi_2 = (7, 5, 8, 1, 6, 3, 4, 2)$$

Figure 4: A permutomino and the two associated permutations. The dotted black vertices represent the elements of $\mathcal{P}_1$, while the circled vertices the elements of $\mathcal{P}_2$.

From the definition any permutomino P of size n has the property that, for each abscissa (ordinate) between 1 and n there is exactly one vertical (horizontal) side in the boundary of P with that coordinate. It is simple to observe that this property is also a sufficient condition for a polyomino to be a permutomino.

Several combinatorial and enumerative properties of convex permutominoes have been determined in [9, 3]. Here, for brevity's sake we just recall that a complete characterization of permutations associated with convex permutominoes has been given in [3]: precisely, the set

$$\{\pi_1(P), \pi_2(P) : P \text{ is a convex permutomino of size } n\}$$

coincides with the set of square permutations of length $n + 1$ [16].

3. Convex permutominoes tiling the plane

In this section we study convex permutominoes which are also pseudo-square polyominoes, that we call *pseudo-square convex permutominoes* (briefly, *psc-permutominoes*). To the authors' opinion, these are interesting objects to be studied since they constitute a first step towards the more general problem of tiling the plane by permutations. Moreover, we will show that these objects

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present remarkable geometrical and combinatorial aspects independent of their relationship with permutations, and in particular some features related to combinatorics on words.

In the study of *psc*-permutominoes we plan to face the following three general problems:

1. characterizing the combinatorial properties of the words X and Y such that $XY\widehat{X}\widehat{Y}$;

2. enumerating *psc*-permutominoes according to the semi-perimeter;

3. characterizing permutations associated with *psc*-permutominoes.

In this paper we will be concerned with the first of these problems, remarking that our results give us an important ideas for tackling the other two problems.

First, we recall that we assume that the word X encoding the boundary of a generic *psc*-permutomino has the form $N^+(E \vee N)^*N^+$. Moreover, we will work indifferently with the word or the path representation of X and Y.

We say that a binary word X is *compatible* with Y if the word $XY\widehat{X}\widehat{Y}$ represents the boundary of a *psc*-polyomino. So, for instance, we have that the word $X = NNENN$ is compatible with $Y = EENEE$ (see Fig. 5 (a)), with $Y' = EESEE$ (see Fig. 5 (b)), and with $Y'' = EESEESSEESSSEESSEESEE$ (see Fig. 5 (c)). A word X of the form $N^+(E \vee N)^*N^+$ is said to be *convex-exact* (briefly, $c - exact$) if there is at least a word Y such that $XY\widehat{X}\widehat{Y}$ is a *psc*-polyomino.

We easily observe that *psc*-permutominoes inherit some basic properties from being pseudo-square convex polyominoes, and in particular we may adapt the statement of Proposition 2, as follows:

Proposition 3. *For each word X of the form $N^+(E \vee N)^*N^+$ there exist at most two different words Y and Y', having minimal length, compatible with X such that:*

1. *neither Y nor Y' is a prefix of the other,*

2. *if there exists Y'' compatible with X, then either Y or Y' is a prefix of Y''.*

It is also clear that, in a pseudo-square convex permutomino, the word X completely determines the related words Y and Y'. For instance referring to Fig. 5, the word $X = NNENN$ is compatible with $Y = EENEE$, and $Y' = EESEE$ while $Y'' = Y'SSEESSSEESSY'$.

We observe that the previous proposition does not hold if we relax the convexity constraint. In Fig. 6, two non convex *ps*-permutominoes are depicted: for each of them, the chosen word Y'' compatible with X, has neither Y nor of Y' as prefix. Furthermore, the word X of the *ps*-permutomino in (a) is c-exact, while that of (b) is not, giving some relevance to the study of the class of words X that are compatible with a generic word Y, and still provide a pseudo square tiling of the plane.

According to Proposition 3, and similarly to what we have done for *psc*-polyominoes, the first classification of *psc*-permutominoes can be given by the concept of *growing direction*. We say that a *psc*-permutomino has *upward* growing direction if the word Y has the form $E^+(N \vee E)^*E^+$. Similarly, we say that it has *downward* growing direction if Y has the form $E^+(S \vee E)^*E^+$. We remark that $X = NNENN$ is the word of minimal length for which there exists at least a *psp*-permutomino in both growing directions (see Fig. 5).
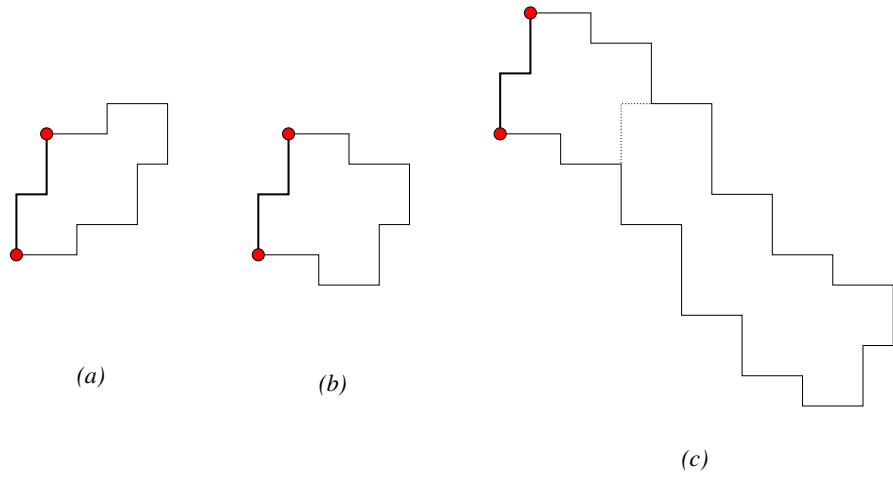
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Figure 5: Three *psc*-permutominoes obtained from the same word $X = NNENN$. The permutomino in (*a*) has an upward growing direction, and it is a *psp*-permutomino, while those in (*b*) and (*c*) have a downward growing direction.
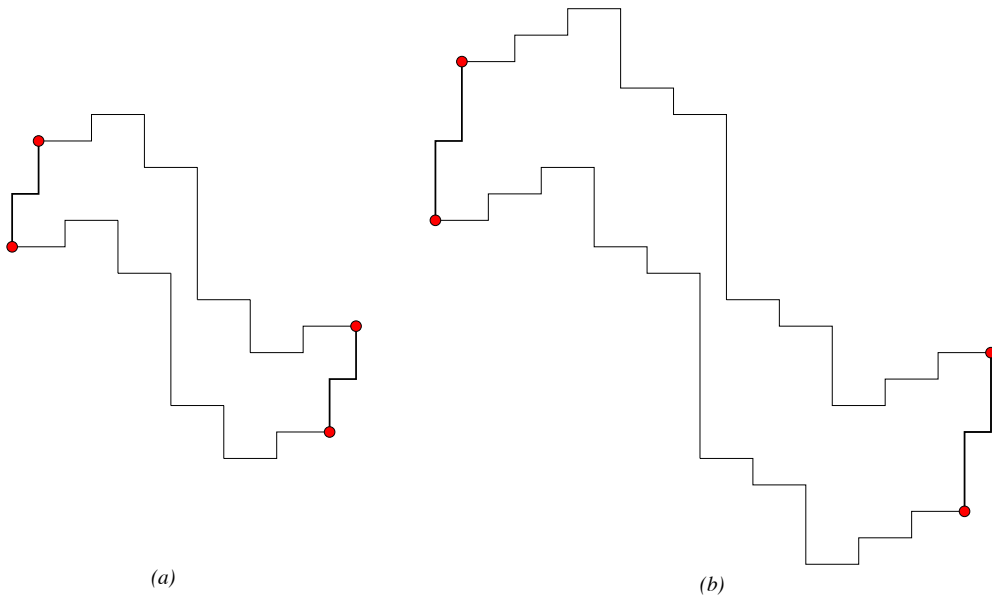

Figure 6: Two non convex *ps*-permutominoes obtained from the c-exact word $X = NNENN$, (*a*), and from the word $X = NNNENNN$, (*b*), that is non c-exact.

In the next sections we will study separately the *psc*-permutominoes having upward and downward growing direction. According to this, we say that X and Y are *up*-compatible (resp. *down-*

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compatible) with X if $XY\widehat{X}\widehat{Y}$ is a *psc*-permutomino having upward (resp. downward) growing direction.

3.1. Upward growing direction: pseudo-square parallelogram permutominoes

Pseudo-square convex permutominoes having upward growing direction have a very simple characterization which they directly inherit from pseudo-square convex polyominoes, precisely from Proposition 2, and is stated by the following.

Proposition 4. *The psc-permutominoes having upward growing direction are precisely the pseudo-square parallelogram permutominoes.*

We observe that, starting from X, and given a positive integer n, there is a unique path $Y(n)$ of length n, and using steps E and N, such that the object $X(n)$ obtained by concatenating $Y(n)$ at the beginning and at the end of X has exactly one side for each abscissa and ordinate between 1 and n. Similarly, letting n tend to infinity, we obtain a unique infinite path Y_∞, and an object $X(\infty)$ having exactly one side for each abscissa and ordinate greater than 1 (see Fig. 7). In $X(n)$, the path Y_∞ starting from the end (resp. beginning) of X will be called the *upper* (resp. *lower*) path of $X(n)$. The following property is straightforward.

Proposition 5. *Every word Y which is up-compatible with X is a prefix of Y_∞.*

In what follows, we prove that the word Y_∞ is periodic, and we provide a simple algorithm to determine a prefix of length n of Y_∞ from X.

In order to study the shape of the compatible paths, we need to introduce some useful functions on permutominoes.

First, we consider a natural function on subsets of $\mathbb{N}$ defined as follows. Let $S \subseteq \mathbb{N}$. We define

$$\operatorname{mex}(S) = \min(\overline{S}),$$

where $\overline{S} = \mathbb{N} - S$. For instance, if $S = \{0, 1, 4\}$ then $\overline{S} = \{2, 3, 5, 6, \ldots\}$ so that $\operatorname{mex}(v) = 2$. This definition appears in some works of Fraenkel and it is related to the Nim game, Sturmian and balanced sequences [10].

A *psc*-permutomino is in particular a permutomino, so that there must be exactly one vertical segment and one horizontal segment per row and per column. Moreover, since X and Y are monotones, the words Y_∞ is completely determined by X and can be constructed iteratively as follows: At each step, we move upward (resp. in the right direction) to the smallest ordinate (resp. abscissa). Therefore, to keep track of the visited rows and visited columns, it is convenient to construct four sequences V_i, H_i, p_i and q_i defined recursively as follows:

Base: (i) V_0 is the set of ordinates where the vertical segments of X start or end;

(ii) H_0 is the set of abscissas where the horizontal segments of X start or end;

(iii) $p_0 = 0$;

(iv) $q_0 = 0$;

Induction step: (i) $V_{i+1} = V_i \cup \{\operatorname{mex}(V_i), \operatorname{mex}(V_i) - p_i + \max(V_i)\}$;

(ii) $H_{i+1} = H_i \cup \{\operatorname{mex}(H_i), \operatorname{mex}(H_i) - q_i + \max(H_i)\}$;
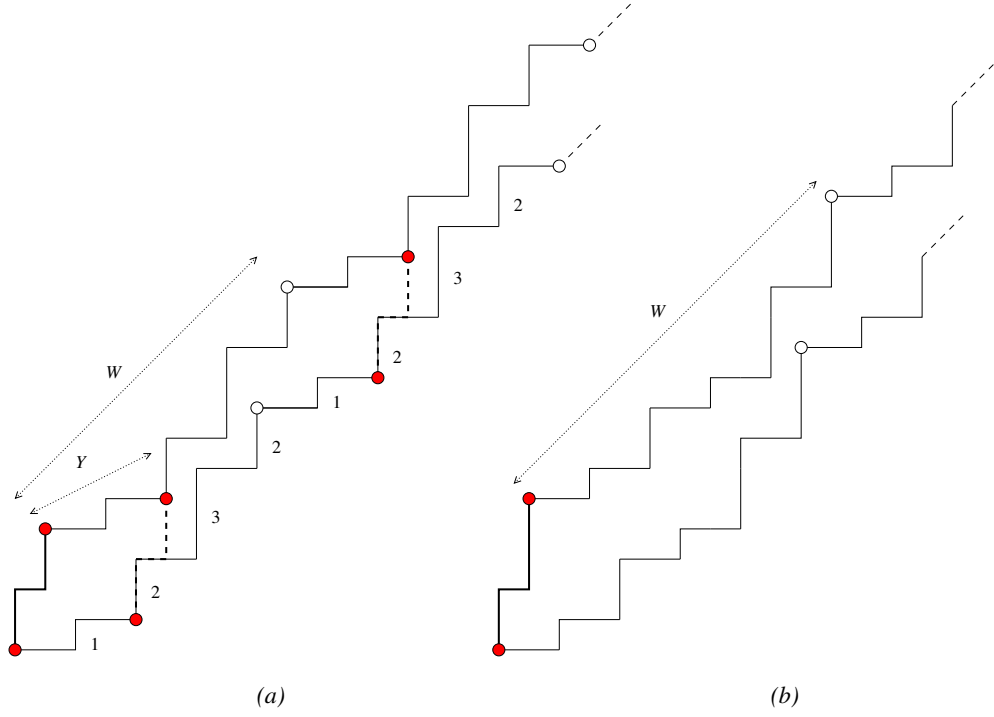
Figure 7: The path $X(\infty)$ does not cross itself: (a) we have no pseudo-square parallelogram permutomino; (b) there is a word Y of length less than the period W of Y_∞ such that X is up-compatible with Y, and we have a pseudo-square parallelogram permutomino.

(iii) $p_{i+1} = \text{mex}(V_i)$;

(iv) $q_{i+1} = \text{mex}(H_i)$;

Example 1. *The path $X = NNENN$ of Figure 7 yields the sets $V_0 = \{0, 2, 4\}$ and $H_0 = \{0, 1\}$. Moreover, $p_0 = q_0 = 0$. Table 1 gives the first values of the sequences p_i, q_i, V_i and H_i. One can reconstruct the vectors V_Y and H_Y from Table 1 using the formulas $V_Y[i] = p_{i+1} - p_i$ and $H_Y[i] = q_{i+1} - q_i$. In this example, $V_Y = (1, 2, 3, 2, 1, 2, 3 \ldots)$ and $H_Y = (2, 2, 2, 2, 2, 2, 2, \ldots)$, which suggests that V_Y has period 4 and H_y has period 1. Therefore,*

$$Y_\infty = (E^2 N^1 E^2 N^2 E^2 N^3 E^2 N^2)(E^2 N^1 E^2 N^2 E^2 N^3 E^2 N^2) \cdots,$$

implying that Y_∞ has period $(1 + 2 + 3 + 2) + (2 + 2 + 2 + 2) = 8 + 8 = 16$.

Surprisingly, the study of *psc*-permutominoes leads naturally to the concept of pseudoperiod. More precisely, let w be some binary word. We say that p is a period of w if $w[i] = w[i + p]$ whenever i and $i + p$ are valid indices. In the same spirit, the number p is called *antiperiod* of w if $w[i] = \overline{w[i + p]}$. This concept has been studied in [1] where the authors prove an extended version of Fine and Wilf's theorem for two periods within some given word. It is easy to verify that

i	p_i	q_i	V_i	H_i
0	0	0	$\{0,2,4\}$	$\{0,1\}$
1	1	2	$\{0,1,2,4,5\}$	$\{0,1,2,3\}$
2	3	4	$\{0,1,2,3,4,5,7\}$	$\{0,1,2,3,4,5\}$
3	6	6	$\{0,1,2,3,4,5,6,7,10\}$	$\{0,1,2,3,4,5,6,7\}$
4	8	8	$\{0,1,2,3,4,5,6,7,8,10,12\}$	$\{0,1,2,3,4,5,6,7,8,9\}$
5	9	10	$\{0,1,2,3,4,5,6,7,8,9,10,12,13\}$	$\{0,1,2,3,4,5,6,7,8,9,10,11\}$
6	11	12	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,15\}$	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13\}$

Table 1: First values of p_i, q_i, V_i and H_i for the path $X = NNENN$.

if the word w overlaps itself with delay d, then d is a period of w. Similarly, if w overlaps with $\overline{w}$ with delay d, then d is an antiperiod of w.

The sequences V_i and H_i satisfy the following properties:

Lemma 1. *Let $i \geq 1$ be an integer.*

(i) $\max(V_{i+1}) - \mathrm{mex}(V_i) = \max(V_i) - \mathrm{mex}(V_{i-1})$;

(ii) $\max(V_i) - \mathrm{mex}(V_{i-1}) = |X|_N$.

(iii) $\max(H_{i+1}) - \mathrm{mex}(H_i) = \max(H_i) - \mathrm{mex}(H_{i-1})$;

(iv) $\max(H_i) - \mathrm{mex}(H_{i-1}) = |X|_E$.

PROOF. (i) Since $\mathrm{mex}(V_i) - p_i + \max(V_i) > \max(V_i), \mathrm{mex}(V_i)$, one obtains

$$
\begin{aligned}
\max(V_{i+1}) &= \mathrm{mex}(V_i) - p_i + \max(V_i) \\
&= \mathrm{mex}(V_i) - \mathrm{mex}(V_{i-1}) + \max(V_i),
\end{aligned}
$$

yielding the equation. (ii) It follows from (i) that

$$
\begin{aligned}
\max(V_i) - \mathrm{mex}(V_{i-1}) &= \max(V_1) - \mathrm{mex}(V_0) \\
&= \mathrm{mex}(V_0) - p_0 + \max(V_0) - \mathrm{mex}(V_0) \\
&= \max(V_0) - p_0 \\
&= |X|_N.
\end{aligned}
$$

(iii) and (iv) are proved similarly. $\square$

Roughly speaking, at each iteration, we add two new elements $\mathrm{mex}(V_i)$ and $\mathrm{mex}(V_i) - p_i + \max(V_i)$ and Lemma 1 states that those elements are apart by distance $|X|_N$. The idea is the same for the horizontal steps. Therefore, it is useful to define auxiliary objects that keep track of which element is obtained from $\mathrm{mex}(V_i)$ and which is obtained from $\mathrm{mex}(V_i) - p_i + \max(V_i)$. For every natural j, let $f(j) = 0$ if $j \in V_0$, otherwise let $f(j)$ be the integer such that $j \in V_{f(j)}$ but $j \notin V_{f(j)-1}$. Clearly, $f(j)$ has exactly two preimages whenever $j \neq 0$. Moreover, for every integer $j \geq 0$, we set

$$
w(j) = \begin{cases} \ell & \text{if } j = 0 \text{ or } j = \min(f^{-1}(f(j))), \\ u & \text{if } j \in V_0 - \{0\} \text{ or } j = \max(f^{-1}(f(j))), \end{cases}
$$

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Informally speaking, the word w may be interpreted as follows: If $w(j) = u$, then the corner at height j in the permutomino is in the upper part XY, while if $w(j) = \ell$, then the corner at height j is in the lower part $\widehat{X}\widehat{Y}$. The function w present antiperiodic properties as well:

Lemma 2. *Let $j \geq 0$ be an integer.*

(i) *If $w(j) = \ell$, then $w(j + |X|_N) = u$.*

(ii) *If $w(j) = u$, then $w(j + |X|_N) = \ell$.*

PROOF. It follows directly from Lemma 1, since, at every step, we add ℓ and u separated by distance $|X|_N$. □

As a consequence, the following statements hold:

Lemma 3. *Let $j \geq 0$ be an integer.*

(i) *If $w(j) = \ell$, then $w(j + 2|X|_N) = \ell$.*

(ii) *If $w(j) = u$, then $w(j + 2|X|_N) = u$.*

(iii) *The word w has antiperiod $|X|_N$.*

(iv) *The word w has period $2|X|_N$.*

PROOF. Follows directly from Lemma 2. □

A last observation is useful:

Lemma 4. *Let w' be a factor of length $2|X|_N$ of w. Then $|w'|_\ell = |w'|_u = |X|_N$.*

PROOF. Since w has antiperiod $|X|_N$, if we take any window w' of length $2|X|_N$, then every occurrence of ℓ in the first half of w' is in correspondence with a u in the second half, while every u is in correspondence with some ℓ in the second half, by Lemma 2. Thus the number of ℓ's and u's must be the same. □

Clearly, Lemmas 2, 3 and 4 are extended when considering horizontal steps.
Using the previously defined tools, we are now ready to prove one of our main results.

Theorem 1. *For any given $X = N^+(E \vee N)^* N^+$, the number $4|X|_N|X|_E$ is a period of Y_∞.*

Remark 1. *The period needs not be the minimal one.*

PROOF. We first prove that the vectors V_Y and H_Y have periods $|X|_N$ and $|X|_E$ respectively.
Let $j \geq 0$ be an integer. By construction, we have

$$V_Y[j] = p_{j+1} - p_j = \text{mex}(V_j) - \text{mex}(V_{j-1}),$$

where $\text{mex}(V_{j-1})$ and $\text{mex}(V_j)$ are two consecutive occurrences of the letter ℓ in w. We know from Lemma 3 that the word w has period $2|X|_N$, so that there are two consecutive occurrences $\text{mex}(V_{j-1}) + 2|X|_N$ and $\text{mex}(V_j) + 2|X|_N$ of the letter u in w. But Lemma 4 guarantees that the

number of ℓ's and u's is the same in any window of length $2|X|_N$ of w, so that there are exactly $|X|_N - 1$ occurrences of ℓ inbetween, i.e. V_Y has period $|X|_N$.

Using a similar argument, one shows that H_Y has period $|X|_E$, so that the run vector of Y_∞ is

$$(H_Y\,[0]\,, V_Y\,[0]\,, H_Y\,[1]\,, V_Y\,[1]\,, H_Y\,[2]\,, V_Y\,[2]\,, \ldots).$$

Hence, Y_∞ has period $2|X|_N \times 2|X|_E = 4|X|_N|X|_E$, as desired. $\qquad\square$

The proof of Theorem 1 gives us a simple recursive way to generate the infinite word Y_∞:

$$Y_\infty(i) = \begin{cases} \varepsilon & \text{if } i = 0, \\ Y_\infty(i-1)\, E^{H_Y(i) \mod |H_Y|}\, N^{V_Y(i) \mod |V_Y|} & \text{otherwise.} \end{cases}$$

Then Y_∞ is the limit of $Y_\infty(i)$ for $i \to \infty$.

From now on, let X be given, let p be the period of Y_∞, and let $W = Y_\infty[1, \ldots, p]$. We remark that the previously described algorithm gives us as a corollary a simple way to check if the boundary of $X(\infty)$ crosses itself or not. Since Y_∞ is periodic, it is sufficient to check it for at most p steps. Summarizing, concerning pseudo-square parallelogram permutominoes, we may have the following possibilities:

1. the boundary of $X(\infty)$ crosses itself. Then by Theorem 1 it crosses itself infinitely many times. In this case, we may have the two possibilities: (a) we have no pseudo-square parallelogram permutomino (see Fig. 8 (a)); (b) there is a prefix Y of W which is up-compatible with X, and we have a pseudo-square parallelogram permutomino (see Fig. 8 (b)).

2. the boundary of $X(\infty)$ does not cross itself; in this case, it may happen that there are no words Y which are up-compatible with X (see Fig. 7 (a)); otherwise, if there is at least a word Y up-compatible with X, then there are infinitely many such words; assuming that Y is the word having minimal length among these ones, then they have the form W^*Y (see Fig. 7 (b), where the black points identify the paths $Y, WY, W^2Y, \ldots$, and the white points identify the paths $W, W^2, \ldots$).

Another remarkable observation is that pseudo-square parallelogram permutominoes are centrosymmetric, i.e. they are fixed by a rotation of angle π. This is the second of our main results:

Theorem 2. *If $XY\widehat{X}\widehat{Y}$ is the boundary word of a pseudo-square parallelogram permutomino, then X and Y are palindromes.*

PROOF. As in the proof of Theorem 1, we study independently the vertical and horizontal steps. Therefore, it is sufficient to show that both V_Y and H_Y are palindromes.

Remark that if X and some prefix Y of Y_∞ yields a *psc*-permutomino, then there exists some prefix w' of w starting with $\ell z u$ and ending with $\ell \bar{z} u$, where z is some word on $\{\ell, u\}$. We show that z is a palindrome. Also, since w has period $2|X|_N$ and antiperiod $|X|_N$, then $w = (\ell z u \bar{z})^\omega$ so that $\ell \bar{z} u$ is a factor of either $\ell z u \bar{z}$ or $u \bar{z} \ell z$

We consider only the case where $\ell \bar{z} u$ is factor of $\ell z u \bar{z}$, the second one being symmetric. The situation is depicted in Figure 10 Let $n = |z| - 1$ and p be the distance between the upper left occurrence of z and the lower occurrence of $\bar{z}$. Then $\bar{z}$ and z have the antiperiod p. Moreover, since $\bar{z}$ overlaps itself, then $q = n + 2 - p$ is a period of z and $\bar{z}$ as well. Thus, z admits an antiperiod p
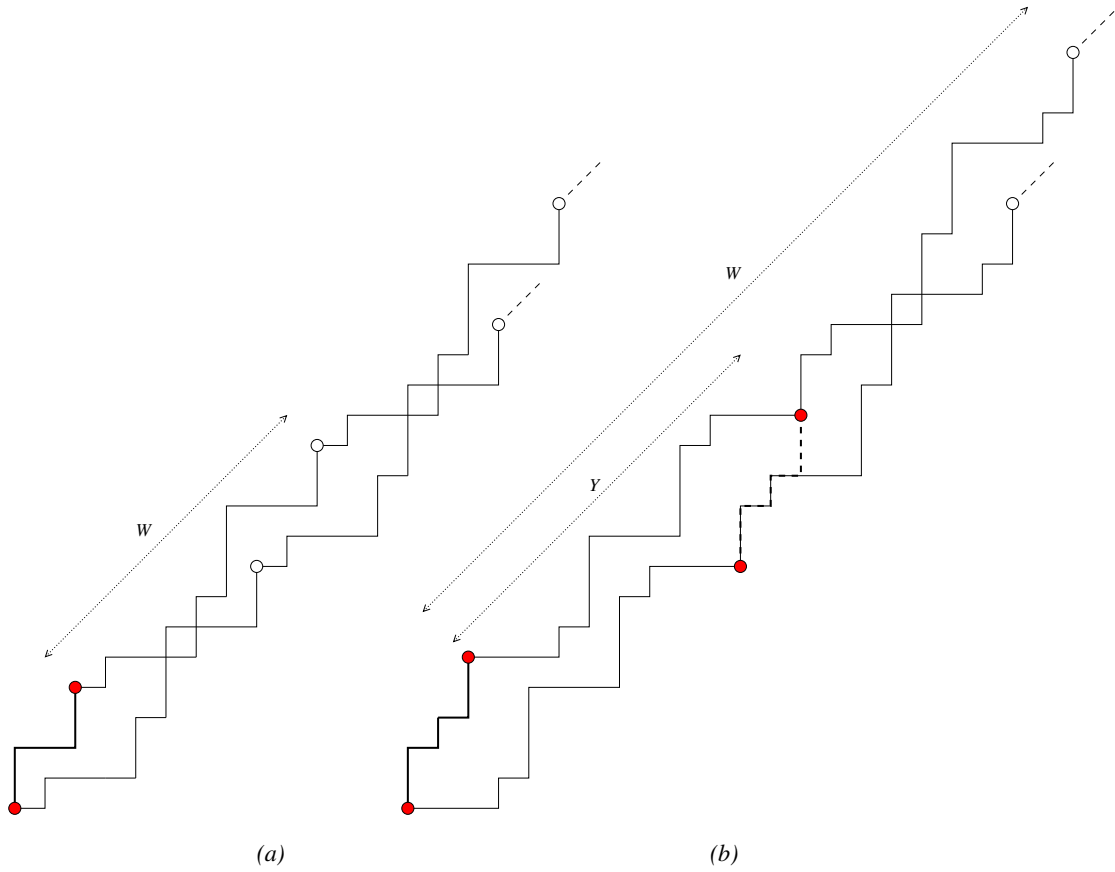
14

Figure 8: The path $X(\infty)$ crosses itself: (a) there is no pseudo-square parallelogram permutomino; (b) there is a word Y of length less than the period W of Y_∞ such that X is up-compatible with Y, and we have a pseudo-square parallelogram permutomino.

and a period q. Also, we know that $p \neq q$, since no word admits a number as both a period and an antiperiod.

Assume that $z = z[0]\,z[1]\,z[2]\cdots z[n]$ (the index starts at 0 and ends at n). First, we notice that the letters at positions $p-1$ and $n-(p-1) = q-1$ in z are the same. Indeed, $z[p-1] = \ell$ and $z[q-1] = \overline{z[q-1]} = \overline{u} = \ell$ (see Figure 10).

Let $I = \{0, 1, 2, \ldots, n\}$ be the set of indices of z and $g = \gcd(p, q)$. We claim that for $k = 1, 2, \ldots, (n+2)/g - 1$, we have $z[kg-1] = z[n-(kg-1)]$, i.e. the palindromic property is satisfied for these indices. Note in particular that p and q are both of the form $kg-1$ for some k, and we have already seen that $z[p-1] = z[q-1]$. Arguing by contradiction, assume that there exists some $k \in \{1, 2, \ldots, (n+2)/g - 1$ such that $z[kg-1] \neq z[n-(kg-1)]$. By Bezout's Lemma, we know that there exist two positive integers x and y such that either $g = xp - yq$ or $g = xq - yp$. Without loss of generality, we assume that $g = xp - yq$, so that $p - (kg-1) = \ell g = x'p - y'q$, where
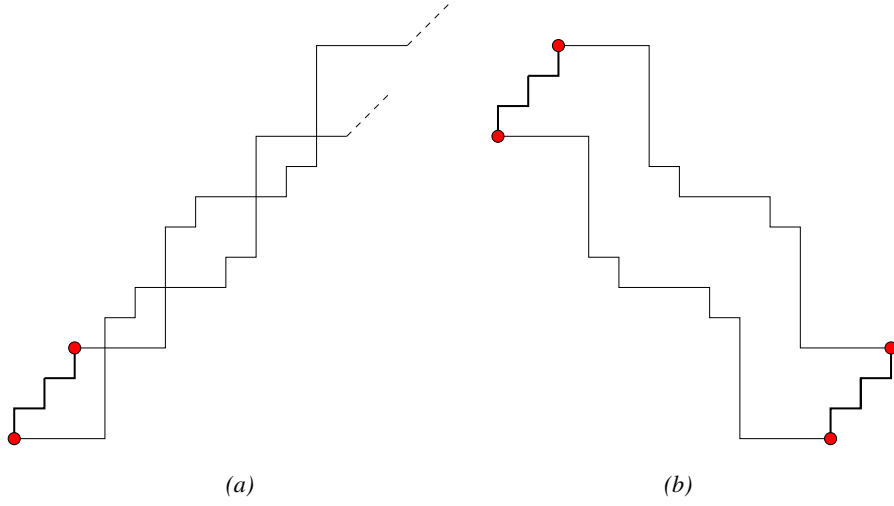
15

Figure 9: (a) the path Y_∞: there is no word Y which is up-compatible with X; (b) a prefix Y' of the path Y'_∞ which is down-compatible with X.
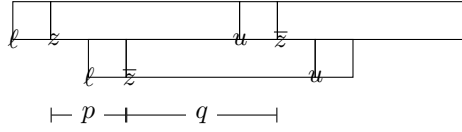

Figure 10: The word $\ell\bar{z}u$ is factor of the word $\ell z u\bar{z}$.

ℓ is some positive integer. Now, we construct a sequence a_i of elements in I defined by $a_0 = kg - 1$ and

$$a_{i+1} = \begin{cases} a_i + p & \text{if } a_i + p \in I, \\ a_i - q & \text{if } a_i - q \in I. \end{cases}$$

There are two possibilities: either the sequence a_i stops when $a_i + p, a_i - q \notin I$ or when it reaches position $p - 1$. But the condition $a_i + p, a_i - q \notin I$ implies $a_i \in \{p - 1, q - 1\}$. Therefore, using the antiperiod p and the period q, we are able to reach from $kg - 1$ the position $p - 1$ or $q - 1$. By symmetry and using exactly the same antiperiod or period, there exists a path from $n - (kg - 1)$ to position $p - 1$ or $q - 1$. This means that $z[kg - 1] = z[n - (kg - 1)]$, contradicting our previous assumption.

Next, we show that q/g is odd. By contradiction, assume that q/g is even, so that p/g is odd (otherwise, g would not be the greatest common divisor). Let $k' = q/(2g)$. Since q/g is even, k' is a valid index in I. Also, since

$$n - (k'g - 1) = n - \left(\frac{q}{2g} \cdot g - 1\right) = \frac{n + p}{2} = \frac{2p + q - 2}{2} = p + \frac{q}{2g} \cdot g - 1 = k'g - 1 + p,$$

one concludes that $z[k'g - 1] \neq z[n - (k'g - 1)]$, a contradiction. Thus, q/g is odd.

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Finally, let $j \in I$. Using the same ideas as above, we construct a path from j to $j + g$. There are two possible cases: either (i) we reach some position of the form $kg - 1$, therefore proving that $z[j] = z[n - j]$ or (ii) we reach the position $j + g$, therefore showing that g is either a period or an antiperiod of z. It cannot be a period since $g \mid p$ and p is an antiperiod. Moreover, it is neither an antiperiod since q/g is odd: this would mean that $z[j] \neq z[j + q]$, which is impossible since q is a period. Hence, z is a palindrome, as desired, and so is the vector V_Y

A similar argument shows that H_Y is a palindrome as well, so that the whole word Y is itself a palindrome, concluding the proof. $\square$

3.2. Downward growing direction

Most of the considerations made for the upward growing direction can be easily reformulated for the downward growing direction. Also in this case, for a given word X of the form $N^+(E \vee N)^*N^+$, we can define an infinite word Y'_∞, and every word Y' which is down-compatible with X is a prefix of Y'_∞. Moreover, we can write down an algorithm – completely analogous to that presented in the previous section – to recursively build the word Y'_∞. There is also a strict relation between the two words Y_∞ and Y'_∞ which is exploited in the next statement.

Proposition 6. *Let X be a word of the form $N^+(E \vee N)^*N^+$. The following properties hold:*

i) Y'_∞ is obtained from Y_∞ by changing all N steps with S steps (see Fig. 9).

ii) The word Y'_∞ is periodic, and its period is a divisor of $4\,|X|_N\,|X|_E$.

iii) If X is up-compatible with Y, then X is down-compatible with the word Y', which is obtained from Y by changing all N steps with S steps (see Fig. 5).

We point out that the converse of iii) does not hold: for instance Figure 5 shows that $X = NENEN$ is downward compatible with a word Y', while it is not up-compatible with the word Y – obtained from Y' by changing all S steps with N steps – and there is no word Y which is up-compatible with X.

4. Further work

Concerning our former objective of studying permutominoes that tile the plane, there are several interesting problems which remain unsolved and should be better investigated. In the sequel we just mention a few of them.

psc-permutominoes. It is worth mentioning that the converse of Theorem 2 does not hold, as one can easily check with the word $X = NNNENNN$. Our main goal would be to characterize the palindromic words which determine *psc*-permutominoes and then possibly enumerate these words according to the length.

One step in this direction follows from an exhaustive experimental investigation of the word X up to length 15, and leads us to the following conjecture, already stated in [7]:

CONJECTURE. If the word X is c-exact, then it has odd length.

As an example, we have three c-exact words of length 5, $NNENN$ (both in the upward growing and in the downward growing directions), $NEEEN$ and $NENEN$ (only in the downward growing direction). These words seem to share some properties with *central words* (or *cut words*), i.e. words admitting two periods such that they miss only one letter so that Fine and Wilf's Theorem applies. Indeed, the first and last author of this paper proved the following theorem in [1]:

Theorem 3. *[1] Let w be a finite word. Let p and q be a periods or antiperiods of w (at least one is an antiperiod). If $|w| \geq p + q$, then $\gcd(p, q)$ is an antiperiod of w.*

It seems reasonable that, once we obtain the characterization of the words X and Y leading to psc-permutominoes, we can tackle their enumeration problem.

Column-convex pseudo-square permutominoes. The problem of studying non convex permutominoes which tile the plane by translation is completely open and difficult to tackle. Also considering a slight relaxation of the convexity constraint, for instance the class of pseudo square column-convex permutominoes, we observe that some basic properties holding for psc-permutominoes do not hold anymore. In Fig. 6, we provide two examples of non convex ps-permutominoes: we notice that the word X of the permutomino (a) is c-exact, and admits a compatible Y both in the downward and in the upward growing direction; on the other side, the word X of the permutomino (b) is not c-exact and all the pseudo square permutominoes where X is one of the components are non convex. So, the psc-permutominoes are properly contained in the class of non convex ps-permutominoes, as expected. It is straightforward to check that an infinite number of words Y compatible with X exist in both cases, since some properties about the period and the antiperiod of Y still hold.

As a first step, it seems possible that some generalization of the results holding for psc-permutominoes can be found (e.g. Propositions 3, 5)

Pseudo-hexagon permutominoes. Another challenging and more general problem concerns the study of pseudo-hexagon convex permutominoes; in this case, as established in [2], many of the basic properties of pseudo-square convex polyominoes do not hold so we do not have unicity of the decomposition.

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