New results on approximate Hilbert pairs of wavelet filters with common factors

Sophie Achard, Marianne Clausel, Irène Gannaz, François Roueff

New results on approximate Hilbert pairs of wavelet filters with common factors

Sophie Achard[a], Marianne Clausel[b], Irène Gannaz[c], and François Roueff[d]

[a]Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-lab, 38000 Grenoble, France
[b]Univ. Grenoble Alpes, CNRS, Grenoble INP, LJK, 38000 Grenoble, France *
[c]Université de Lyon, CNRS UMR 5208, INSA de Lyon, Institut Camille Jordan, France
[d]LTCI, Télécom Paris, Institut polytechnique de Paris, France

June 13, 2019

Abstract

In this paper, we consider the design of wavelet filters based on the *Thiran common-factor* approach proposed in Selesnick [2001]. This approach aims at building finite impulse response filters of a *Hilbert-pair* of wavelets serving as real and imaginary part of a complex wavelet. Unfortunately it is not possible to construct wavelets which are both finitely supported and analytic. The wavelet filters constructed using the *common-factor* approach are then approximately analytic. Thus, it is of interest to control their analyticity. The purpose of this paper is to first provide precise and explicit expressions as well as easily exploitable bounds for quantifying the analytic approximation of this complex wavelet. Then, we prove the existence of such filters enjoying the classical perfect reconstruction conditions, with arbitrarily many vanishing moments.

Keywords. Complex wavelet, Hilbert-pair, orthonormal filter banks, common-factor wavelets

1 Introduction

Wavelet transforms provide efficient representations for a wide class of signals. In particular signals with singularities may have a sparser representation compared to the representation in Fourier basis. Yet, an advantage of Fourier transform is its analyticity, which enables to exploit both the magnitude and the phase in signal analysis. In order to combine both advantages of Fourier and real wavelet transform, one possibility is to use a complex wavelet transform. The analyticity can be obtained by choosing properly the wavelet filters. This may offer a true enhancement of real wavelet transform for example in singularity extraction purposes.

*At present Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France

We refer to Selesnick et al. [2005], Tay [2007] and references therein for an overview of the motivations for analytic wavelet transforms. A wide range of applications can be addressed using such wavelets as image analysis [Chaux et al., 2006], signal processing [Wang et al., 2010], molecular biology [Murugesan et al., 2015], neuroscience [Whitcher et al., 2005].

Several approaches have been proposed to design a pair of wavelet filters where one wavelet is (approximately) the Hilbert transform of the other. Using this pair as real and imaginary part of a complex wavelet allows the design of (approximately) analytic wavelets. The simplest complex analytic wavelets are the generalized Morse wavelets, which are used in continuous wavelet transforms in Lilly and Olhede [2010]. The approximately analytic Morlet wavelets can also be used for the same purpose, see Selesnick et al. [2005]. However, for practical or theoretical reasons, it is interesting to use discrete wavelet transforms with finite filters, in which case it is not possible to design a perfectly analytic wavelets. In addition to the finite support property, one often requires the wavelet to enjoy sufficiently many vanishing moments, perfect reconstruction, and smoothness properties. Among others, linear-phase biorthogonal filters were proposed in Kingsbury [1998a,b] or q-shift filters in Kingsbury [2000]. We will focus here on the *common-factor* approach, developed in Selesnick [2001, 2002]. In Selesnick [2002] a numerical algorithm is proposed to compute the FIR filters associated to an approximate Hilbert pair of orthogonal wavelet bases. Improvements of this method have been proposed recently in Tay [2010], Murugesan and Tay [2014]. The approach of Selesnick [2001] is particularly attractive as it builds upon the usual orthogonal wavelet base construction by solving a Bezout polynomial equation. Nevertheless, to the best of our knowledge, the validity of this specific construction has not been proved. Moreover the quality of the analytic approximation has not been thoroughly assessed. The main goal of this paper is to fill these gaps. We also provide a short simulation study to numerically evaluate the quality of analyticity approximation for specific *common-factor* wavelets.

After recalling the definition of Hilbert pair wavelet filters, the construction of the Thiran's *common-factor* wavelets following Thiran [1971], Selesnick [2002] is summarized in Section 2. Theoretical results are then developed to evaluate the impact of the Thiran's *common-factor* degree L on the analytic property of the derived complex wavelet. In Section 3, an explicit formula to quantify the analytic approximation is derived. In addition, we provide a bound demonstrating the improvement of the analytic property as L increases. These results apply to all wavelets obtained from FIR filters with Thiran's *common-factor*. Of particular interest are the orthogonal wavelet bases with perfect reconstruction. Section 4 is devoted to proving the existence of such wavelets arising from filters with Thiran's *common-factor*, which correspond to the wavelets introduced in Selesnick [2001, 2002]. Finally, in Section 5, some numerical examples illustrate our findings. All proofs are given in the Appendices.

2 Approximate Hilbert pair wavelets

2.1 Wavelet filters of a Hilbert pair

Let ψ_G and ψ_H be two real-valued wavelet functions. Denote by $\widehat{\psi}_G$ and $\widehat{\psi}_H$ their Fourier transform,

$$\widehat{\psi}_G(\omega) = \int \psi_G(t)\, \mathrm{e}^{-\mathrm{i}t\omega}\, \mathrm{d}t\ .$$

We say that (ψ_G, ψ_H) forms a Hilbert pair if

$$\widehat{\psi}_G(\omega) = -\mathrm{i}\ \mathrm{sign}(\omega)\widehat{\psi}_H(\omega)\ ,$$

where $\mathrm{sign}(\omega)$ denotes the sign function taking values $-1, 0$ and 1 for $\omega < 0$, $\omega = 0$ and $\omega > 0$, respectively. Then the complex-valued wavelet $\psi_H(t) + \mathrm{i}\psi_G(t)$ is analytic since its Fourier transform is only supported on the positive frequency semi-axis.

Suppose now that the two above wavelets are obtained from the (real-valued) low-pass filters $(g_0(n))_{n\in\mathbb{Z}}$ and $(h_0(n))_{n\in\mathbb{Z}}$, using the usual multi-resolution scheme (see Daubechies [1992]). We denote their z-transforms by $G_0(\cdot)$ and $H_0(\cdot)$, respectively. In Selesnick [2001] and Ozkaramanli and Yu [2003], it is established that a necessary and sufficient condition for (ψ_G, ψ_H) to form a Hilbert pair is to satisfy, for all $\omega \in (-\pi, \pi)$,

$$G_0(\mathrm{e}^{\mathrm{i}\omega}) = H_0(\mathrm{e}^{\mathrm{i}\omega})\mathrm{e}^{-\mathrm{i}\omega/2}\ . \tag{1}$$

Since $\mathrm{e}^{-\mathrm{i}\omega/2}$ takes different values at $\omega = \pi$ and $\omega = -\pi$, we see that this formula cannot hold if both G_0 and H_0 are non-zero polynomials or rational fractions, which indicates that the construction of Hilbert pairs cannot be obtained with finite impulse response (FIR) filters or recursive filters. In particular a strict analytic property for the wavelet is not achievable for a compactly supported wavelet, which is also a direct consequence of the Paley-Wiener theorem.

However, in certain cases, it is preferable to preserve the compact support property of the wavelet and the corresponding FIR property of the filters. We will hereafter focus on this case, although Selesnick [2002] also proposed quasi-analytic recursive filters with infinite impulse responses by adapting the construction in Herley and Vetterli [1993] to common-factor paired filters.

Under the FIR constraint, the strict analytic condition (1) has to be relaxed into an approximation around the zero frequency,

$$G_0(\mathrm{e}^{\mathrm{i}\omega}) \sim H_0(\mathrm{e}^{\mathrm{i}\omega})\mathrm{e}^{-\mathrm{i}\omega/2} \quad \text{as} \quad \omega \to 0\ . \tag{2}$$

Several constructions have then been proposed to define approximate Hilbert pair wavelets, that is, pairs of wavelet functions satisfying the quasi analytic condition (2) [Tay, 2007]. The *common-factor* procedure proposed in Selesnick [2002], is giving one solution to the construction of approximate Hilbert pair wavelets. This is the focus of the following developments.

2.2 The common-factor procedure

The *common-factor* procedure [Selesnick, 2002] is designed to provide approximate Hilbert pair wavelets driven by an integer $L \geqslant 1$ and additional properties relying on a *common factor* transfer function F. Namely, the solution reads

$$H_0(z) = F(z)D_L(z) \ , \tag{3}$$

$$G_0(z) = F(z)D_L(1/z)z^{-L} \ , \tag{4}$$

where D_L is the z transform of a causal FIR filter of order L, $D_L(z) = 1 + \sum_{\ell=1}^{L} d(\ell)z^{-\ell}$, such that

$$\frac{\mathrm{e}^{-\mathrm{i}\omega L} D_L(\mathrm{e}^{-\mathrm{i}\omega})}{D_L(\mathrm{e}^{\mathrm{i}\omega})} = \mathrm{e}^{-\mathrm{i}\omega/2} + O(\omega^{2L+1}) \quad \text{as} \quad \omega \to 0 \ . \tag{5}$$

In Thiran [1971], a causal FIR filter satisfying this constraint is defined, the so-called *maximally flat* solution given by (see also [Selesnick, 2002, Eq (2)]):

$$d(\ell) = (-1)^n \binom{L}{\ell} \prod_{k=0}^{\ell-1} \frac{1/2 - L + k}{3/2 + k}, \quad \ell = 1, \ldots, L. \tag{6}$$

The cornerstone of our subsequent results is the following simple expression for $D_L(z)$, which appears to be new, up to our best knowledge.

Proposition 1. *Let L be a positive integer and $D_L(z) = 1 + \sum_{n=1}^{L} d(n)z^{-n}$ where the coefficients $(d(n))_n$ are defined by (6). Then, for all $z \in \mathbb{C}^*$, we have*

$$D_L(z) = \frac{1}{2(2L+1)} z^{-L} \left[(1 + z^{1/2})^{2L+1} + (1 - z^{1/2})^{2L+1} \right] , \tag{7}$$

where $z^{1/2}$ denotes either of the two complex numbers whose squares are equal to z.

Here $\mathbb{C}^*$ denotes the set of all non-zero complex numbers.

Remark 1. In spite of the ambiguity in the definition of $z^{1/2}$, the right-hand side in (7) is unambiguous because, when developing the two factors in the expression between square brackets, all the odd powers of $z^{1/2}$ cancel out.

Remark 2. It is interesting to note that the closed form expression (7) of D_L directly implies the approximation (5). Indeed, the right-hand side of (7) yields

$$\begin{aligned} D_L(\mathrm{e}^{\mathrm{i}\omega}) &= \frac{1}{2(2L+1)} \mathrm{e}^{-\mathrm{i}\omega(L-1/2)/2} \left[(2\cos(\omega/4))^{2L+1} + (-2\mathrm{i}\sin(\omega/4))^{2L+1} \right] \\ &= \frac{2^{2L}}{2L+1} \mathrm{e}^{-\mathrm{i}\omega(L-1/2)/2} \cos^{2L+1}(\omega/4) + O(\omega^{2L+1}) \ . \end{aligned}$$

It is then straightforward to obtain (5).

Proof. See Section A. □

To summarize the *common-factor* approach, we use the following definition.

Definition (*Common-factor* wavelet filters)**.** For any positive integer L and FIR filter with transfer function F, a pair of wavelet filters $\{H_0, G_0\}$ is called an L-approximate Hilbert wavelet filter pair with common factor F if it satisfies (3) and (4) with $H_0(1) = G_0(1) = \sqrt{2}$.

Condition $H_0(1) = G_0(1) = \sqrt{2}$ is equivalent to

$$F(1) = \frac{\sqrt{2}}{D_L(1)} = \sqrt{2}(2L+1)2^{-2L} \ . \tag{8}$$

A remarkable feature in the choice of the common filter F is that it can be used to ensure additional properties such as an arbitrary number of vanishing moments, perfect reconstruction or smoothness properties.

First an arbitrary number M of vanishing moments is set by writing

$$F(z) = Q(z)(1 + 1/z)^M, \tag{9}$$

with $Q(z)$ the z-transform of a causal FIR filter (hence a real polynomial of z^{-1}).

An additional condition required for the wavelet decomposition is perfect reconstruction. It is acquired when the filters satisfy the following conditions (see Vetterli [1986]):

$$G_0(z)G_0(1/z) + G_0(-z)G_0(-1/z) = 2 \ , \tag{PR-G}$$

$$H_0(z)H_0(1/z) + H_0(-z)H_0(-1/z) = 2 \ . \tag{PR-H}$$

This condition is classically used for deriving wavelet bases $\psi_{Gj,k} = 2^{j/2}\psi_G(2^j \cdot -k)$ and $\psi_{Hj,k} = 2^{j/2}\psi_H(2^j \cdot -k)$, $j, k \in \mathbb{Z}$, which are orthonormal bases of $L^2(\mathbb{R})$. This will be investigated in Section 4.

3 Quasi-analyticity of *common-factor* wavelets

We now investigate the quasi-analyticity properties of the complex wavelet obtained from Hilbert pairs wavelet filters with the *common-factor* procedure.

Let $(\phi_H(\cdot), \psi_H(\cdot))$ be respectively the father and the mother wavelets associated with the (low-pass) wavelet filter H_0. The transfer function H_0 is normalized so that $H_0(1) = \sqrt{2}$ (this is implied by (3) and (8)). The father and mother wavelets can be defined through their Fourier transforms as

$$\widehat{\phi}_H(\omega) = \prod_{j=1}^{\infty} \left[2^{-1/2} H_0(\mathrm{e}^{\mathrm{i}2^{-j}\omega})\right] , \tag{10}$$

$$\widehat{\psi}_H(\omega) = 2^{-1/2} H_1(\mathrm{e}^{\mathrm{i}\,\omega/2})\, \widehat{\phi}_H(\omega/2), \tag{11}$$

where H_1 is the corresponding high-pass filter transfer function defined by $H_1(z) = z^{-1}H_0(-z^{-1})$ (see *e.g.* Selesnick [2001]). We also denote by (ϕ_G, ψ_G) the father and the mother wavelets associated with the wavelet filter G_0. Equations similar to (10) and (11) hold for $\widehat{\phi}_G$, and $\widehat{\psi}_G$ using G_0 and G_1 in place of H_0 and H_1 (see *e.g.* Selesnick [2001]).

We first give an explicit expression of $\widehat{\phi}_G$ and of $\widehat{\psi}_G$ with respect to $\widehat{\phi}_H$ and $\widehat{\psi}_H$.

Theorem 2. *Let L be a positive integer. Let $\{H_0, G_0\}$ be an L-approximate Hilbert wavelet filter pair. Let (ϕ_H, ψ_H) denote the father and mother wavelets defined by (10) and (11) and denote (ϕ_G, ψ_G) the wavelets defined similarly from the filter G_0. Then, we have, for all $\omega \in \mathbb{R}$,*

$$\widehat{\phi}_G(\omega) = \mathrm{e}^{\mathrm{i}\beta_L(\omega)}\, \widehat{\phi}_H(\omega)\mathrm{e}^{-\mathrm{i}\omega/2} \;, \tag{12}$$

$$\widehat{\psi}_G(\omega) = \mathrm{i}\; \mathrm{e}^{\mathrm{i}\eta_L(\omega)}\, \widehat{\psi}_H(\omega) \;. \tag{13}$$

where

$$\alpha_L(\omega) = 2(-1)^L \arctan\left(\tan^{2L+1}(\omega/4)\right) \;, \tag{14}$$

$$\beta_L(\omega) = \sum_{j=1}^{\infty} \alpha_L(2^{-j}\omega) \;, \tag{15}$$

$$\eta_L(\omega) = -\alpha_L(\omega/2+\pi) + \beta_L(\omega/2) \;. \tag{16}$$

In (14), we use the convention $\arctan(\pm\infty) = \pm\pi/2$ *so that α_L is well defined on $\mathbb{R}$.*

Proof. See Section A. □

Following Theorem 2, we can write, for all $\omega \in \mathbb{R}$,

$$\widehat{\psi}_H(\omega) + \mathrm{i}\,\widehat{\psi}_G(\omega) = \left(1 - \mathrm{e}^{\mathrm{i}\eta_L(\omega)}\right) \widehat{\psi}_H(\omega) \;. \tag{17}$$

This formula shows that the quasi-analytic property and the Fourier localization of the complex wavelet $\psi_H + \mathrm{i}\,\psi_G$ can be respectively described by

(a) how close the function $1-\mathrm{e}^{\mathrm{i}\eta_L}$ is to the step function $2\mathbb{1}_{\mathbb{R}_+}$ (or $-\mathrm{e}^{\mathrm{i}\eta_L}$ to the sign function);

(b) how localized the (real) wavelet ψ_H is in the Fourier domain.

Property (b) is a well known feature of wavelets usually described by the behavior of the wavelet at frequency 0 (e.g. M vanishing moments implies a behavior in $O(|\omega|^M)$) and by the polynomial decay at high frequencies. This behavior depends on the wavelet filter (see Villemoes [1992], Eirola [1992], Ojanen [2001]) and a numerical study of property (b) is provided in Section 5.

Note that, remarkably, property (a), **only depends on** L. Figure 1 displays the function $1 - \mathrm{e}^{\mathrm{i}\eta_L}$ for various values of L. It illustrates the fact that as L grows, $1 - \mathrm{e}^{\mathrm{i}\eta_L}$ indeed gets closer and closer to the step function $2\mathbb{1}_{\mathbb{R}_+}$. We can actually prove the following result which bounds how close the Fourier transform of the wavelet $\psi_H + \mathrm{i}\,\psi_G$ is to $2\mathbb{1}_{\mathbb{R}_+}\widehat{\psi}_H$.

Denote, for all $\omega \in \mathbb{R}$ and $A \subset \mathbb{R}$, the distance of ω to A by

$$\delta(\omega, A) = \inf\left\{|\omega - x| \;:\; x \in A\right\} \;. \tag{18}$$

Theorem 3. *Under the same assumptions as Theorem 2, we have, for all $\omega \in \mathbb{R}$,*

$$\left|\widehat{\psi}_H(\omega) + \mathrm{i}\,\widehat{\psi}_G(\omega) - 2\mathbb{1}_{\mathbb{R}_+}(\omega)\,\widehat{\psi}_H(\omega)\right| = U_L(\omega)\left|\widehat{\psi}_H(\omega)\right| ,$$

where U_L is a $\mathbb{R} \to [0, 2]$ function satisfying, for all $\omega \in \mathbb{R}$,

$$U_L(\omega) \leqslant 2\sqrt{2}\left(\log_2\left(\frac{\max(4\pi, |\omega|)}{2\pi}\right) + 2\right)\left(1 - \frac{\delta(\omega, 4\pi\mathbb{Z})}{\max(4\pi, |\omega|)}\right)^{2L+1} . \tag{19}$$

Proof. See Section A. □

This result provides a control over the difference between the Fourier transform $\widehat{\psi}_H + \mathrm{i}\,\widehat{\psi}_G$ of the complex wavelet and the Fourier transform $2\mathbb{1}_{\mathbb{R}_+}\widehat{\psi}_H$ of the analytic signal associated to ψ_H. In particular, as $L \to \infty$, the relative difference $U_L = \left|\widehat{\psi}_H + \mathrm{i}\,\widehat{\psi}_G - 2\mathbb{1}_{\mathbb{R}_+}\widehat{\psi}_H\right| / \left|\widehat{\psi}_H\right|$ converges to zero exponentially fast on any compact subsets that do not intersect $4\pi\mathbb{Z}$.

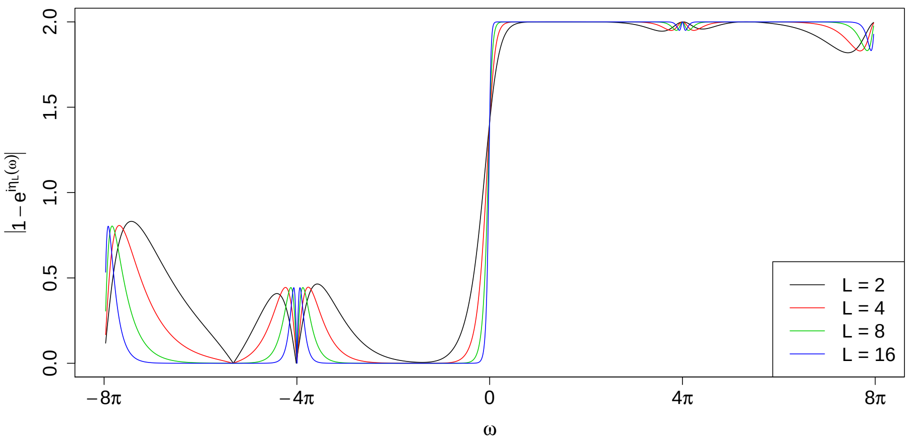

Figure 1: Plots of the function $\omega \mapsto |1 - \mathrm{e}^{\mathrm{i}\eta_L(\omega)}|$ for $L = 2, 4, 8, 16$.

4 Solutions with perfect reconstruction

Let us now follow the path paved by Selesnick [2002] to select Q appearing in the factorization (9) of the common factor F to impose M vanishing moments. First observe that, under (3), (4) and (9), the perfect reconstruction conditions (PR-G) and (PR-H) both follow from

$$R(z)S(z) + R(-z)S(-z) = 2 , \tag{20}$$

where we have set $R(z) = Q(z)Q(1/z)$ and $S(z) = (2 + z + 1/z)^M D_L(z)D_L(1/z)$.

To achieve (20), the following procedure is proposed in Selesnick [2002], which follows the approach in Daubechies [1992] adapted to the common factor constraint in (3).

Step 1 Find R with finite, real and symmetric impulse response satisfying (20).

Step 2 Find a real polynomial $Q(1/z)$ satisfying the factorization $R(z) = Q(z)Q(1/z)$.

However, in Selesnick [2002], the existence of solutions R and Q is not proven, although numerical procedures indicate that solutions can be exhibited. We shall now fill this gap and show the existence of such solutions for any integers $M, L \geqslant 1$.

We first establish the set of solutions for R.

Proposition 4. *Let L and M be two positive integers. Let D_L be defined as in Proposition 1 and let $S(z) = (2 + z + 1/z)^M D_L(z)D_L(1/z)$. Then the two following assertions hold.*

(i) There exists a unique real polynomial r of degree at most $M + L - 1$ such that $R(z) = r\left(\frac{2+z+1/z}{4}\right)$ satisfies (20) for all $z \in \mathbb{C}^*$.

(ii) For any real polynomial p, the function $R(z) = p\left(\frac{2+z+1/z}{4}\right)$ satisfies (20) on $z \in \mathbb{C}^*$ if and only if it satisfies

$$p(y) = r(y) + s(1-y)\, q(y) \;, \tag{21}$$

where

$$s(y) = y^M \sum_{n=0}^{L} \binom{2L+1}{2n} y^n \;, \tag{22}$$

and q is any real polynomial satisfying $q(1-y) = -q(y)$.

Proof. See Section B. □

Proposition 4 provides a justification of **Step 1**. In particular, a natural candidate for **Step 1** is $R(z) = r\left(\frac{2+z+1/z}{4}\right)$. Now, by the Riesz Lemma (see e.g. [Daubechies, 1992, Lemma 6.1.3]), the factorization of **Step 2** holds if and only if $R(z)$ takes its values in $\mathbb{R}_+$ on the unit circle $\{z \in \mathbb{C} \;:\; |z| = 1\}$. This is equivalent to check that $r(y) \geqslant 0$ for all $y \in [0, 1]$. In practice, this property can be checked as follows:

1. compute the coefficients of the polynomial r using a numerical procedure (see Section 5);
2. derive a numerical evaluation of its roots;
3. check that these roots lie away of $(0, 1)$.

Since $r(1/2) > 0$ (see Appendix B), this numerical procedure guaranties that $r(y) \geqslant 0$ for all $y \in [0, 1]$. It turned out to be successful for all values of L and M tested in Section 5. Whether $r(y) \geqslant 0$ for all $y \in [0, 1]$ for any values of the integers L and M remains an open question, which we were unable to answer. Nevertheless we next prove that **Step 2** can always be carried out for any $L, M \geqslant 1$, if not by $R(z) = r\left(\frac{2+z+1/z}{4}\right)$, at least by replacing r by a polynomial p of the form (21) with a conveniently chosen q.

Theorem 5. *Let L and M be two positive integers and let r and s be the polynomials defined as in Proposition 4. Then there exists a real polynomial q such that $R(z) = [r + s\,q]\left(\frac{2+z+1/z}{4}\right)$ is a solution of (20) and satisfies the factorization $R(z) = Q(z)Q(1/z)$ where $Q(1/z)$, real polynomial of z, does not vanish on the unit circle.*

Proof. See Section B. □

Proposition 4 and Theorem 5 allows one to carry out the usual program to the construction of compactly supported orthonormal wavelet bases, as described in Daubechies [1992]. Hence we get the following.

Corollary 6. *Let L and M be two positive integers. Let Q be as in Theorem 5. Define F as in (9) and let $\{H_0, G_0\}$ be the L-approximate Hilbert wavelet filter pair associated to F. Then the wavelet bases $(\psi_{H,j,k})$ and $(\psi_{G,j,k})$ are orthonormal bases of $L^2(\mathbb{R})$.*

Observe that Theorem 5 states the existence of the polynomial Q but does not define it in a unique way. We explain why in the following remark.

Remark 3. Since r in Proposition 4 is defined uniquely, it follows that, if we require that all the roots of Q are inside the unit circle, there is *at most one solution* for Q with degree at most $K = M + L - 1$, which corresponds to the case $q = 0$. This solution, when it exists, is usually called the *minimal phase, minimal degree* solution. However we were not able to prove that r does not vanish on $[0, 1]$, which is a necessary and sufficient condition to obtain such a minimal degree solution for Q. Hence we instead prove the existence of solutions for Q by allowing q to be non-zero.

5 Numerical computation of approximate Hilbert wavelet filters

5.1 State of the art

Let M and L be positive integers. Then, by Theorem 5, we can define the polynomial Q and derive from its coefficients the impulse response of the corresponding L-approximate Hilbert wavelet filter pair with M vanishing moments and perfect reconstruction.

We now discuss the numerical computation of the coefficients of Q in the case where the polynomial r defined by Proposition 4 does not vanish on $[0, 1]$. Indeed suppose that one can obtain a numerical computation of this polynomial r. Then the roots of r can also be computed by a numerical solver and, as explained in Remark 3, if they do not belong on $[0, 1]$ (which has to be checked taking into account the possible numerical errors), it only remains to factorize $R(z) = r((2 + z + 1/z)/4)$ into $Q(z)Q(1/z)$ by separating the roots conveniently. Taking all roots of modulus inferior to 1 leads to "min-phase" wavelets. There are other ways of factorizing R, namely "mid-phase" wavelets, see Selesnick [2002], leading to wavelets with

Fourier transform of the same magnitude but with different phases. This difference can be useful in some multidimensional applications where the phase is essential.

Hence the computation of the wavelet filters boils down to the numerical computation of the polynomial r defined by Proposition 4. In Selesnick [2002], this computation is achieved by using the following algorithm.

- Let $s_1 = (\binom{k}{2M})_{k=0,\dots,2M}$ and $s_2 = \begin{pmatrix} d_L(0) & \dots & d_L(L) \end{pmatrix} \star \begin{pmatrix} d_L(L) & \dots & d_L(0) \end{pmatrix}$, where $\star$ denotes the convolution for sequences. Then $S(z) = (2+z+1/z)^M D_L(z) D_L(1/z) = \sum_{n=0}^{2(M+L)} s(n) z^{n-(M+L)}$ with $s = s_1 \star s_2$. The filter s has length $2(M+L)+1$.
- The filter r is such that $s \star r$ is half-band. Let T denote the Toeplitz matrix associated with $\begin{pmatrix} 0 \dots 0 & s \end{pmatrix}$, vector of length $4(M+L)+1$, that is, $T_{k,j} = s(1+(k-j))$ if $0 \leqslant k-j \leqslant 2(M+L)$ and $T_{k,j} = 0$ else. We introduce C the matrix obtained by keeping only the even rows of T, which has size $(2(M+L)-1) \times (2(M+L)-1)$. Then r is the solution of the equation
$$Cr = b \tag{23}$$
with $b = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}$ a $2(M+L)-1$-vector with a 1 at the middle (*i.e.* at $(M+L)$-th position).

We implemented this linear inversion method but it turned out that the corresponding linear equation is ill-conditioned for too high values of M and L (for instance $M = L = 7$). For smaller values of L and M, we recover the wavelet filters of the `hilbert.filter` program of the `R`-package `waveslim` computed only for (M, L) equal to (3,3), (3,5), (4,2) and (4,4), see Whitcher [2015].

5.2 A recursive approach to the computation of the Bezout minimal degree solution

We propose now a new method for computing the L-approximate *common-factor* wavelet pairs with M vanishing moments under the perfect reconstruction constraint. As explained previously, this computation reduces to determining the coefficients of the polynomial r defined in Proposition 4. Our approach is intended as an alternative to the linear system resolution step of the approach proposed in Selesnick [2002]. Since our algorithm is recursive, to avoid any ambiguity, we add the subscripts L, M for denoting the polynomials r and s appearing in 4. That is, we set
$$s_{L,M}(y) = y^M \sum_{n=0}^{L} \binom{2L+1}{2n} y^n$$
and $r_{L,M}$ is the unique polynomial of degree at most $M+L-1$ satisfying the Bezout equation

$$[B(L,M)] \qquad r_{L,M}(1-y) s_{L,M}(1-y) + r_{L,M}(y) s_{L,M}(y) = (2L+1)^2\, 2^{-2L-2M+1} \,.$$

We propose to compute $r_{L,M}$ for all $L \geqslant 1$, $M \geqslant 0$ by using the following result.

Proposition 7. *Let $L \geqslant 1$. Define*

$$y_{k,L} = -\tan^2\left(\frac{\pi(2k+1)}{2(2L+1)}\right) , \qquad k \in \{0, \cdots, L-1\} . \tag{24}$$

Then the solution $r_{L,0}$ of the Bezout equation $[B(L,0)]$ is given by

$$r_{L,0}(y) = (2L+1)^2\, 2^{-2L+1} \sum_{k=0}^{L-1} \frac{\prod_{m\neq k}(y-(1-y_{m,L}))}{s_{L,0}(1-y_{k,L})\prod_{m\neq k}(y_{m,L}-y_{k,L})} . \tag{25}$$

Moreover, for all $M \geqslant 1$, we have the following relation between the solution of $[B(L,M)]$ and that of $[B(L,M-1)]$:

$$4\,y\,r_{L,M}(y) = r_{L,M-1}(y) \;-\; 2^{-2L}\, r_{L,M-1}(0)\,(1-2y)\,s_{L,M-1}(1-y) . \tag{26}$$

Proof. See Appendix C. □

This result provides a recursive way to compute $r_{L,M}$ by starting with $r_{L,0}$ using the interpolation formula (25) and then using the recursive formula (26) to compute $r_{L,1}, r_{L,2}, \ldots$ up to $r_{L,M}$. In contrast to the method of Selesnick [2002] which consists in solving a (possibly ill-conditioned) linear system, this method is only based on product and composition of polynomials.

5.3 Some numerical result on smoothness and analyticity

We now provide some numerical results on the quality of the analyticity of the L-approximated Hilbert wavelet. All the numerical computations have been carried out by the method of Selesnick [2002] which seems to be the one used by practitioners (as in the software of Whitcher [2015]). Recall that as established in Theorem 3, for all $\omega \in \mathbb{R}$,

$$\left|\widehat{\psi}_H(\omega) + \mathrm{i}\,\widehat{\psi}_G(\omega) - 2\mathbb{1}_{\mathbb{R}_+}(\omega)\,\widehat{\psi}_H(\omega)\right| = U_L(\omega)\left|\widehat{\psi}_H(\omega)\right| ,$$

where U_L is displayed in Figure 2. Thus the quality of analyticity relies on the behavior of U_L but also of $\widehat{\psi}_H(\omega)$. First, $\widehat{\psi}_H(\omega)$ goes to 0 when $\omega \to 0$ thanks to the property of M vanishing moments given by (9). Secondly, $|\widehat{\psi}_H(\omega)|$ decays to zero as $|\omega|$ goes to infinity. This last point is verified numerically, by the estimation of the Sobolev exponents of ψ_H using Ojanen [2001]'s algorithm. Values are given in Table 1. For $M > 1$ Sobolev exponents are greater than 1. Notice that "min-phase" and "mid-phase" factorizations of R have the same exponents since the methods do not change the magnitude of $\widehat{\psi}_H + \mathrm{i}\widehat{\psi}_G$.

Figure 2 displays the overall shapes of the Fourier transforms $\widehat{\psi}_H$ of orthonormal wavelets with *common-factor* for various values of M and L. Their quasi-analytic counterparts $\widehat{\psi}_H(\omega) + \mathrm{i}\,\widehat{\psi}_G(\omega)$ are plotted below in the same scales. It illustrates the satisfactory quality of analytic approximation.

Table 1: Sobolev exponent estimated for ψ_H functions. Dots correspond to configurations where numerical instability occurs in the numerical inversion of (23).

M \ L	1	2	3	4	5	6	7	8
1	0.60	0.72	0.81	0.89	0.94	0.98	0.99	1.00
2	1.11	1.23	1.34	1.44	1.54	1.63	1.73	1.82
3	1.52	1.64	1.74	1.83	1.92	2.01	2.09	2.17
4	1.87	1.98	2.07	2.16	2.24	2.32	2.40	2.48
5	2.19	2.29	2.37	2.45	2.53	2.60	2.68	·
6	2.48	2.57	2.65	2.72	2.80	2.87	·	·
7	2.74	2.83	2.91	2.98	3.05	3.12	·	·
8	3.00	3.09	3.16	3.23	3.29	·	·	·

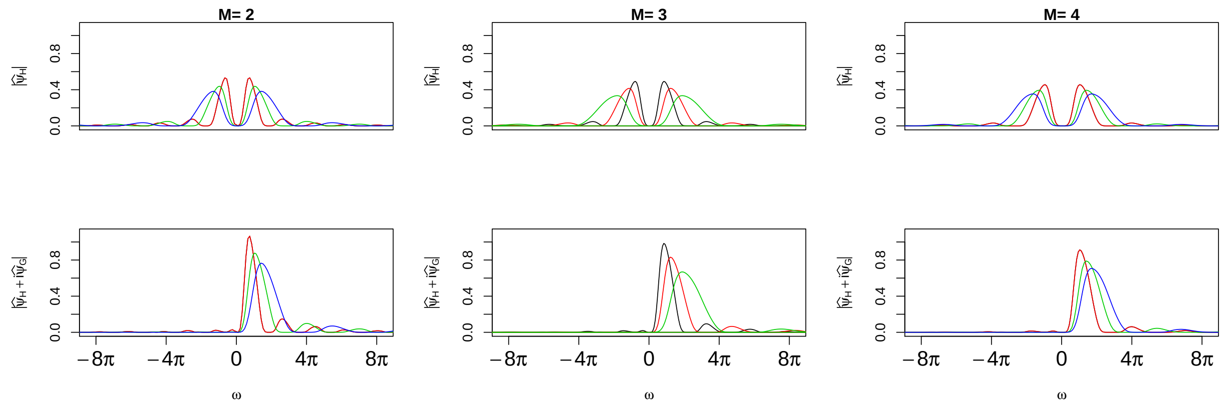

Figure 2: Top row: Plots of $|\widehat{\psi}_H|$ for $M = 2$(left), 3 (center), 4 (right) and $L = 2$ (black), 4 (red), 8 (green). Bottom row: same for $|\widehat{\psi}_H + \mathrm{i}\widehat{\psi}_G|$.

Tay et al. [2006] propose two objective measures of quality based on the spectrum,

$$E_1 = \frac{\max\{|\widehat{\psi}_H(\omega) + \mathrm{i}\widehat{\psi}_G(\omega)|,\, \omega < 0\}}{\max\{|\widehat{\psi}_H(\omega) + \mathrm{i}\widehat{\psi}_G(\omega)|,\, \omega > 0\}} \text{ and } E_2 = \frac{\int_{\omega<0} |\widehat{\psi}_H(\omega) + \mathrm{i}\widehat{\psi}_G(\omega)|^2\, d\omega}{\int_{\omega>0} |\widehat{\psi}_H(\omega) + \mathrm{i}\widehat{\psi}_G(\omega)|^2\, d\omega}.$$

Numerical values of E_1 and E_2 are computed using numerical evaluations of $\widehat{\psi}_H$ on a grid, and, concerning E_2, using Riemann sum approximations of the integrals. Such numerical computations of E_1 and E_2 are displayed in Figure 3 for various values of M and L. The functions E_1 and E_2 are decreasing with respect to L (which corresponds to the behaviour of U_L). They are also decreasing with respect to M (through the faster decay of $\widehat{\psi}_H$ around zero and infinity). Moreover, the values illustrate the good analyticity quality of *common-factor* wavelets. For example, values appear to be lower than those of approximate analytic wavelets based on Bernstein polynomials given in Tay et al. [2006].

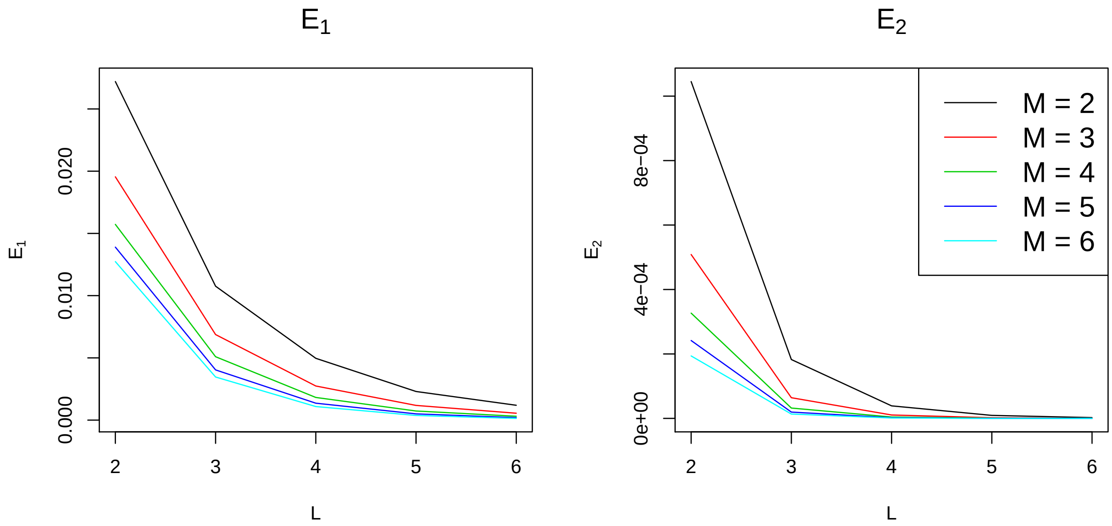

Figure 3: Plot of E_1 and E_2 with respect to L for different values of M.

6 Conclusion

Approximate Hilbert pairs of wavelets are built using the *common-factor* approach. Specific filters are obtained under perfect reconstruction conditions. They depend on two integer parameters L and M which correspond respectively to the order of the analytic approximation and the number of null moments. We demonstrate that the construction of such wavelets is valid by proving their existence for any parameters $L, M \geqslant 1$. Our main contribution in this paper is to provide an exact formula of the relation between the Fourier transforms of the two real wavelets associated to the filters. This expression allows us to evaluate the analyticity approximation of the wavelets, *i.e.* to control the presence of energy at the negative frequency. This result may be useful for applications, where the approximated analytic properties of the wavelet have to be optimized, in addition to the usual localization in time and frequency. Numerical simulations show that these wavelets are easy to compute for not too large values of L and M, and confirm our theoretical findings, namely, that the analytic approximation quickly sharpens as L increases.

7 Acknowledgements

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. The authors would like to thank the referee for his careful reading.

A Proofs of Section 3

A.1 Proof of Proposition 1

Proof of (7). Notice that $d(L)^{-1}z^L D_L(z) = \sum_{n=0}^{L} \frac{d(L-n)}{d(L)} z^n$ and that for all $n = 0, \dots, L-1$,

$$\begin{aligned}
\frac{d(L-n)}{d(L)} &= \binom{L}{n} \prod_{\ell=L-n}^{L-1} \frac{2\ell+3}{2L-2\ell-1} \\
&= \binom{L}{n} \left(\prod_{k=1}^{n} (2k-1) \right)^{-1} \prod_{\ell=L-n+1}^{L} (2\ell+1) \\
&= \frac{L!}{n!(L-n)!} \frac{2^n n!}{(2n)!} \frac{(2L+1)!}{2^L L!} \frac{2^{L-n}(L-n)!}{(2L-2n+1)!} \\
&= \binom{2L+1}{2n}
\end{aligned}$$

It is then easy to check that

$$d(L)^{-1} z^L D_L(z) = \frac{1}{2} \left((1+z^{1/2})^{2L+1} + (1-z^{1/2})^{2L+1} \right).$$

The fact that $d(L) = 1/(2L+1)$ concludes the proof. □

A.2 Technical results on D

We first establish the following result, which will be useful to handle ratios with $D_L(\mathrm{e}^{i\omega})$.

Lemma 8. *Let L be a positive integer. Define D_L as in Proposition 1. Then $D_L(z)$ does not vanish on the unit circle ($|z| = 1$) and*

$$\min_{z\in\mathbb{C}\,:\,|z|=1} |D_L(z)| = |D_L(-1)| = \frac{2^L}{2L+1} < \max_{z\in\mathbb{C}\,:\,|z|=1} |D_L(z)| = |D_L(1)| = \frac{2^{2L}}{2L+1}.$$

Proof. Since $D_L(1/z)$ is a real polynomial of z, we have for all $z \in \mathbb{C}$ such that $|z| = 1$, $|D_L(z)|^2 = D_L(z)D_L(1/z)$. Moreover, as shown in the proof of Proposition 4, if $z = \mathrm{e}^{2\mathrm{i}\theta}$ with $\theta \in \mathbb{R}$, then $|D_L(z)|^2$ reads as in (43), which is minimal and maximal for $\cos(\theta) = 0$ and 1, respectively. □

We now study $z^{-L}\frac{D_L(1/z)}{D_L(z)}$ on the circle.

Lemma 9. *For all $z = \mathrm{e}^{\mathrm{i}\omega}$ with $\omega \in \mathbb{R}$, we have*

$$\mathrm{e}^{-\mathrm{i}\omega L} \frac{D_L(\mathrm{e}^{-\mathrm{i}\omega})}{D_L(\mathrm{e}^{\mathrm{i}\omega})} = \mathrm{e}^{-\mathrm{i}\omega/2 + \mathrm{i}\alpha_L(\omega)}, \tag{27}$$

where α_L is the function defined on $\mathbb{R}$ by (14).

Proof. Observe that, for all $z \in \mathbb{C}^*$, denoting by $z^{1/2}$ any of the two roots of z,

$$z^{-L}\frac{D_L(1/z)}{D_L(z)} = z^L\frac{(1+z^{-1/2})^{2L+1} + (1-z^{-1/2})^{2L+1}}{(1+z^{1/2})^{2L+1} + (1-z^{1/2})^{2L+1}}$$
$$= z^{-1/2}\frac{(1+z^{1/2})^{2L+1} + (z^{1/2}-1)^{2L+1}}{(1+z^{1/2})^{2L+1} - (z^{1/2}-1)^{2L+1}}$$

Set now $z = \mathrm{e}^{\mathrm{i}\omega}$. We deduce that

$$\mathrm{e}^{-\mathrm{i}\omega L}\frac{D_L(\mathrm{e}^{-\mathrm{i}\omega})}{D_L(\mathrm{e}^{\mathrm{i}\omega})} = \mathrm{e}^{-\mathrm{i}\omega/2}\frac{\mathrm{e}^{\mathrm{i}\omega(2L+1)/4}\cos(\omega/4)^{2L+1}(1+\mathrm{i}(-1)^L\tan(\omega/4)^{2L+1})}{\mathrm{e}^{\mathrm{i}\omega(2L+1)/4}\cos(\omega/4)^{2L+1}(1-\mathrm{i}(-1)^L\tan(\omega/4)^{2L+1})}.$$

The result then follows from the classical result $\frac{1+\mathrm{i}a}{1-\mathrm{i}a} = \mathrm{e}^{2\,\mathrm{i}\arctan(a)}$ with here $a = (-1)^L\tan(\omega/4)^{2L+1}$. □

A.3 Proof of Theorem 2

Proof of equality (12). Equation (10) provides the relation between $\widehat{\phi}_H$ and H_0. The same relation holds between $\widehat{\phi}_G$ and G_0. It follows with Lemma 8, (3) and (4), that, for all $\omega \in \mathbb{R}$,

$$\widehat{\phi}_G(\omega) = \widehat{\phi}_H(\omega)\prod_{j=1}^{\infty}\left[\mathrm{e}^{-\mathrm{i}\omega 2^{-j}L}\frac{D_L(\mathrm{e}^{-\mathrm{i}\omega 2^{-j}})}{D_L(\mathrm{e}^{\mathrm{i}\omega 2^{-j}})}\right].$$

Applying Lemma 9, we get that, for all $\omega \in \mathbb{R}$,

$$\widehat{\phi}_G(\omega) = \widehat{\phi}_H(\omega)\prod_{j=1}^{\infty}\mathrm{e}^{-\mathrm{i}\omega 2^{-j}/2+\mathrm{i}\alpha_L(\omega 2^{-j})}$$
$$= \widehat{\phi}_H(\omega)\exp\left(-\mathrm{i}\omega/2\sum_{j=1}^{\infty}2^{-j} + \mathrm{i}\sum_{j=1}^{\infty}\alpha_L(\omega 2^{-j})\right).$$

We thus obtain (12) using the definition of β_L given by (15). □

Proof of equality (13). First observe that the relation between the high-pass filters G_1 and H_1 follows from that between the low-pass filter G_0 and H_0, namely

$$G_1(z) = (-z)^L\frac{D_L(-z)}{D_L(-1/z)}H_1(z).$$

The relationship between $\widehat{\psi}_G$ and $\widehat{\phi}_G$ is given by (11) (exchanging G and H), yielding, for all $\omega \in \mathbb{R}$,

$$\widehat{\psi}_G(\omega) = 2^{-1/2}\,(-1)^L\mathrm{e}^{\mathrm{i}\omega L/2}\frac{D_L(-\mathrm{e}^{\mathrm{i}\omega/2})}{D_L(-\mathrm{e}^{-\mathrm{i}\omega/2})}\,H_1(\mathrm{e}^{\mathrm{i}\omega/2})\widehat{\phi}_G(\omega/2).$$

We now replace $\widehat{\phi}_G$ by the expression obtained in (12) and thanks to (11),

$$\widehat{\psi}_G(\omega) = (-1)^L\mathrm{e}^{\mathrm{i}\omega L/2}\frac{D_L(-\mathrm{e}^{\mathrm{i}\omega/2})}{D_L(-\mathrm{e}^{-\mathrm{i}\omega/2})}\mathrm{e}^{-\mathrm{i}\omega/4}\mathrm{e}^{\mathrm{i}\beta_L(\omega/2)}\widehat{\psi}_H(\omega).$$

Since D_L has a real impulse response and $-1 = \mathrm{e}^{\mathrm{i}\pi} = \mathrm{e}^{-\mathrm{i}\pi}$, Lemma 9 gives that, for all $\omega \in \mathbb{R}$,

$$(-1)^L \mathrm{e}^{\mathrm{i}\omega L/2} \frac{D_L(-\mathrm{e}^{\mathrm{i}\omega/2})}{D_L(-\mathrm{e}^{-\mathrm{i}\omega/2})} = \overline{\mathrm{e}^{-\mathrm{i}L(\omega/2+\pi)} \frac{D_L(\mathrm{e}^{-\mathrm{i}(\omega/2+\pi)})}{D_L(\mathrm{e}^{\mathrm{i}(\omega/2+\pi)})}} = \mathrm{i}\ \mathrm{e}^{\mathrm{i}\omega/4 - \mathrm{i}\alpha_L(\omega/2+\pi)} \ .$$

Hence, we finally get that, for all $\omega \in \mathbb{R}$,

$$\widehat{\psi}_G(\omega) = \mathrm{i}\ \mathrm{e}^{-\mathrm{i}\alpha_L(\omega/2+\pi)+\mathrm{i}\beta_L(\omega/2)} \widehat{\psi}_H(\omega) \ .$$

(13) is proved. □

A.4 Proof of Theorem 3

Approximation of $1 - \mathrm{e}^{\mathrm{i}\eta_L}$

We first state a simple result on the function $\mathrm{e}^{\mathrm{i}\alpha_L}$.

Lemma 10. *Let L be a positive integer. The function α_L defined by (14) is (4π)-periodic. Moreover $\mathrm{e}^{\mathrm{i}\alpha_L}$ is continuous on $\mathbb{R}$ and we have, for all $\omega \in \mathbb{R}$,*

$$\left| \mathrm{e}^{\mathrm{i}\alpha_L(\omega)} - \mathcal{I}(\omega) \right| \leqslant 2\sqrt{2}\ \Delta^{2L+1}(\omega) \ , \tag{28}$$

where

$$\mathcal{I}(\omega) = \begin{cases} 1 & \text{if } \omega \in [-\pi, \pi) + 4\pi\mathbb{Z} \\ -1 & \text{otherwise,} \end{cases} \tag{29}$$

and

$$\Delta(\omega) := \min\left(|\tan(\omega/4)|, |\tan(\omega/4)|^{-1} \right) \tag{30}$$

Proof. By definition (14), α_L is (4π)-periodic and continuous on $\mathbb{R} \setminus (2\pi + 4\pi\mathbb{Z})$. Moreover, at any of its discontinuity points in $2\pi + 4\pi\mathbb{Z}$, α_L jumps have height 2π. Hence $\mathrm{e}^{\mathrm{i}\alpha_L}$ is continuous over $\mathbb{R}$.

We now prove (28). We will in fact show the following more precise bounds, valid for all $\omega \in \mathbb{R}$.

$$|\cos(\alpha_L(\omega)) - 1| \leqslant 2|\tan(\omega/4)|^{2(2L+1)} \ , \tag{31}$$

$$|\cos(\alpha_L(\omega)) + 1| \leqslant 2|\tan(\omega/4)|^{-2(2L+1)} \ , \tag{32}$$

$$|\sin(\alpha_L(\omega))| \leqslant 2\Delta^{2L+1}(\omega) \ . \tag{33}$$

The bounds (31) and (32) easily follow from the identity

$$\cos(\alpha_L(\omega)) = \cos(2\arctan(\tan(\omega/4)^{2L+1})) = \frac{1 - \tan^{2(2L+1)}(\omega/4)}{1 + \tan^{2(2L+1)}(\omega/4)} \ .$$

The bound (33) follows from the identity

$$\sin(\alpha_L(\omega)) = \sin(2\arctan(\tan^{2L+1}(\omega/4))) = \frac{2\tan^{2L+1}(\omega/4)}{1 + \tan^{2(2L+1)}(\omega/4)} \ .$$

The proof is concluded. □

Observe that by (15) and the definition of η_L in (16), $\mathrm{e}^{\mathrm{i}\eta_L}$ can be expressed directly from $\mathrm{e}^{\mathrm{i}\alpha_L}$, namely as

$$\mathrm{e}^{\mathrm{i}\eta_L(\omega)} = \mathrm{e}^{-\mathrm{i}\alpha_L(\omega/2+\pi)} \prod_{j\geqslant 1} \mathrm{e}^{\mathrm{i}\alpha_L\left(2^{-j-1}\omega\right)} \; .$$

A quite natural question is to determine the function $1 - \mathrm{e}^{\mathrm{i}\eta_L}$ obtained when $\mathrm{e}^{\mathrm{i}\alpha_L}$ is replaced by its large L approximation $\mathcal{I}$. This is done in the following result.

Lemma 11. *Define the (4π)-periodic rectangular function $\mathcal{I}$ by (29). Then, for all $\omega \in \mathbb{R}\backslash\{0\}$, we have*

$$1 - \mathcal{I}(\omega/2+\pi) \prod_{j\geqslant 1} \mathcal{I}\left(2^{-j-1}\omega\right) = 2\mathbb{1}_{\mathbb{R}_+}(\omega) \; . \tag{34}$$

Proof. Note that the function $\omega \mapsto \mathcal{I}(\omega+\pi)$ is the right-continuous, (4π)-periodic function that coincides with the sign of ω on $\omega \in [-2\pi, 2\pi) \setminus \{0\}$. It is then easy to verify that, by definition of $\mathcal{I}$, we have, for all $\omega \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{I}(\omega) &= \mathcal{I}(2\omega+\pi)\,\mathcal{I}(\omega+\pi) \\
&= \frac{\mathcal{I}(2\omega+\pi)}{\mathcal{I}(\omega+\pi)} \qquad (35)\\
&= \frac{-\mathcal{I}(2\omega+\pi)}{-\mathcal{I}(\omega+\pi)} \; . \qquad (36)
\end{aligned}$$

(By periodicity of $\mathcal{I}$, it only suffices to check the first equality on $\omega \in [-2\pi, 2\pi)$, the two other equalities follow, since $\mathcal{I}$ takes values in $\{-1, 1\}$.) Now, from the previous assertion, we have, for all $\omega < 0$, that $\mathcal{I}(\omega 2^{-j} + \pi) = 1$ for large enough j, and thus (35) implies

$$\prod_{j\geqslant 1} \mathcal{I}(2^{-j}\omega) = \prod_{j\geqslant 1} \frac{\mathcal{I}(2^{-(j-1)}\omega+\pi)}{\mathcal{I}(2^{-j}\omega+\pi)} = \mathcal{I}(\omega+\pi) \; ,$$

while, for all $\omega > 0$, since $-\mathcal{I}(\omega 2^{-j} + \pi) = 1$ for large enough j, (36) implies

$$\prod_{j\geqslant 1} \mathcal{I}(2^{-j}\omega) = \prod_{j\geqslant 1} \frac{-\mathcal{I}(2^{-(j-1)}\omega+\pi)}{-\mathcal{I}(2^{-j}\omega+\pi)} = -\mathcal{I}(\omega+\pi) \; .$$

Identity (34) follows. $\square$

We can now derive the main result of this section.

Proposition 12. *Let L be a positive integer. The function η_L defined by (14), (15) and (16) satisfies the following bound, for all $\omega \in \mathbb{R}$,*

$$\left|1 - \mathrm{e}^{\mathrm{i}\eta_L(\omega)} - 2\mathbb{1}_{\mathbb{R}_+}(\omega)\right| \leqslant 2\sqrt{2}\left(\Delta^{2L+1}(\omega/2+\pi) + \sum_{k=1}^{\infty} \Delta^{2L+1}(2^{-k-1}\omega)\right) \; , \tag{37}$$

where Δ is defined by (30).

Proof. We have, for all $\omega \in \mathbb{R}$ and $J \geqslant 1$,

$$\prod_{j=1}^{J} \mathrm{e}^{\mathrm{i}\alpha_L(2^{-j}\omega)} - \prod_{j=1}^{J} \mathcal{I}(2^{-j}\omega) = \sum_{k=1}^{J} a_{k,J}(\omega) \ ,$$

where we denote

$$a_{k,J}(\omega) = \left(\prod_{j=1}^{k-1} \mathrm{e}^{\mathrm{i}\alpha_L(2^{-j}\omega)}\right) \cdot \left(\mathrm{e}^{\mathrm{i}\alpha_L(2^{-k}\omega)} - \mathcal{I}(2^{-k}\omega)\right) \cdot \left(\prod_{j=k+1}^{J} \mathcal{I}(2^{-j}\omega)\right) \ ,$$

with the convention $\prod_1^0(\dots) = \prod_{J+1}^J(\dots) = 1$. Since α_L is real valued and $\mathcal{I}$ is valued in $\{-1,1\}$, it follows that, for all $\omega \in \mathbb{R}$ and $J \geqslant 1$,

$$\left|\prod_{j=1}^{J} \mathrm{e}^{\mathrm{i}\alpha_L(2^{-j}\omega)} - \prod_{j=1}^{J} \mathcal{I}(2^{-j}\omega)\right| \leqslant \sum_{k=1}^{J} |a_{k,J}(\omega)| \leqslant \sum_{k=1}^{J} \left|\mathrm{e}^{\mathrm{i}\alpha_L(2^{-k}\omega)} - \mathcal{I}(2^{-k}\omega)\right| \ .$$

Applying Lemma 10 yields for all $\omega \in \mathbb{R}$ and $J \geqslant 1$,

$$\left|\prod_{j=1}^{J} \mathrm{e}^{\mathrm{i}\alpha_L(2^{-j}\omega)} - \prod_{j=1}^{J} \mathcal{I}(2^{-j}\omega)\right| \leqslant 2\sqrt{2}\left(\sum_{k=1}^{J} \Delta^{2L+1}(2^{-k}\omega)\right) \ .$$

Letting $J \to \infty$ and applying the definition of β_L, we deduce that, for all $\omega \in \mathbb{R}$,

$$\left|\mathrm{e}^{\mathrm{i}\beta_L(\omega)} - \prod_{j=1}^{\infty} \mathcal{I}(2^{-j}\omega)\right| \leqslant 2\sqrt{2}\sum_{k=1}^{\infty} \Delta^{2L+1}(2^{-k}\omega) \ . \tag{38}$$

By definition of η_L, since α_L and β_L are real valued and $\mathcal{I}$ is valued in $\{-1,1\}$, we have, for all $\omega \in \mathbb{R}$

$$\left|\mathrm{e}^{\mathrm{i}\eta_L(2\omega)} - \mathcal{I}(\omega+\pi)\prod_{j=1}^{\infty} \mathcal{I}(2^{-j}\omega)\right| \leqslant \left|\mathrm{e}^{\mathrm{i}\alpha_L(\omega+\pi)} - \mathcal{I}(\omega+\pi)\right| + \left|\mathrm{e}^{\mathrm{i}\beta_L(\omega)} - \prod_{j=1}^{\infty} \mathcal{I}(2^{-j}\omega)\right| \ .$$

Hence, with Lemma 10 and (38), we conclude, for all $\omega \in \mathbb{R}$,

$$\left|\mathrm{e}^{\mathrm{i}\eta_L(2\omega)} - \mathcal{I}(\omega+\pi)\prod_{j=1}^{\infty} \mathcal{I}(2^{-j}\omega)\right| \leqslant 2\sqrt{2}\left(\Delta^{2L+1}(\omega+\pi) + \sum_{k=1}^{\infty} \Delta^{2L+1}(2^{-k}\omega)\right) \ .$$

The bound (37) then follows from Lemma 11. □

Study of the upper bound

The objective is to simplify the right-hand side of (37) to obtain the form given in (19). The following lemma essentially gives some interesting properties of the function Δ.

Lemma 13. *Let Δ be the function defined by (30). Then Δ is an even (2π)-periodic function, increasing and bijective from $[0,\pi]$ to $[0,1]$. It follows that, for all $\omega \in \mathbb{R}$,*

$$\Delta(\omega) = \tan\left(\frac{\pi}{4}(1 - \frac{1}{\pi}\delta(\omega, \pi + 2\pi\mathbb{Z}))\right) \leqslant 1 - \frac{1}{\pi}\delta(\omega, \pi + 2\pi\mathbb{Z}) , \tag{39}$$

where δ is the function defined in equation (18).

Proof. The proof is straightforward and thus omitted. □

Note that the upper bound in (39) decreases from 1 to 0 as $\delta(\omega, \pi + 2\pi\mathbb{Z})$ increases from 0 to π. Since Δ takes its values in $[0,1)$ on $\mathbb{R} \setminus (\pi + 2\pi\mathbb{Z})$, Lemma 10 shows that, out of the set $\pi + 2\pi\mathbb{Z}$, $\mathrm{e}^{\mathrm{i}\alpha_L}$ uniformly converges to the (4π)-periodic rectangular function $\mathcal{I}$ as $L \to \infty$.

We will use the following bound.

Lemma 14. *For all $\omega \in (-\pi/4, \pi/4)$ and $L \geqslant 0$, we have*

$$\sum_{j=0}^{\infty} |\tan|^{2L+1}(2^{-j}\omega) \leqslant 2\,|\tan|^{2L+1}(\omega) .$$

Proof. It suffices to prove the inequality for $\omega \in (0, \pi/4)$. By convexity of tan, the slope $x \mapsto x^{-1}\tan(x)$ is increasing on $[0, \pi/4)$ and so is $x \mapsto x^{-1}\tan^{2L+1}(x)$ for $L \geqslant 0$. Hence we have, for all $\omega \in (0, \pi/4)$,

$$\begin{aligned}
\sum_{j=0}^{\infty} \tan^{2L+1}(2^{-j}\omega) &= \sum_{j=0}^{\infty} 2^{-j}\omega\left(2^{-j}\omega\right)^{-1}\tan^{2L+1}(2^{-j}\omega) \\
&\leqslant \sum_{j=0}^{\infty} 2^{-j}\omega\,(\omega)^{-1}\tan^{2L+1}(\omega) \\
&= 2\,\tan^{2L+1}(\omega) .
\end{aligned}$$

The proof is concluded. □

We also have the following lemma.

Lemma 15. *For all $\omega \in \mathbb{R}$, we have*

$$\delta(\omega/2 + \pi, \pi + 2\pi\mathbb{Z}) \geqslant 2^{-1}\delta(\omega, 4\pi\mathbb{Z}) , \tag{40}$$

$$\delta(2^{-j}\omega, \pi + 2\pi\mathbb{Z}) \geqslant 2^{-j}\,\delta(\omega, 4\pi\mathbb{Z}) \qquad \textit{for all integer } j \geqslant 2 . \tag{41}$$

Proof. The bound (40) is obvious. To show (41), take $x \in \pi + 2\pi\mathbb{Z}$. Then, for all $\omega \in \mathbb{R}$ and $j \geqslant 2$, we have $\left|2^{-j}\omega - x\right| = 2^{-j}\left|\omega - 2^j x\right|$ and, since $2^j x \in 4\pi\mathbb{Z}$, we get (41). □

We are now able to give a more concise upper bound.

Lemma 16. *Let Δ be defined by (30). Then, for all $\omega \in \mathbb{R}$, we have*

$$\sum_{j=1}^{\infty} \Delta^{2L+1}(2^{-j-1}\omega) \leqslant \left(\log_2\left(\frac{\max(4\pi,|\omega|)}{2\pi}\right)+1\right)\left(1-\frac{\delta(\omega,4\pi\mathbb{Z})}{\max(4\pi,|\omega|)}\right)^{2L+1} . \tag{42}$$

Proof. Denote

$$\iota(\omega) = \min\left\{j \geqslant 1 \;:\; |\omega|\,2^{-j-1} < \pi\right\} \leqslant \log_2\left(\frac{\max(4\pi,|\omega|)}{2\pi}\right) .$$

$$\sum_{j=1}^{\infty} \Delta^{2L+1}(2^{-j-1}\omega) \leqslant \sum_{j=1}^{\iota(\omega)-1} \Delta^{2L+1}(2^{-j-1}\omega) + \sum_{j\geqslant\iota(\omega)} |\tan|^{2L+1}(2^{-j-3}\omega)$$

Lemma 14 gives that, for all $\omega \in \mathbb{R}$,

$$\sum_{j\geqslant\iota(\omega)} |\tan|^{2L+1}(2^{-j-3}\omega) \leqslant 2|\tan|^{2L+1}(2^{-\iota(\omega)-3}\omega) = 2\Delta^{2L+1}(2^{-\iota(\omega)-1}\omega) .$$

The last two bounds yield, for all $\omega \in \mathbb{R}$,

$$\sum_{j=1}^{\infty} \Delta^{2L+1}(2^{-j-1}\omega) \leqslant (\iota(\omega)+1)\left(\sup_{1\leqslant j\leqslant\iota(\omega)} \Delta(2^{-j-1}\omega)\right)^{2L+1} .$$

Note that Lemma 13 and (41) imply

$$\sup_{1\leqslant j\leqslant\iota(\omega)} \Delta(2^{-j-1}\omega) \leqslant 1 - \frac{1}{\pi}2^{-\iota(\omega)-1}\,\delta(\omega,4\pi\mathbb{Z}) .$$

The above bound on $\iota(\omega)$ then gives (42). □

We can now conclude with the proof of the main result.

Proof of Theorem 3. By Lemma 13 and (40), we have, for all $\omega \in \mathbb{R}$,

$$\Delta(\omega/2+\pi) \leqslant 1 - \frac{1}{2\pi}\delta(\omega,4\pi\mathbb{Z}) \leqslant 1 - \frac{\delta(\omega,4\pi\mathbb{Z})}{\max(4\pi,|\omega|)} .$$

Using this bound, (17), Proposition 12 and Lemma 16, we get (19). □

B Proofs of Section 4

The following lemma will be useful.

Lemma 17. *Let L be a positive integer. The complex roots of the polynomial $\widetilde{s}(x) = \sum_{n=0}^{L}\binom{2L+1}{2n}x^n$ belong to $\mathbb{R}_-$.*

Proof. Observe that for all $z \in \mathbb{C}$, $\widetilde{s}(z^2) = \frac{1}{2}((1+z)^{2L+1} + (1-z)^{2L+1})$. Thus if $\widetilde{s}(z^2) = 0$ with $z = x + \mathrm{i}y$ and $(x,y) \in \mathbb{R}^2$, we necessarily have that $|1+z|^2 = (1+x)^2 + y^2$ is equal to $|1-z|^2 = (1-x)^2 + y^2$, and thus $x = 0$ and $z^2 \in \mathbb{R}_-$. □

Proof of Proposition 4. By Proposition 1, we have, for all $\theta \in \mathbb{R}$,

$$\begin{aligned}
D_L(\mathrm{e}^{2\mathrm{i}\theta}) D_L(\mathrm{e}^{-2\mathrm{i}\theta}) &= \frac{1}{4(2L+1)^2} \left| (1+\mathrm{e}^{\mathrm{i}\theta})^{2L+1} + (1-\mathrm{e}^{\mathrm{i}\theta})^{2L+1} \right|^2 \\
&= \frac{2^{2(2L+1)}}{4(2L+1)^2} \left[\cos^{2(2L+1)}(\theta/2) + \sin^{2(2L+1)}(\theta/2) \right] \\
&= \frac{2^{2L}}{(2L+1)^2} \left[\frac{(1+\cos(\theta))^{2L+1} + (1-\cos(\theta))^{2L+1}}{2} \right] \\
&= \frac{2^{2L}}{(2L+1)^2} \sum_{n=0}^{L} \binom{2L+1}{2n} \cos^{2n}(\theta) \,. \qquad (43)
\end{aligned}$$

Note that if $z = \mathrm{e}^{2\mathrm{i}\theta}$ with $\theta \in \mathbb{R}$, then $2 + z + 1/z = 2(1+\cos(2\theta)) = (2\cos(\theta))^2$. By definition of S, we obtain that, for all $\theta \in \mathbb{R}$,

$$\begin{aligned}
S(\mathrm{e}^{2\mathrm{i}\theta}) &= 2^{2M} \cos^{2M}(\theta) \times \frac{2^{2L}}{(2L+1)^2} \sum_{n=0}^{L} \binom{2L+1}{2n} \cos^{2n}(\theta) \\
&= \frac{2^{2M+2L}}{(2L+1)^2} \, s(\cos^2(\theta)) \,,
\end{aligned}$$

where s is the polynomial defined by (22). Looking for a solution R of (20) in the form $R(z) = U\left(\frac{2+z+1/z}{4}\right)$ with U real polynomial and focusing on $z = \mathrm{e}^{2\mathrm{i}\theta}$ with $\theta \in \mathbb{R}$, we obtain the equation

$$U(1-y)s(1-y) + U(y)s(y) = C(L,M)^2. \qquad (44)$$

where we have denoted $y = \sin^2(\theta) = 1 - (2 + z + 1/z)/4 \in [0,1]$ and $C(L,M) = (2L+1)2^{-M-L+1/2}$. Reciprocally, any such polynomial U provides a solution $R(z) = U\left(\frac{2+z+1/z}{4}\right)$ of (20) for $z = \mathrm{e}^{2\mathrm{i}\theta}$ with $\theta \in \mathbb{R}$ and then for all $z \in \mathbb{C}^*$ by analytic extension.

Since the complex roots of s are valued in the set $\mathbb{R}_-$ of non-positive real numbers (see Lemma 17), we get that $s(1-y)$ and $s(y)$ are prime polynomials of degree $L+M$. Thus the Bezout Theorem allows us to describe the couples of real polynomials (U,V) solutions of the equation

$$V(y)s(1-y) + U(y)s(y) = C(L,M)^2 \,.$$

Note that U is a solution of (44) if and only if (U,V) is a solution of the Bezout equation with $V(y) = U(1-y)$. Now, by uniqueness of the solution of the Bezout equation such that both U and V have degrees at most $L+M-1$, we see that this solution must satisfy $V(y) = U(1-y)$ (since otherwise $(V(1-y), U(1-y))$ would provide a different solution). Hence we obtain a unique solution $U = r$ of (44) of degree at most $L+M-1$, which proves Assertion (i).

Other solutions (U,V) of the Bezout equation are obtained by taking $U(y) = r(y) + s(1-y)\,q(y)$ with q any polynomial. Looking for such a solution of (44), we easily get that it is one if and only if q satisfies $q(1-y) = -q(y)$. The proof of Assertion (ii) is concluded. □

Proof of Theorem 5. By Proposition 4, the given R is a solution of (20) provided that q satisfies $q(1-y) = -q(y)$, which we assume in the following. As explained above, the factorization holds if and only if R is non-negative on the unit circle, or, equivalently, if $r(y) + s(1-y)\,q(y)$ is non-negative for y in $(0,1)$. By antisymmetry of q around $1/2$, this is equivalent to have, for all $y \in [1/2, 1)$,

$$r(y) + s(1-y)\,q(y) \quad \text{and} \quad r(1-y) - s(y)\,q(y) \geqslant 0\,.$$

Since $s(y) > 0$ for all $y \in (0,1)$ and using that (44) holds with $U = r$, we finally obtain that the claimed factorization holds if and only if, for all $y \in [1/2, 1)$,

$$-\frac{r(y)}{s(1-y)} \leqslant q(y) \leqslant -\frac{r(y)}{s(1-y)} + \frac{C(L,M)^2}{s(y)s(1-y)}\,. \tag{45}$$

We first note that for all $y \in [1/2, 1)$, $1/(s(y)s(1-y)) \geqslant 1/(s(1)s(1/2)) > 0$. Hence the upper bound condition in (45) is away from the lower bound by at least a positive constant over $y \in [0, 1/2)$. Second, using again (44) with $U = r$ at $y = 1/2$ we have

$$2r(1/2)s(1/2) = C(L,M)^2\,.$$

It follows that, for $y = 1/2$, (45) reads

$$-\frac{C(L,M)^2}{2s^2(1/2)} \leqslant q(1/2) \leqslant \frac{C(L,M)^2}{2s^2(1/2)}\,,$$

This is compatible with $q(1/2) = 0$ inherited by the antisymmetric property of q around $1/2$. We conclude by applying the Stone-Weierstrass theorem to obtain the existence of a real polynomial q satisfying (45) for all $y \in [1/2, 1)$, $q(y) = -q(1-y)$ for all $y \in \mathbb{R}$. ☐

C Proofs of Section 5

We start with a result more precise than Lemma 17.

Lemma 18. *Let L be a positive integer. The complex roots of the polynomial $\widetilde{s}(y) = \sum_{n=0}^{L} \binom{2L+1}{2n} y^n$ are the $y_{0,L}, \dots, y_{L-1,L}$ defined in (24).*

Proof. Recall that $\widetilde{s}(z^2) = \frac{1}{2}\left[(1+z)^{2L+1} + (1-z)^{2L+1}\right]$ and that $\widetilde{s}(z^2) = 0$ is equivalent to $z \neq 1$ and $\left(\frac{1+z}{1-z}\right)^{2L+1} = -1$, that is

$$\frac{1+z}{1-z} = \mathrm{e}^{\mathrm{i}(\pi + 2k\pi)/(2L+1)}\,, \qquad k \in \{-L, \cdots, L\}\,.$$

There is no such z for $k = L$ and for $k \in \{-L, \cdots, L-1\}$, this is the same as

$$z = \frac{\mathrm{e}^{\mathrm{i}(\pi+2k\pi)/(2L+1)} - 1}{1 + \mathrm{e}^{\mathrm{i}(\pi+2k\pi)/(2L+1)}} = \mathrm{i}\,\tan\left(\frac{\pi(2k+1)}{2(2L+1)}\right)\,.$$

Taking the square and keeping only $k = 0, \dots, L$ to get distinct roots we get the result. ☐

Proof of Proposition 7. Using Lemma 18 and the Bezout equation $[B(L,0)]$, we have that for any $k \in \{0, \cdots, L-1\}$

$$r_{L,0}(1-y_{k,L}) = \frac{(2L+1)^2\, 2^{-2L+1}}{s_{L,0}(1-y_{k,L})} \ .$$

Since the polynomial $r_{L,0}$ has degree at most $L-1$ with known values in the L distinct points $1-y_{k,L}$, we deduce its explicit expression given in Proposition 7 by a standard interpolation formula.

We conclude with the proof of the recursive relation (26). Since for any $M \geqslant 1$, $s_{L,M}(y) = y s_{L,M-1}$, the polynomial $r_{L,M}$ satisfies

$$r_{L,M}(1-y) \times [(1-y)s_{L,M-1}(1-y)] + r_{L,M}(y) \times [y s_{L,M-1}(y)] = (2L+1)^2\, 2^{-2M-2L+1} \ . \quad (46)$$

We deduce that $2^2 y r_{L,M}(y)$ satisfies equation $[B(L, M-1)]$ and by Proposition 4,

$$2^2 y r_{L,M}(y) = r_{L,M-1}(y) + s_{L,M-1}(1-y)\, q(y) \ , \quad (47)$$

where q is a polynomial satisfying $q(1-y) = -q(y)$. Since the degree of $r_{L,M}$ is at most $L+M-1$ and that of $s_{L,M-1}$ is $L+M-1$, we get that q takes the form $q(y) = q(0)(1-2y)$ and it only remains to determine $q(0)$. Note that $s_{L,M}(1) = 2^{2L}$, so (47) yields $q(0) = -2^{-2L} r_{L,M-1}(0)$ and we finally obtain (26). □

References

C. Chaux, L . Duval, and J.C. Pesquet. Image analysis using a dual-tree m-band wavelet transform. *IEEE Transactions on Image Processing*, 15(8):2397–2412, 2006.

I. Daubechies. *Ten lectures on wavelets*, volume 61. SIAM, 1992.

T. Eirola. Sobolev characterization of solutions of dilation equations. *SIAM Journal on Mathematical Analysis*, 23(4):1015–1030, 1992.

Cormac Herley and Martin Vetterli. Wavelets and recursive filter banks. *IEEE Transactions on Signal Processing*, 41(8):2536–2556, 1993.

N.G. Kingsbury. The dual-tree complex wavelet transform: A new efficient tool for image restoration and enhancement. In *Proc. European Signal Processing Conf.*, page 319–322, 1998a.

N.G. Kingsbury. The dual-tree complex wavelet transform: A new technique for shift invariance and directional filters. In *Proc. 8th IEEE DSP Workshop*, number 86, 1998b.

N.G. Kingsbury. A dual-tree complex wavelet transform with improved orthogonality and symmetry properties. In *Proc. IEEE Int. Conf. Image Processing*, volume 2, pages 375–378, 2000.

J. Lilly and S. Olhede. On the analytic wavelet transform. *IEEE Transactions on Information Theory*, 56(8):4135–4156, 2010.

S. Murugesan and D. Tay. A new class of almost symmetric orthogonal Hilbert pair of wavelets. *Signal Processing*, 95:76–87, 2014.

S. Murugesan, D. Tay, I. Cooke, and P. Faou. Application of dual tree complex wavelet transform in tandem mass spectrometry. *Computers in biology and medicine*, 63:36–41, 2015.

H. Ojanen. Orthonormal compactly supported wavelets with optimal sobolev regularity: Numerical results. *Applied and Computational Harmonic Analysis*, 10(1):93–98, 2001.

H. Ozkaramanli and R. Yu. On the phase condition and its solution for Hilbert transform pairs of wavelet bases. *IEEE Trans. Signal Processing*, 51(12):3293–3294, 2003.

I.W. Selesnick. Hilbert transform pairs of wavelet bases. *Signal Processing Letters, IEEE*, 8 (6):170–173, 2001.

I.W. Selesnick. The design of approximate Hilbert transform pairs of wavelet bases. *Signal Processing, IEEE Transactions on*, 50(5):1144–1152, 2002.

I.W. Selesnick, R.G. Baraniuk, and N.C. Kingsbury. The dual-tree complex wavelet transform. *Signal Processing Magazine, IEEE*, 22(6):123–151, 2005.

D. Tay. Designing Hilbert-pair of wavelets: recent progress and future trends. In *Information, Communications & Signal Processing, 2007 6th International Conference on*, pages 1–5. IEEE, 2007.

D. Tay. A new approach to the common-factor design technique for Hilbert-pair of wavelets. *IEEE Signal Processing Letters*, 17(11):969–972, 2010.

D. Tay, N.G. Kingsbury, and M. Palaniswami. Orthonormal Hilbert-pair of wavelets with (almost) maximum vanishing moments. *IEEE Signal Processing Letters*, 13(9):533–536, 2006.

J.-P. Thiran. Recursive digital filters with maximally flat group delay. *Circuit Theory, IEEE Transactions on*, 18(6):659–664, 1971.

M. Vetterli. Filter banks allowing perfect reconstruction. *Signal processing*, 10(3):219–244, 1986.

L.F. Villemoes. Energy moments in time and frequency for two-scale difference equation solutions and wavelets. *SIAM journal on mathematical analysis*, 23(6):1519–1543, 1992.

Y. Wang, Z. He, and Y. Zi. Enhancement of signal denoising and multiple fault signatures detecting in rotating machinery using dual-tree complex wavelet transform. *Mechanical Systems and Signal Processing*, 24(1):119–137, 2010.

B. Whitcher, P.F. Craigmile, and P. Brown. Time-varying spectral analysis in neurophysiological time series using Hilbert wavelet pairs. *Signal Processing*, 85(11):2065–2081, 2005.

Brandon Whitcher. *Waveslim*. R project, 2015. URL https://cran.r-project.org/web/packages/waveslim/waveslim.pdf.