Continuum percolation in proximity graphs

Jean-Michel Billiot, Franck Corset, Eric Fontenas

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J-M. Billiot, F. Corset, E. Fontenas *†

Université de Grenoble, laboratoire Jean Kuntzmann, UMR 5224

Abstract

We establish a sufficient condition to obtain continuum percolation on a class of proximity graphs when their vertices are distributed under a stationary Poisson point process with unit intensity in the plane. We apply this result on a family of graphs which generalizes β-skeleton graphs.

Keywords: percolation; β-skeleton graphs; Poisson point process.

1 Introduction

A proximity graph is a graph defined as follows: the vertex set V is a set of points in the plane $\mathbb{R}^2$, each pair $\{u, v\}$ of vertices is assigned a region $N(u, v)$, and u and v are adjacent if and only if $N(u, v)$ contains no other vertices in V. For example, in the Gabriel Graph, $N(u, v)$ is the disk with diameter uv. Delaunay triangulation, Relative Neighborhood Graph and Gabriel Graph are most known examples of proximity graphs. Proximity graphs find their applications in fields of science and engineerings: geographical variational analysis [8], evolutionary biology [14], spatial analysis in biology [7], simulation of epidemics [17].

β-skeleton graphs, proposed in [13], form a family of proximity graphs monotonously parametrized by a parameter β. It consists of two distinct families called circle-based skeleton graphs and lune-based skeleton graphs. For $\beta = 1$, the circle-based skeleton graph is the Gabriel graph, and, for $\beta = 2$, the lune-based skeleton maps to the relative neighborhood graph (RNG). Following [12], the RNG is almost surely connected when the vertices are distributed under a stationary Poisson point process. It is well known that, for $\beta < 1$, the Gabriel graph is a subgraph of the circle-based skeleton graphs and, for $\beta < 2$, the RNG is a subgraph of the lune-based skeleton graph: in both cases, the skeleton graphs are thus connected when their vertex set is finite. In an other way, the circle-based skeleton graph is not always connected for $\beta > 1$ and the lune-based skeleton graph for $\beta > 2$. Thus, in both cases, the problem of percolation arises due to the fact that the neighborhood associated with these proximity graphs increases in function of the parameter β.

*Corresponding author : Eric.Fontenas@upmf-grenoble.fr

†Postal address: Figal Team, BSHM, 1251 avenue centrale, B.P 47, 38040 Grenoble cedex 9, France

So, in this paper, we establish a sufficient condition on N, the neighborhood associated with a family of proximity graphs, to obtain percolation. Our result has several advantages: first, as we only need a condition on the area and on the diameter of the neighborhood N, we can choose several forms of N. Moreover, this allows to choose some particular direction problem of oriented percolation.

As one interesting application of this result, we introduce a new family of proximity graphs which we call generalized β-skeleton graphs. The feature of these graphs is that the form of the neighborhood N of each pair of vertices of these graphs changes according to the distance between both vertices: that means that β is not constant but is a function of the distance between the vertices. Our primary focus is to give a class of function β for which the continuum percolation occurs in the associated generalized β-skeleton graph (its vertices are distributed under a stationary Poisson point process in the plane). It is particularly interesting to choose a function β with the maximum that is greater than one in the case of the circle based skeleton and than two for the lune-based skeleton graph. In fact our result can provide a partial answer of the percolation problem described before.

One way to prove continuum percolation is to compare with an independent site percolation on the square lattice $\mathbb{Z}^2$: see [10] for continuum percolation problems for the k-nearest neighbor graph under Poisson process and [3] in the case of Gabriel graph. An other way is to compare with an 1-dependent bond percolation. In this paper, we adopt this approach. According to this idea, Bollobás and Riordan [5] establish an exact result on the critical probability for random Voronoi percolation, a similar result of Kesten's theorem in continuous setting. In other contexts, Balister and Bollobás [2] give bounds on k in the k-nearest neighbor graph for percolation with several possible definitions (see [6] and the references therein for results on continuum percolation).

The paper is organized as follows. We recall the stationary Poisson point process. We present and prove our main result on the continuum percolation for a family of proximity graphs and we discuss possible perspectives and motivations of this work. We then apply this result on generalized β-skeleton graphs and give some examples for which the continuum percolation occurs.

2 The stationary Poisson point process

We consider a stationary Poisson point process Φ of intensity one on $\mathbb{R}^2$, a measurable mapping of the probability space $[\Omega, \mathcal{A}, \mathbf{P}]$ into $[N, \mathcal{N}]$. An element φ of N can be regarded as a measure on $\mathbb{R}^2$ so that $\varphi(B)$, B a bounded Borel set, is the number of points of φ in B. The σ- algebra $\mathcal{N}$ is defined as the smallest σ-algebra on N to make measurable all mappings $\varphi \to \varphi(B)$ for B running through the bounded Borel sets. The distribution P of the point process Φ is determined by the probabilities

$$P(Y) \,=\, \mathbf{P}(\Phi \in Y) \,=\, \mathbf{P}(\{\omega \in \Omega / \, \Phi(\omega) \in Y\}) \quad \text{for}\, Y \in \mathcal{N}.$$

We introduce the Palm distribution of the process Φ. The Palm distribution probabilities (see [16]) denoted by P_x are the conditional probabilities of point process events when the point $x \in \mathbb{R}^2$ is observed at a specific location. If Y is some point process property, then $P_x(Y) = \mathbf{P}(\Phi \text{ has property } Y || x)$. In the same way, the reduced Palm distribution denoted by $P^!_x$ is defined as $P^!_x(Y) = \mathbf{P}(\Phi\backslash\{x\} \in Y||x)$ for Y in $\mathcal{N}$. In the case of a Poisson point process Φ, the Slivnyak's theorem (see [16]) gives

$$\forall x \in \mathbb{R}^2,\ P_x(Y) = \mathbf{P}(\Phi \cup \{x\} \text{ has property } Y) \text{ and } P^!_x(Y) = P(Y).$$

3 Main result: continuum percolation in proximity graphs

3.1 The rolling ball statement and the percolation model

To prove that continuum percolation occurs in some proximity graphs, we shall compare the process to various bond percolation models on $\mathbb{Z}^2$. In these models, the states of the edges will not be independent. However they will satisfy the following definition:

Definition 1 *A bond percolation model is 1-dependent if whenever E_1 and E_2 are sets of edges at graph distance at least 1 from each another (i.e., if no edge of E_1 is incident to an edge of E_2) then the state of the edges in E_1 is independent of the state of the edges in E_2.*

We shall use the following result of Balister et al. [1]:

Theorem 1 *If every edge in a 1-dependent bond percolation model on $\mathbb{Z}^2$ is open with probability at least* 0.8639*, then almost surely there is an infinite open component.*

Consider a proximity graph $G = (V, E)$: the vertex set V is a set of points in the plane $\mathbb{R}^2$, each pair $\{u, v\}$ of vertices is assigned a region $N(u, v)$, and uv is an edge of G (an element of E) if and only if $N(u, v)$ contains no other vertices in V.
For percolation we need to find an infinite path, i.e., a sequence $u_1, u_2 \dots$ with $u_iu_{i+1} \in E$ for all i. Consider the rectangular region consisting of two adjacent squares S_1, S_2 shown in figure 1. Both S_1 and S_2 have side length $2r + 2q$, where r is chosen later and q depends on r, on a real s, $0 \le s \le 2r$, and on the diameter of the neighborhood N: q is chosen so that every neighborhood N of two different vertices of the graph G inside $D_1 \cup L$ stays inside the rectangular zone $S_1 \cup S_2$. We define $\mathcal{E}_{S_1,S_2}$ to be the event that there is at least one point in D_1 and that every vertex u_1 in the central disk D_1 of S_1 is joined to at least one vertex in the central disk D_2 of S_2 by a $G-$path, regardless of the state of the Poisson point process outside of S_1 and S_2.

Now consider the following percolation model on $\mathbb{Z}^2$. Each vertex $(i, j) \in \mathbb{Z}^2$ corresponds to a square $[Ri, R(i + 1)] \times [Rj, R(j + 1)] \in \mathbb{R}^2$, where $R = 2r + 2q$. An edge is open between adjacent vertices (corresponding to squares S_1 and S_2) if both the corresponding events $\mathcal{E}_{S_1,S_2}$ and $\mathcal{E}_{S_2,S_1}$ hold. Note that this is indeed a 1-dependent model on $\mathbb{Z}^2$. Any open path in $\mathbb{Z}^2$ corresponds to a sequence of events $\mathcal{E}_{S_1,S_2}, \mathcal{E}_{S_2,S_3} \dots$ that occur, where S_i is the square associated with a site in $\mathbb{Z}^2$. Every vertex u_1 of the original Poisson point process that lies in the central disk D_1 of S_1 now has an infinite path leading away from it: one can find points u_i in the central disk of S_i and paths from u_{i-1} to u_i inductively for every $i > 1$. In particular, each such u_1 lies in an infinite component.

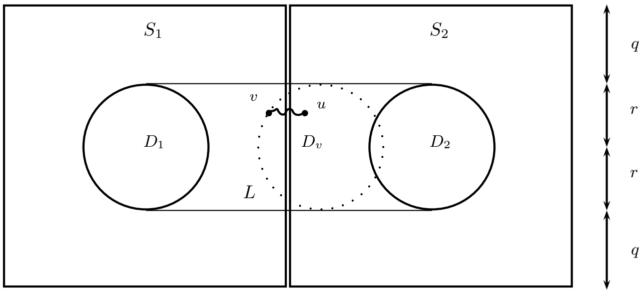


Figure 1: The rolling ball D_v (dotted circle)

3.2 Main result

Let r, D_1, D_2, D_v and L be as in the figure 1 (L is the region between the two disks D_1 and D_2). For each point v in $D_1 \cup L$,
a) D_v is the dotted disk of radius r inside $D_1 \cup L \cup D_2$ with v on its D_1-side boundary,
b) $D(v, s)$ the disk of center v and radius s,
c) $R(r, s)$ is the area of the region $R(v, r, s) = D_v \cap D(v, s)$.
The point O denotes the origin of $\mathbb{R}^2$ and, so, $R(r, s)$ is the area of the region $R(O, r, s) = D_O \cap D(O, s)$.
We consider the following class $\mathcal{P}$ of proximity graphs:

Definition 2 *A proximity graph $G = (V, E)$ is in the class $\mathcal{P}$ if:*
1) the set of vertices $V = \varphi$ is a realization of the Poisson point process Φ on the plane $\mathbb{R}^2$
2) each pair $\{u, v\}$ of vertices set V is assigned a region $N(u, v)$, and uv is an edge of G if and only if $N(u, v)$ contains no other vertices in V.
3) for each pair of vertices $\{u, v\}$ in $D_1 \cup L$, the diameter of $N(u, v)$ is bounded
4) there exist a non negative real q and a non negative real r and a real s, $0 \leq s \leq 2r$, as

$$\int_{R(O,r,s)} \left(1 - e^{-|N(O,u)|}\right) du \quad \leq \frac{0.06805 - e^{-\pi r^2}}{4r(2r + 2q)} - \frac{e^{-R(r,s)}}{2} \tag{1}$$

with $R(r, s) = -rs\sqrt{1 - \frac{s^2}{4r^2}} + (2r^2 - s^2) \arcsin\left(\frac{s}{2r}\right) + \frac{\pi}{2}s^2$.
$|N(O, u)|$ is the area of the neighboorhood associated with the points O and u when u moves in $R(O, r, s)$.

We now present the main result of this paper:

Theorem 2 *For all proximity graph G of $\mathcal{P}$, we have*

$$P(\mathcal{E}_{S_1,S_2} \cap \mathcal{E}_{S_2,S_1}) \geq 0.8639. \tag{2}$$

So, there is almost surely an infinite component in the proximity graph G under the stationary Poisson point process. The continuum percolation occurs almost surely in these proximity graphs.

To prove this result, we have to obtain the inequality (2) and apply the theorem 1.

3.3 Proof of the theorem 2

To obtain the inequality (2), we study the complementary of the event $\mathcal{E}_{S_1,S_2} \cap \mathcal{E}_{S_2,S_1}$. In order to bound the probability that the intersection of the events $\mathcal{E}_{S_1,S_2}$ and $\mathcal{E}_{S_2,S_1}$ fails, we use the idea of the rolling ball method introduced by Balister and Bollobás [2] in the case of the k-nearest neighbor graph. We adapt this method to our class $\mathcal{P}$ of graphs by using classical tools of point processes like Slivnyak's theorem and the reduced Campbell's theorem (see [16]).
Let Φ be a stationary Poisson point process of intensity one on $\mathbb{R}^2$, a measurable mapping of the probability space $[\Omega, \mathcal{A}, \mathbf{P}]$ into $[N, \mathcal{N}]$. Consider φ of N a realization of the Poisson point process Φ on the plane $\mathbb{R}^2$. We introduce the following notations:
- $\varphi - \delta_v$ is the notation for the point pattern φ with the point $v \in \varphi$ deleted.
- φ_C is the notation for the point-pattern φ restricted to the set C.
- $\varphi(C)$ denotes the number of points of φ in the set C.
- φ^{+x} (or φ^{-x}) is the configuration φ translated of vector x(or -x) $\in \mathbb{R}^2$.
Consider $G = (V = \varphi, E)$ a proximity graph of the class $\mathcal{P}$.
We recall: for all points v in $D_1 \cup L$, D_v is the dotted disk of radius r inside $D_1 \cup L \cap D_2$ with v on its C_1-side boundary, $D(v, s)$ the disk of center v and radius s and $R(r, s)$ the area of the region $R(v, r, s) = D_v \cap D(v, s)$. The point O denotes the origin of $\mathbb{R}^2$ and, so, $R(r, s)$ is the area of the region $R(O, r, s) = D_O \cap D(O, s)$.
Given a non negative real $s \in]0, 2r]$, we introduce the event A_{S_1,S_2} as

$$A_{S_1,S_2} = \{\varphi \in N / \forall v \in \varphi_{D_1 \cup L}, \exists u \in \varphi_{R(v,r,s)}, (\varphi - \delta_v - \delta_u)(N(u, v)) = 0\}.$$

This event is the set of configurations φ for which, for all points v of these configurations in $D_1 \cup L$, we can find an other point u of φ in $R(v, r, s)$ so that uv is an edge of the graph G. Moreover, we only consider G-path in $D_1 \cup L$ with edges with a length of not more than s.
Note in particular that the existence of u, a point of the realization φ, in $R(v, r, s)$ implies that the edge uv in G is independent of the Poisson process outside of $S_1 \cup S_2$. This is because both u and v are at distance at least q from the exterior of $S_1 \cup S_2$. Recall that the parameter q depends on the graph G through its neighborhood N and on the parameters s and r. So, the existence of an edge of G only depends on the points within $S_1 \cup S_2$.
Let $\bar{A}_{S_1,S_2}$ be the complementary of A_{S_1,S_2}. We get that

$$\bar{A}_{S_1,S_2} \cup A_1 \subset A_1 \cup A_2 \cup A_3$$

where:
- A_1 is the set of the realisations φ which contain no points in D_1.
- A_2 is the set of the realisations φ for which there exists a point v of φ in $D_1 \cup L$ so

that $R(v,r,s)$ contains no other points of φ.
- A_3 is the set of the realisations φ for which there exists $v \in \varphi \cap (D_1 \cup L)$ such for every point of u of φ in $R(v,r,s)$, there is another point of φ in $N(u,v)$.
Remark that if none of A_1, A_2 and A_3 holds, the condition we want does, as given any point v_1 in D_1, one can find a neighbour v_2 in $R(v_1,r,s)$, then a neighbour v_3 of v_2 in $R(v_2,r,s)$ and so on...(the moving circle argument). Remark that the probability that the intersection of basic good events $\mathcal{E}_{S_1,S_2}$ and $\mathcal{E}_{S_2,S_1}$ fails is bounded as

$$P(\bar{\mathcal{E}}_{S_1,S_2} \cup \bar{\mathcal{E}}_{S_2,S_1}) \leq P(\bar{A}_{S_1,S_2} \cup \bar{A}_{S_2,S_1} \cup A_1)$$

and we have

$$P(\bar{A}_{S_1,S_2} \cup A_1) \leq P(A_1) + P(A_2) + P(A_3).$$

The first probability is $P(A_1) = e^{-\pi r^2}$. To bound $P(A_2)$ and $P(A_3)$, we use the reduced Campbell's Theorem and Slivnyak's theorem ([16]). Given $A_{R_v} = \{\varphi \in N/\varphi(R(v,r,s)) = 0\}$ and $A_{R_O} = \{\varphi \in N/\varphi(R(O,r,s)) = 0\}$, it comes

$$\mathbb{1}_{A_2}(\varphi) \leq \sum_{v\in\varphi} \mathbb{1}_{D_1\cup L}(v)\mathbb{1}_{A_{R_v}}(\varphi - \delta_v).$$

We deduce that

$$\begin{aligned} P(A_2) &\leq \int_{D_1\cup L} dv \int_N \mathbb{1}_{A_{R_v}}(\varphi)P_v^!(d\varphi) \\ &= \int_{D_1\cup L} dv \int_N \mathbb{1}_{A_{R_O}}(\varphi^{-v})P_O^!(d\varphi) \\ &= |D_1 \cup L|\, P_O^!(A_{R_O}) = |D_1 \cup L| P(A_{R_O}) = 2r(2r+2q)e^{-|R(r,s)|}. \end{aligned}$$

For the last probability, by introducing the following events

$$\begin{aligned} &A_v = \{\varphi \in N/\forall u \in \varphi_{R(v,r,s)},\, (\varphi - \delta_u)(N(u,v)) > 0\}, \\ &A_O = \{\varphi \in N/\forall u \in \varphi_{R(0,r,s)},\, (\varphi - \delta_u)(N(u,O)) > 0\}, \\ &A_{Ou} = \{\varphi \in N/\varphi(N(O,u)) > 0\}, \end{aligned}$$

we have

$$1_{A_3}(\varphi) = \max_{v\in\varphi} \mathbb{1}_{D_1\cup L}(v)\mathbb{1}_{A_v}(\varphi - \delta_v) \leq \sum_{v\in\varphi} \mathbb{1}_{D_1\cup L}(v)\mathbb{1}_{A_v}(\varphi - \delta_v).$$

Thus, we can bound $P(A_3)$ as

$$\begin{aligned} P(A_3) &\leq \int_{D_1\cup L} dv \int_N \mathbb{1}_{A_v}(\varphi)P_v^!(d\varphi) \\ &= \int_{D_1\cup L} dv \int_N \mathbb{1}_{A_O}(\varphi^{-v})P_O^!(d\varphi) \\ &= |D_1 \cup L|\, P_O^!(A_O) = 2r(2r+2q)\, P(A_O). \end{aligned}$$

Moreover, from the following inequality

$$\mathbb{1}_{A_O}(\varphi) \le \sum_{u\in\varphi} \mathbb{1}_{R(O,r,s)}(u)\mathbb{1}_{A_{Ou}}(\varphi - \delta_u),$$

we obtain

$$\begin{aligned} P(A_O) &\le \int_{R(O,r,s)} du \int_N \mathbb{1}_{A_{Ou}}(\varphi) P_u^!(d\varphi) = \int_{R(O,r,s)} P_u^!(A_{Ou})\, du, \\ &= \int_{R(O,r,s)} P(A_{Ou})\, du = \int_{R(O,r,s)} \left(1 - e^{-|N(O,u)|}\right) du. \end{aligned}$$

This leads to the following result:

$$P(A_3) \le 2r(2r+2q) \int_{R(O,r,s)} \left(1 - e^{-|N(O,u)|}\right) du.$$

Now, we can compute an upper bound of the probability of the event $\bar{A}_{S_1,S_2} \cup A_1$:

$$\begin{aligned} P(\bar{A}_{S_1,S_2} \cup A_1) &\le e^{-\pi r^2} + 2r(2r+2q)e^{-R(r,s)} \\ &+ 4r(2r+2q) \int_{R(O,r,s)} \left(1 - e^{-|N(O,u)|}\right) du. \end{aligned} \quad (3)$$

The third term in the inequality (3) is the probability to obtain at least one point in the neighboorhood $N(O,u)$ associated with the points O and u when u moves in $R(O,r,s)$. Following the inequalities (3) and (1),

$$\begin{aligned} P(\bar{\mathcal{E}}_{S_1,S_2} \cup \bar{\mathcal{E}}_{S_2,S_1}) &\le P(\bar{A}_{S_1,S_2} \cup \bar{A}_{S_2,S_1} \cup A_1) \\ &\le 2e^{-\pi r^2} + 4r(2r+2q)e^{-R(r,s)} \\ &\quad + 8r(2r+2q) \int_{R(O,r,s)} \left(1 - e^{-|N(O,u)|}\right) du \\ &\le 0.1361 \end{aligned}$$

With the result of the theorem 1, this last inequality concludes the proof of the theorem 2.

3.4 Remarks and comments

1. In our context, the vertices of graphs of the class $\mathcal{P}$ are a realization of a point Poisson process. It could be interesting as in [3] to study continuum percolation for graph in the class $\mathcal{P}$ when the points are distributed under a Gibbs point process. As our proof relies on classical tools namely Campbell's theorem and Slivnyak's theorem, one way consists in using Georgii-Nguyen-Zessin formula and suitable bounds of the partition function.

2. First, we point out that the method proposed by [11] and used for k-nearest neighbor graph and Delaunay graph, as well as the adaptation by [3] to the Gabriel

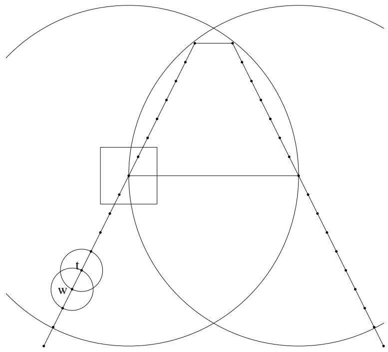


Figure 2: Example of construction of configuration with arbitrary large RNG- path

graph does not apply to the Relative Neighborhood Graph (RNG). More precisely, by comparison with independent site percolation on the square lattice, it is sufficient to deal with an event on a fixed box. We just have to control the probability of having at least one point and less than a fixed number of points in each small box K. Then it is possible to obtain a suitable path with edges of the graph connecting two boxes with a significant probability. Indeed, we may choose some configurations of points such that the length and the number of points of a path in a proximity graph between two points are arbitrarily large: for example (see figure 2), wt is an edge of the RNG but uv is not because the point q belongs to the vacuity region of this edge. Now compare with 1-dependent bond percolation model, the events are more flexible because it's defined in the union of two adjacent boxes.

3. The theorem 2 gives a class of graphs for which the percolation occurs. It should be interesting to study the site and bond percolation as well as phase transition for several statistical mechanics models for these graphs.
In [4], the phase transition in the Delaunay continuum Potts model is established. It is a generalization of [9] where the soft repulsion between several species of particles acts on the Delaunay graph. The paper [3] gives an answer for the Gabriel graph on which the repulsion is strong enough to maintain a phase transition. So, if we consider some proximity graphs of the class $\mathcal{P}$, in terms of percolation, it means: is there bond percolation for these graphs ?

4. Recently, [15] proposed sufficient conditions on infinite graphs to deduce a non trivial bond percolation threshold. Among these assumptions, they assume that the dual graph is bounded degree. It is interesting to relax this condition for proximity graphs which in general have a dual not bounded degree.

4 Examples of proximity graphs: generalized β-skeleton graphs

4.1 Definition

Let V be a locally finite subset of $\mathbb{R}^2$ and $\beta_0 > 0$ a constant. Given two distinct points u and v of V, $\alpha = d(u,v)$ is the euclidean distance between u and v. $D(x_0, r)$ denotes the disk of center x_0 with radius r. We consider two neighborhoods of the points u and v (see Figure 3): the first one $L_{\beta_0}(u,v)$, $\beta_0 \in [1; +\infty[$, is defined as the intersection of two disks:

$$L_{\beta_0}(u,v) = D\left(a_1 = u + \frac{\beta_0}{2}(v-u), \alpha\frac{\beta_0}{2}\right) \cap D\left(a_2 = v + \frac{\beta_0}{2}(u-v), \alpha\frac{\beta_0}{2}\right).$$

For the second one, we calculate an angle γ using the formulas

$$\gamma = \begin{cases} \arcsin(1/\beta_0), & \text{if } \beta_0 \geq 1 \\ \pi - \arcsin(\beta_0), & \text{if } \beta_0 \leq 1 \end{cases}$$

For any two points u and v in the plane, let $C_{\beta_0}(u,v)$ be the set of points w for which angle uwv is greater than γ. Then, $C_{\beta_0}(u,v)$ takes the form of a union of two open disks with radius $\beta_0\alpha/2$ for $\beta_0 \geq 1$ and $\gamma \leq \pi/2$, and it takes the form of the intersection of two open disks with radius $\alpha/(2\beta_0)$ for $\beta_0 \leq 1$ and $\gamma \geq \pi/2$. When $\beta_0 = 1$ the two formulas give the same value $\gamma = \pi/2$, and $C_{\beta_0}(u,v)$ takes the form of a single open disk with $\alpha/2$ as its radius. We can write that

$$C_{\beta_0}(u,v) = \begin{cases} D\left(c_1, \alpha\frac{\beta_0}{2}\right) \cup D\left(c_2, \alpha\frac{\beta_0}{2}\right) & \\ \text{with } d(c_1,u) = d(c_1,v) = d(c_2,u) = d(c_2,v) = \alpha\beta_0/2 & if \beta_0 \geq 1. \\ D\left(c_1, \frac{\alpha}{2\beta_0}\right) \cap D\left(c_2, \frac{\alpha}{2\beta_0}\right) & \\ \text{with } d(c_1,u) = d(c_1,v) = d(c_2,u) = d(c_2,v) = \alpha/(2\beta_0) & if \beta_0 \leq 1. \end{cases}$$

We state that uv is an edge of the graph $G_{\beta_0} = (V, E_{\beta_0})$ (where E_{β_0} is the set of edges of G_{β_0}) if the intersection between the neighborhood associated with the vertices u and v in V and the set of vertices $V \backslash \{u,v\}$ is empty. The graph $G^L_{\beta_0}$ associated with the neighborhood L_{β_0} is usually called lune-based skeleton graph and the other $G^C_{\beta_0}$ associated with the neighborhood C_{β_0} called circle-based skeleton graph. In [18], we can find some results on the computation of the β_0-skeleton graphs and the resulting computational complexities. Note that both neighborhoods are similar for $\beta_0 = 1$ which corresponds to the Gabriel Graph.

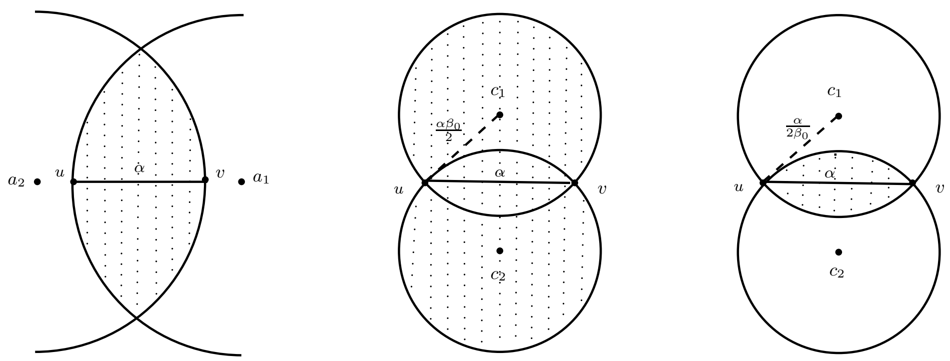
We now suggest a new family of graphs denoted by $G^\star_\beta = (V, E_\beta)$ which generalizes the β_0-skeleton graphs: given two vertices u and v in V of G_β, the form of the neighborhood - chosen among the previous neighborhoods L_β and C_β- associated with both vertices should rely on the distance $\alpha = d(u,v)$. That means that the function β should not be constant but be a function of the distance of the vertices of the graph.

Definition 3 *Let β be a bounded continuous function on $[0,+\infty[$ with values in $]0,+\infty[$. A generalized β-skeleton graph $G^\star_\beta = (V, E_\beta)$ is defined as:*
- V is the set of vertices, a locally finite subset of $\mathbb{R}^2$.
- E_β is the set of edges and N_β the associated neighborhood: it corresponds to one of the two following sets:
1. $\forall (u,v) \in V^2, uv \in E_\beta \Leftrightarrow N_\beta(u,v) \cap V \backslash \{u,v\} = \emptyset$ with

$$\begin{cases} N_\beta(\cdot) = L_\beta(\cdot) \text{ if } \beta(d(u,v)) \geq 1 \\ N_\beta(\cdot) = C_\beta(\cdot) \text{ if } \beta(d(u,v)) \leq 1 \end{cases}$$

and this graph denoted by G^L_β is a generalization of the lune-based skeleton graph.
2. $\forall (u,v) \in V^2, uv \in E_\beta \Leftrightarrow N_\beta(u,v) \cap V \backslash \{u,v\} = \emptyset$ with $N_\beta(\cdot) = C_\beta(\cdot)$ and this graph denoted by G^C_β is the circle-based skeleton graph.

Remarks : -$G^\star_\beta$-graphs include the β-skeletons graphs [13] when β is constant (and so independent of the distance between two points of V). When β is constant greater than 1, G^L_β gives the lune-based skeleton graph and the point 2 maps to the circle-based skeleton graph G^C_β. Likewise, when β is lower than one, the points 1 and 2 correspond to G^C_β .
-As in the case of the β_0-skeleton graphs, the property of "monoticity" holds: the generalized β-skeleton graphs $G^\star_\beta = (V, E_\beta)$ family (when we choose the same definition of the set of edges) is monotonic with respect to the function β, i.e. $G^\star_{\beta_1} = (V, E_{\beta_1}) \subset G^\star_{\beta_2} = (V, E_{\beta_2})$ $(E_{\beta_1} \subset E_{\beta_2})$ for all functions $\beta_1(\cdot) > \beta_2(\cdot)$.
-The choice of the function β allows to build some $G^\star_\beta$-graphs with different vacuity regions.



$L_{\beta_0}(u,v)$ with $\beta_0 \geq 1$ $\quad$ $C_{\beta_0}(u,v)$ with $\beta_0 \geq 1$ $\quad$ $C_{\beta_0}(u,v)$ with $0 < \beta_0 \leq 1$

Figure 3: Various forms of neighborhoods of β_0-skeleton graphs

4.2 The choice of the parameter q

We give the areas of the neighborhoods and explicit values and bounds of q according to the different definitions of the $G^\star_\beta$-graph.

Lemma 1 *Following the definition 3, given a* $G^\star_\beta = (V, E_\beta)$ *graph, we denote by* $\beta_M = \max_{\alpha, 0 \le \alpha \le s}(\beta(\alpha), 1)$ *and, for convenience, we replace the notation* $\beta(\alpha)$ *by* β.

1. *For the first case of the definition 1, when* $\begin{cases} N_\beta(\cdot) = L_\beta(\cdot) \text{ if } \beta(\cdot) \ge 1 \\ N_\beta(\cdot) = C_\beta(\cdot) \text{ if } \beta(\cdot) \le 1 \end{cases}$,
   if $N_\beta(\cdot) = L_\beta(\cdot)$ *then* $|N_\beta(\alpha)| = \frac{\alpha^2}{2}\left[\beta^2 \arcsin\left(\frac{\sqrt{2\beta - 1}}{\beta}\right) - (\beta - 1)\sqrt{2\beta - 1}\right]$.
   if $N_\beta(\cdot) = C_\beta(\cdot)$ *then* $|N_\beta(\alpha)| = \frac{\alpha^2}{2}\left[\frac{\pi}{2\beta^2} - \frac{\arccos(\beta)}{\beta^2} - \sqrt{\frac{1}{\beta^2} - 1}\right]$.
   It comes that

$$q = \max(q_1, q_2) \; with \begin{cases} q_1 = \max_{\alpha, 0 \le \alpha \le s, \beta(\alpha) < 1}\left[\frac{\alpha}{2}\left(\frac{1}{\beta} - \sqrt{\frac{1}{\beta^2} - 1}\right) - \left(r - \sqrt{r^2 - \frac{\alpha^2}{4}}\right)\right] \\ q_2 = \max_{\alpha, 0 \le \alpha \le s, \beta(\alpha) \ge 1}\left[\frac{\alpha}{2}\sqrt{2\beta - 1} - \left(r - \sqrt{r^2 - \frac{\alpha^2}{4}}\right)\right] \end{cases}$$

   and

$$q \le q_M = r(\sqrt{2\beta_M} - 1).$$

2. *For the second case, when* $\beta(\cdot) > 1$ *and* $N_\beta(\cdot) = C_\beta(\cdot)$,
   $|N_\beta(\alpha)| = \frac{\alpha^2}{2}\left[\frac{\pi\beta^2}{2} + \beta^2 \arccos\left(\frac{1}{\beta}\right) + \sqrt{\beta^2 - 1}\right]$. *It comes that*

$$q = \max(q_1, q_3) \; with \begin{cases} q_1 = \max_{\alpha, 0 \le \alpha \le s, \beta(\alpha) < 1}\left[\frac{\alpha}{2}\left(\frac{1}{\beta} - \sqrt{\frac{1}{\beta^2} - 1}\right) - \left(r - \sqrt{r^2 - \frac{\alpha^2}{4}}\right)\right] \\ q_3 = \max_{\alpha, 0 \le \alpha \le s, \beta(\alpha) \ge 1}\left[\frac{\alpha}{2}\left(\sqrt{\beta^2 - 1} + \beta\right) - \left(r - \sqrt{r^2 - \frac{\alpha^2}{4}}\right)\right] \end{cases}$$

   and

$$q \le q_M = r\left(\sqrt{2\beta_M}\sqrt{\beta_M + \sqrt{\beta_M^2 - 1}} - 1\right).$$

Remark that, in both cases in the lemma 1, when β goes to 0, the area of the corresponding neighborhood tends to 0.

4.3 Percolation in a class of generalized β-skeleton graphs

We consider the following class $\mathcal{P}$ of $G^\star_\beta$-graphs:

Definition 4 *The class* $\mathcal{P}$ *is the set of generalized* β*-skeleton graphs* $G = (V, E_\beta)$ *where:*
- the set of vertices $V = \varphi$ *is a realization of the Poisson point process* Φ *on the plane* $\mathbb{R}^2$

- β *is a bounded continuous function on* $[0,+\infty[$ *with values in* $]0,+\infty[$ *for which there exist a non negative real* r *and a real* s*,* $0 \leq s \leq 2r$*, as*

$$\int_0^s \alpha \arccos\left(\frac{\alpha}{2r}\right)\left(1-e^{-|N_\beta(\alpha)|}\right) d\alpha \quad \leq \frac{0.06805 - e^{-\pi r^2}}{4r(2r+2q)} - \frac{e^{-R(r,s)}}{2} \qquad (4)$$

with $R(r,s) = -rs\sqrt{1-\frac{s^2}{4r^2}} + (2r^2 - s^2)\arcsin\left(\frac{s}{2r}\right) + \frac{\pi}{2}s^2$*.*
$N_\beta(\alpha)$ *is the neighboorhood associated with the points* O *and* u *with* $\alpha = d(u,O)$ *when* u *moves in* $R(O,r,s)$*.*

The following theorem is a consequence of the theorem 2.

Theorem 3 *For all generalized* β*-skeleton graphs* $G^\star_\beta$ *of* $\mathcal{P}$*, we have*

$$P(\mathcal{E}_{S_1,S_2} \cap \mathcal{E}_{S_2,S_1}) \geq 0.8639. \qquad (5)$$

So, there is almost surely an infinite component in $G^\star_\beta$ *under the stationary Poisson point process. The continuum percolation occurs almost surely in these graphs.*

Moreover, in particular, the lune-based skeleton graph is not always connected for $\beta > 2$ and the circle-based skeleton graph for $\beta > 1$. The two following results give an answer for generalized β- skeleton graph when the maximum of the function β is greater than 2 in the first case and than 1 for the other:

Corollary 1 *Consider a generalized* β*-skeleton graphs* $G^\star_\beta$ *of* $\mathcal{P}$ *as*

$$\beta_M = \max_{\alpha\in[0;+\infty[} \beta(\alpha) > 2.$$

The continuum percolation occurs in $G^\star_\beta$ *(point 1 of the definition 3).*

And, in the same way,

Corollary 2 *Consider a generalized* β*-skeleton graphs* $G^\star_\beta$ *of* $\mathcal{P}$ *as*

$$\beta_M = \max_{\alpha\in[0;+\infty[} \beta(\alpha) > 1.$$

The continuum percolation occurs in $G^\star_\beta$ *(point 2 of the definition 3).*

Some examples of graphs of $\mathcal{P}$

1. We have a sufficient condition to obtain inequality (4). For all function β so that it exists a non negative real r as

$$\forall \alpha \in [0;2r],\ |N_\beta(\alpha)| \leq \ln\left[\frac{8r(2r+2q_M)\pi r^2}{8r(2r+2q_M)\pi r^2 - 0.1361 + 2e^{-\pi r^2} + 4r(2r+2q_M)e^{-\pi r^2}}\right],$$

the inequality (4) holds (q_M is given in the lemma 1). Thus, the continuous percolation occurs for the $G^\star_\beta$-graph associated with the previous function β.

2. To compute the integral in the inequality (4), a simple way is to consider a function β constant equal to some β_M on an interval $[0, t]$ and function of α on the interval $[t, s]$ so that $|N_\beta(\alpha)| = |N_{\beta_M}(t)|$ for all α in $[t, s]$. In table 1, we give particular values of β_M and the associated values of s, r and t for which the inequality (4) holds.

Table 1: Some numerical results given β_M

β_M	Form of $N_\beta(\cdot)$ given in the definition 1	r	s	a ($t = a/100 \times s$)
2	$L_\beta(\cdot)$ followed by $C_\beta(\cdot)$	1.491	2.731	0.631
3	$L_\beta(\cdot)$ followed by $C_\beta(\cdot)$	1.515	2.824	0.484
2	$C_\beta(\cdot)$	1.6	2.882	0.176
3	$C_\beta(\cdot)$	1.7	2.862	0.087

3. Using the same idea, we can obtain a similar result as the theorem 3 when the $G^\star_\beta$-graph is directed as follows: given a vertex u of the graph with polar coordinates (α, θ), the neighborhood $N_\beta(O, u)$ depends on $\alpha = d(O, u)$ and of θ. That means the function β is function of the length of the edges and of the angle. In the proof of the theorem 3 and, so, in the definition 4, only the integral in the inequality (3) is replaced as following:

$$2 \int_0^s \alpha \arccos\left(\frac{\alpha}{2r}\right) \left(1 - e^{-|N_\beta(\alpha)|}\right) d\alpha \to \int_0^s \int_{-\arccos(\alpha/(2r))}^{\arccos(\alpha/(2r))} \alpha \left(1 - e^{-|N_\beta(\alpha,\theta)|}\right) d\theta \, d\alpha.$$

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