On the growth behaviour of Hironaka quotients

Hélène Maugendre, Françoise Michel

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H. Maugendre * F. Michel†

Abstract. We consider a finite analytic morphism $\phi = (f, g) : (X, p) \longrightarrow (\mathbb{C}^2, 0)$ where (X, p) is a complex analytic normal surface germ and f and g are complex analytic function germs. Let $\pi : (Y, E_Y) \to (X, p)$ be a good resolution of ϕ with exceptional divisor $E_Y = \pi^{-1}(p)$. We denote $G(Y)$ the dual graph of the resolution π. We study the behaviour of the Hironaka quotients of (f, g) associated to the vertices of $G(Y)$. We show that there exists maximal oriented arcs in $G(Y)$ along which the Hironaka quotients of (f, g) strictly increase and they are constant on the connected components of the closure of the complement of the union of the maximal oriented arcs.

Mathematics Subject Classifications (2000). 14B05, 14J17, 32S15,32S45, 32S55, 57M45.

Key words. Normal surface singularity, Resolution of singularities, Hironaka quotients, Discriminant.

1 Introduction

Let $\phi = (f, g) : (X, p) \longrightarrow (\mathbb{C}^2, 0)$ be a finite analytic morphism which is defined on a complex analytic normal surface germ (X, p) by two complex analytic function germs f and g.

We say that $\pi : (Y, E_Y) \longrightarrow (X, p)$ is a *good resolution of* ϕ if it is a resolution of the singularity (X, p) in which the total transform $E_Y^+ = ((fg) \circ \pi)^{-1}(0)$ is a normal crossings divisor and such that the irreducible components of the exceptional divisor $E_Y = \pi^{-1}(p)$ are non-singular. An irreducible component E_i of E_Y is called a *rupture component* if it is not a rational curve or if it intersects at least three other components of the total transform. By definition a *curvetta* c_i of an irreducible component E_i of E_Y is a smooth curve germ that intersects transversaly E_i at a smooth point of the total transform. To each irreducible component E_i of E we associate the rational number:

$$q_{E_i} = \frac{V_{f\circ\pi}(c_i)}{V_{g\circ\pi}(c_i)}$$

*Address: Institut Fourier, Université Grenoble-Alpes, France. E-mail: helene.maugendre@univ-grenoble-alpes.fr

†Address: Université Paul Sabatier, Toulouse, France. E-mail: fmichel@math.univ-toulouse.fr

where $V_{f\circ\pi}(c_i)$ (resp. $V_{g\circ\pi}(c_i)$) is the order of $f \circ \pi$ (resp. $g \circ \pi$) on c_i. This quotient is called the *Hironaka quotient* of (f, g) on E_i.

This set of rational numbers associated to a complex analytic normal surface germ has been first introduced in [6]. It is shown, in particular, that if (u, v) are local coordinates of $(\mathbb{C}^2, 0)$ and if π is the minimal good resolution of ϕ, then the subset of Hironaka quotients associated to the rupture components of E_Y are topological invariants of (ϕ, u, v). Another proof of this result is given in [12].

In this paper we study the growth behaviour of the Hironaka quotients of $\phi = (f, g)$ in the dual graph of a good resolution $R : (X', E_{X'}) \to (X, p)$ of the pair $(X, \{fg = 0\})$, obtained (see Section 2) using Hirzebruch-Jung's method. We also consider $\rho : (\tilde{X}, E_{\tilde{X}}) \to (X, p)$, the minimal good resolution of (X, p) such that the total transform of $\{fg = 0\}$ (by ρ) is a normal crossings divisor. By definition ρ is *the minimal resolution of* ϕ. But, X' dominates $\tilde{X}$ by $\beta : X' \to \tilde{X}$ which is a sequence of blowing-downs of some specific irreducible components of $E_{X'}$. Then, we obtain similar results, on the growth behaviour of the Hironaka quotients of (f, g), for the minimal resolution of ϕ and we can generalize them to any good resolution of ϕ.

Let $\pi : (Y, E_Y) \longrightarrow (X, p)$ be a good resolution of ϕ. The weighted dual graph associated to π, denoted $G(Y)$, is constructed as follows. To each irreducible component E_i of the exceptional divisor E_Y we associate a vertex (i) weighted by its Hironaka quotient q_{E_i}. When two irreducible components of E_Y intersect, we join their associated vertices by edges which number is equal to the number of intersection points. When k $(k \geq 0)$ irreducible components of the strict transform of $\{fg = 0\}$ meet E_i, we add to the vertex (i) k edges. If an edge represents the intersection point of an irreducible component of the strict transform of f (resp. g) with E_i, we endow the edge with a going-out arrow (resp. a going-in arrow (it means a reverse arrow)). By convention the Hironaka quotient of a going-in arrow is 0 and the Hironaka quotient of a going-out arrow is infinite.

Moreover by construction the graph $G(Y)$ is partially oriented as follows.

Let (e_{ij}) be an edge which represents an intersection point $E_i \cap E_j$. When $q_{E_i} = q_{E_j}$ the edge (e_{ij}) is not oriented. When $q_{E_i} < q_{E_j}$ then (e_{ij}) is oriented from (i) to (j) and we say that the edge e_{ij} is positively oriented.

Definition 1 *A maximal arc in $G(Y)$ is a subgraph which is homeomorphic to a segment and which satisfies the following conditions:*

1. *it begins with a going-in arrow and ends with a going-out arrow,*
2. *it is a sequence of positively oriented edges,*
3. *the orientation of the edges induces a compatible positive orientation on the whole segment.*

Figure 1: The two possible shapes of a maximal arc in $G(Y)$.

We denote $A(Y)$ the union of all maximal arcs in $G(Y)$.

Remark 1 *A vertex (i) of $G(Y)$ is in $A(Y)$ if and only if there exists at least one going-in arrow or edge arriving at (i) and at least one going-out arrow or edge leaving (i).*

Our main result is :

Theorem 1 *Let $\pi : (Y, E_Y) \longrightarrow (X, p)$ be a good resolution of ϕ. The Hironaka quotients of ϕ on the vertices of a connected component of the closure of $G(Y) \backslash A(Y)$ are constant.*

Moreover $G(Y) \backslash A(Y)$ doesn't contain any arrow.

Remark 2 *A consequence of Theorem 1 is that $G(Y) \backslash A(Y)$ does not contain any oriented edge.*

Let (X, p) be an irreducible complex analytic surface germ (in particular, p is not necessarily an isolated singular point) and let $\phi : (X, p) \longrightarrow (\mathbb{C}^2, 0)$ be a finite analytic morphism defined on (X, p). Theorem 1 will also be true for the resolutions of ϕ which begin by the normalization $\nu : (\bar{X}, \bar{p}) \to (X, p)$. More precisely if $\bar{R} : (\bar{Y}, E_{\bar{Y}}) \to (\bar{X}, \bar{p})$ is a good resolution of $\phi \circ \nu$, we apply theorem 1 to the finite morphism $\phi \circ \nu$ and the resolution $\bar{R}$. Using the notation $G(\bar{Y})$ for weighted dual graph associated to $\bar{R}$, we have:

Theorem (generalized) *Let $\phi : (X, p) \longrightarrow (\mathbb{C}^2, 0)$ be a finite morphism defined on an irreducible complex analytic surface germ (X, p). Let $\nu \circ \bar{R} : (\bar{Y}, E_{\bar{Y}}) \to (X, p)$ be a good resolution of ϕ. The Hironaka quotients of ϕ on the vertices of a connected component of the closure of $G(\bar{Y}) \backslash A(\bar{Y})$ are constant.*

Moreover $G(\bar{Y}) \backslash A(\bar{Y})$ doesn't contain any arrow.

One motivation to study Hironaka quotients is their relations with the Puiseux expansion of the branches of the discriminant of ϕ. The first Puiseux exponents of the discriminant of ϕ are the Hironaka quotients on the rupture vertices of the minimal resolution of ϕ (see [6] and [12]). Moreover, as proved in [3], it is possible to express all the Puiseux exponents of the discriminant curve of a finite morphism ϕ as some Hironaka quotients of the minimal resolution of finite morphisms $\phi_i : (X, p) \longrightarrow (\mathbb{C}^2, 0)$ defined by an iterative process which begins with ϕ. It is illustrated in example 3.

We give two other examples to express the interest of Theorem 1. In example 1, $(X, p) = (\mathbb{C}^2, 0)$ and Theorem 1 is applied to show the growth of the Hironaka quotients of $\phi = (f, g)$. This behaviour of Hironaka quotients can not be obtained using the previous results of [11], because $\{f = 0\}$ and have $\{g = 0\}$ many branches with high contact. In example 2, (X, p) is singular and we have non trivial subgraphs $A(\tilde{X})$ and $\overline{G(\tilde{X}) \backslash A(\tilde{X})}$.

Theorem 1 will be proved using Theorem 2 (Section 6.2) which precises the behaviour of Hironaka quotients in the Hirzebruch-Jung resolution of ϕ described in section 2 (Definition 3). To prove Theorem 2, we relate the Hironaka quotients with the first Puiseux exponents of plane curve germs as follows.

Let (c, p) be a germ of irreducible curve on (X, p) which is not a branch of $\{fg = 0\}$. Let $\pi : (Y, E_Y) \longrightarrow (X, p)$ be a good resolution of ϕ. Let (c_Y, z) be an irreducible component of the strict transform of (c, p) in (Y, E_Y), in particular $z \in E_Y$. As explain in Section 3, the first Puiseux exponent q_c of the plane curve germ $((\phi \circ \pi)(c), 0) \subset (\mathbb{C}^2, 0)$ has the following behaviour:

if z is a smooth point of the total transform E_Y^+ and if E_i is the irreducible component of E_Y which contains z, then q_c is equal to the Hironaka quotient q_{E_i} of E_i.

if $z \in E_i \cap E_j$ and $q_{E_i} = q_{E_j}$, then $q_{E_i} = q_c = q_{E_j}$.

if $z \in E_i \cap E_j$ and $q_{E_i} < q_{E_j}$, then $q_{E_i} < q_c < q_{E_j}$.

This allows us to describe in Lemma 4 (in section 5), the growth behaviour of the Hironaka quotients associated to the minimal resolution of a quasi-ordinary normal surface germ.

In sections 2 we define the Hirzebruch-Jung resolution of ϕ and we describe its topological properties used in sections 4 to 7.

In section 7 we show how Theorem 1 can be deduced from Theorem 2.

Acknowledgments We thank Patrick Popescu-Pampu for useful discussions.

2 The Hirzebruch-Jung's resolution of (X, p) associated to ϕ

Let $\phi = (f, g) : (X, p) \longrightarrow (\mathbb{C}^2, 0)$ be a finite analytic morphism which is defined on a complex analytic normal surface germ (X, p) by two complex analytic function germs f and g.

The discriminant curve of ϕ is the image by ϕ of the critical locus $C(\phi)$ of ϕ. We denote Δ the union of the irreducible components of the discriminant curve which are not included in $\{uv = 0\}$.

We denote by $r : (Z, E_Z) \to (\mathbb{C}^2, 0)$ the minimal embedded resolution of $\Delta^+ = \Delta \cup \{uv = 0\}$ and $G(Z)$ its dual graph constructed as described in the introduction where f is replaced by u, g by v and we add an edge ended by a star for each irreducible component of the strict transform of Δ. Moreover we weight the vertex associated to an irreducible component D_i of E_Z by its Hironaka quotient q_{D_i} defined as follows:

$$q_{D_i} = \frac{V_{u \circ r}(c_i)}{V_{v \circ r}(c_i)}$$

where c_i is a curvetta of D_i.

We construct as in [8] and [14] a Hirzebruch-Jung resolution of $\phi : (X, p) \to (\mathbb{C}^2, 0)$. Here, we begin with the minimal resolution r of Δ^+. The pull-back of ϕ by r is a finite morphism $\phi_r : (Z', E_{Z'}) \to (Z, E_Z)$ which induces an isomorphism from $E_{Z'}$ to E_Z. We denote r_ϕ the pull-back of r by ϕ, $r_\phi : (Z', E_{Z'}) \to (X, p)$.

In general Z' is not normal. Let $n : (\bar{Z}, E_{\bar{Z}}) \to (Z', E_{Z'})$ be its normalization.

Remark 3 *1. By construction, the discriminant locus of $\phi_r \circ n$ is included in $E_Z^+ = r^{-1}(\Delta^+)$ which is the total transform of $\Delta^+ = \Delta \cup \{uv = 0\}$ in Z.*

2. Let $E_{Z'}^0$ be the open set of the points of $E_{Z'}$ which are smooth points in the total transform $E_{Z'}^+ = \phi_r^{-1}(E_Z^+)$. If $z' \in E_{Z'}^0$, there exists a small neighbourhood U' of z' in Z' which is a $\mu -$constant family of curves parametrized by $U' \cap E_{Z'}$. Of course U' can be chosen such that $U' \cap E_{Z'}$ is a smooth disc in $E_{Z'}^0$. Therefore $n^{-1}(U')$ is a finite disjoint union of smooth germs of surface.

$$\begin{tikzcd}
(Z', E_{Z'}) \arrow[r, "r_\phi"] \arrow[d, "\phi_r"'] & (X, p) \arrow[d, "\phi"] \\
(Z, E_Z) \arrow[r, "r"] & (\mathbb{C}^2, 0)
\end{tikzcd}$$

Figure 2: The first step to construct the Hirzebruch-Jung resolution of ϕ.

3. *The restriction of the map $\phi_r \circ n$ to $E_{\bar{Z}}$ induces a finite morphism from $E_{\bar{Z}}$ to E_Z which is a regular covering on $E^0_{\bar{Z}} = n^{-1}(E^0_{Z'})$.*

Definition 2 *A Hirzebruch-Jung singularity is a quasi-ordinary singularity of normal surface germ.*

Lemma 1 *Let P be a double point of E^+_Z and $\bar{P}$ a point of $(\phi_r \circ n)^{-1}(P)$. Then $\bar{P}$ is a double point of $E^+_{\bar{Z}}$. Moreover, if $\bar{P}$ is not a smooth point of $\bar{Z}$ then $\bar{P}$ is a Hirzebruch-Jung singularity of $\bar{Z}$.*

Proof. Let P be a double point of E^+_Z and $U(P)$ be a regular neighbourhood of P in Z. As Z is a smooth surface, $\partial U(P)$ is a 3-dimensional sphere. Let us show that $(\phi_r \circ n)^{-1}(P)$ is a union of double points of $E^+_{\bar{Z}}$. If $\bar{P} \in (\phi_r \circ n)^{-1}(P)$, let U be the connected component of $(\phi_r \circ n)^{-1}(U(P))$ that contains $\bar{P}$. Let $\phi_r \circ n_|$ be the restriction of $\phi_r \circ n$ to the boundary ∂U of U. So, $\phi_r \circ n_| : \partial U \to \partial U(P)$ is a finite ramified covering with ramification locus in $\partial U(P) \cap E^+_Z$. As Z is smooth and P is a double point of E_Z, then $\partial U(P) \cap E^+_Z$ is a Hopf link in the 3-sphere $\partial U(P)$. Therefore ∂U is a lens space that contains the link $\partial U \cap E^+_{\bar{Z}}$ included in two distinct irreducible components of $E^+_{\bar{Z}}$. Hence, $\bar{P}$ is a double point of $E^+_{\bar{Z}}$. As the ramification locus of $\phi_r \circ n_| : U \to U(P)$ is included in a normal crossing divisor, if $\bar{P}$ is not a smooth point of $\bar{Z}$ it is a Hirzebruch-Jung singularity.

As explain in lemma 1, if $\bar{z}$ is a singular point of $\bar{Z}$, then $(\phi_r \circ n)(\bar{z})$ is a double point of E^+_Z . In particular, there are finitely many isolated singular points in $\bar{Z}$. The singularities of $\bar{Z}$ are Hirzebruch-Jung singularities. More precisely, let $\bar{z}_i, 1 \le i \le n$, be the finite set of the singular points of $\bar{Z}$ and $(\bar{Z}_i, \bar{z}_i)$ a sufficiently small neighbourhood of $\bar{z}_i$ in $\bar{Z}$. We have the following result (see [14] or [8] for a proof):

Theorem. *The exceptional divisor of the minimal resolution of $(\bar{Z}_i, \bar{z}_i)$ is a normal crossings divisor, each irreducible component of its exceptional divisor is a smooth rational curve, and its resolution dual graph is a bamboo (it means is homeomorphic to a segment).*

Remark 4 *In $\bar{Z}$ an irreducible component of the strict transform of $\{fg = 0\}$ is not necessarily a curvetta of an irreducible component of the exceptional divisor. But the normalization morphism n has separated the irreducible components of $\{f = 0\}$ from the ones of $\{g = 0\}$.*

Let $\bar{\rho}_i : (Z''_i, E_{Z''_i}) \to (\bar{Z}_i, \bar{z}_i)$ be the minimal resolution of the singularity $(\bar{Z}_i, \bar{z}_i)$. From [8] (corollary 1.4.3), see also [14] (paragraph 4), the spaces Z''_i and the maps $\bar{\rho}_i$ can be gluing for $1 \leq i \leq n$, in a suitable way to give a smooth space X' and a map $\bar{\rho} : (X', E_{X'}) \to (\bar{Z}, E_{\bar{Z}})$ satisfying the following property :

Proposition 1 *The map $r_\phi \circ n \circ \bar{\rho} : (X', E_{X'}) \to (X, p)$ is a good resolution of the singularity (X, p) in which the strict transform of $\{fg = 0\}$ is a normal crossings divisor.*

Proof. The resolution r separates the strict transform of $\{u = 0\}$ from the one of $\{v = 0\}$. All the branches of the strict transform of $\{g = 0\}$ (resp. $\{f = 0\}$) by r_ϕ meet $E_{Z'}$ at the same point P' (resp. Q') and $P' \neq Q'$ because $\phi_r(P') \neq \phi_r(Q')$. The normalization morphism n separates the irreducible components of $f = 0$ and those of $g = 0$. In $\bar{Z}$, let $\bar{P}$ be the intersection point of an irreducible component of the strict transform of $\{g = 0\}$ (resp. $\{f = 0\}$) with $E_{\bar{Z}}$. Let $P = (\phi_r \circ n)(\bar{P})$, $U(P)$ a regular neighbourhood of P in Z and U the connected component of $(\phi_r \circ n)^{-1}(U(P))$ that contains $\bar{P}$. $U(P)$ is a smooth surface germ that contains the double point P. Then from lemma 1, $\bar{P}$ is either a smooth point of $\bar{Z}$, either a Hirzebruch-Jung singularity of $\bar{Z}$. In the second case, $\bar{\rho}$ is a resolution of $\bar{P}$.

Let us denote $R := r_\phi \circ n \circ \bar{\rho}$.

As R is the composition of three well defined morphisms, we can use the following definition which is a relative (to $\{fg = 0\}$) version of the classical Hirzebruch-Jung resolution of a normal germ of surface.

Definition 3 *The morphism $R : (X', E_{X'}) \to (X, p)$ is the Hirzebruch-Jung resolution associated to ϕ.*

Now we can use the following result (for a proof see [4], Theorem 5.9, p.87):

Theorem . *Let $\rho : (\tilde{X}, E_{\tilde{X}}) \to (X, p)$ be the minimal resolution of ϕ. There exists $\beta : (X', E_{X'}) \to (\tilde{X}, E_{\tilde{X}})$ such that $\rho \circ \beta = R$ and the map β consists in a composition of blowing-downs of irreducible components, of the successively obtained exceptional divisors, of self-intersection -1, genus 0 and which are not rupture components.*

$$
\begin{tikzcd}
(X', E_{X'}) \arrow[drr, "\beta"] \arrow[dd, "\bar{\rho}"'] \arrow[dddrr, "R"] & & \\
& & (\tilde{X}, E_{\tilde{X}}) \arrow[dd, "\rho"] \\
(\bar{Z}, E_{\bar{Z}}) \arrow[dr, "n"] & & \\
& (Z', E_{Z'}) \arrow[r, "r_\phi"] \arrow[d, "\phi_r"'] & (X, p) \arrow[d, "\phi"] \\
& (Z, E_Z) \arrow[r, "r"] & (\mathbb{C}^2, 0)
\end{tikzcd}
$$

Figure 3: The commutative diagram of the morphisms involved in the Hizebruch-Jung resolution of ϕ.

3 The quotients associated to the morphism ϕ

Let (c, p) be a germ of irreducible curve on (X, p) which is not a branch of $\{fg = 0\}$. Let $V_f(c)$ (resp. $V_g(c)$) be the order of f (resp. g) on c.

Definition 4 *The* **contact quotient** q_c *of* (c, p) *associated to the morphism* $\phi = (f, g)$ *is equal to:*

$$q_c = \frac{V_f(c)}{V_g(c)}.$$

The following remark relates the contact quotient of a germ (c, p) in (X, p) with the first Puiseux exponent of the direct image of (c, p) by ϕ.

Remark 5 *For local coordinates* (u, v) *of* $(\mathbb{C}^2, 0)$ *such that* $u \circ \phi = f$ *and* $v \circ \phi = g$*, let* $u = a_0 v^{m/n} + a_1 v^{(m+1)/n} + \ldots$*,* $a_0 \neq 0$*, be a Puiseux expansion of* $\phi(c)$*. The definition of* q_c *implies that* q_c *is equal to the first Puiseux exponent of* $\phi(c)$*:*

$$q_c = \frac{m}{n} = \frac{V_u(\phi(c))}{V_v(\phi(c))}.$$

When the strict transform of a germ (c, p) in a good resolution $\pi : (Y, E_Y) \longrightarrow (X, p)$ of (X, p) meets E_Y^+ at a smooth point, then q_c is an Hironaka quotient. More precisely, we recall proposition 2.1 of [12].

Proposition 2.1 of [12]. *Let* $\pi : (Y, E_Y) \longrightarrow (X, p)$ *be a good resolution of* ϕ *and* E_i *be an irreducible component of* E_Y *of Hironaka quotient* q_{E_i}*. We denote by* E_i^0 *the smooth points of* E_i *in the total transform by* π *of* $\{fg = 0\}$ *.*

Let x *be a point of* E_i^0 *and* (ξ_i, x) *be an irreducible germ of curve at* x*. Then the contact quotient* $q_{\pi(\xi_i)}$ *of* $(\pi(\xi_i), p)$ *is equal to the Hironaka quotient on* E_i*:*

$$q_{\pi(\xi_i)} = q_{E_i}.$$

Let $r : (Z, E_Z) \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of $\Delta^+ = \Delta \cup \{uv = 0\}$. Let D_i be an irreducible component of E_Z and c_i a curvetta of D_i.

Definition 5 *The Hironaka quotient of* D_i*, denoted* q_{D_i}*, is equal to:*

$$q_{D_i} = \frac{V_{u \circ r}(c_i)}{V_{v \circ r}(c_i)}.$$

Remark 6 *Let* $(\gamma, 0)$ *be an irreducible curve germ in* $(\mathbb{C}^2, 0)$ *which admits* $u = a_0 v^{m/n} + a_1 v^{(m+1)/n} + \ldots$*,* $a_0 \neq 0$ *as Puiseux expansion. Let* (C, z) *be the strict transform by* r *of* $(\gamma, 0)$ *in* (Z, E_Z)*. Then* $\dfrac{m}{n} = \dfrac{V_{u \circ r}(C)}{V_{v \circ r}(C)}$*.*

Hence, if z *is a smooth point of an irreducible component* D_i *of* E_Z*, we have* $\dfrac{m}{n} = q_{D_i}$*.*

The following lemma is quite obvious, but very useful for computation of Hironaka quotients.

Lemma 2 *Let* (c', p') *be a germ of curve (at* p'*). Let* $\alpha : (c', p') \to (X, p)$ *be a holomorphic germ which is a ramified covering over* (c, p) *of generic degree* k *and ramification locus* p'*. We have:*

$$q_c = \frac{V_f(c)}{V_g(c)} = \frac{V_{f \circ \alpha}(c')}{V_{g \circ \alpha}(c')}.$$

Proof. We have the following orders of functions:

$$V_{f\circ\alpha}(c') = k(V_f(c))\, and\ V_{g\circ\alpha}(c') = k(V_g(c)).$$

As in Section 2, we denote by $\rho : (\tilde{X}, E_{\tilde{X}}) \to (X, p)$ the minimal resolution of ϕ and by $R : (X', E_{X'}) \to (X, p)$ the Hirzebruch-Jung resolution of ϕ.

Using the above remark 5 and lemma 2 we obtain the following behaviour of the Hironaka quotients for the divisors and morphisms involved in the diagram of Figure 1.

Lemma 3 *Let E'_i be an irreducible component of $E^+_{X'}$.*
If $(\phi_r \circ n \circ \bar{\rho})(E'_i)$ is an irreducible component D_i in E^+_Z, then $q_{E'_i} = q_{D_i}$.
If $(\phi_r \circ n \circ \bar{\rho})(E'_i) \in D_i \cap D_j$ with $q_{D_i} = q_{D_j}$, then $q_{E'_i} = q_{D_i} = q_{D_j}$.
If $(\phi_r \circ n \circ \bar{\rho})(E'_i) \in D_i \cap D_j$ with $q_{D_i} < q_{D_j}$, then $q_{D_i} \leq q_{E'_i} \leq q_{D_j}$.
When $\beta(E'_i)$ is an irreducible component $\tilde{E}_i$ of $E_{\tilde{X}}$, we have $q_{E'_i} = q_{\tilde{E}_i}$.

4 The maximal arc for the minimal resolution of Δ^+

As in Section 2, $r : (Z, E_Z) \to (\mathbb{C}^2, 0)$ is the minimal embedded resolution of $\Delta^+ = \Delta \cup \{uv = 0\}$. Let $G(Z)$ be its dual graph constructed as described in the introduction where f is replaced by u, g by v. We add an edge ended by a star for each irreducible component of the strict transform of Δ. Moreover we weight the vertex associated to an irreducible component D_i of E_Z by its Hironaka quotient q_{D_i} (see definition 5).

From remark 5 , the quotient $q_{D_i} = \dfrac{V_{u\circ r}(c_i)}{V_{v\circ r}(c_i)}$ is equal to the first Puiseux exponent of $(r(c_i), 0)$.

As Δ^+ is a plane curve germ, $G(Z)$ is a tree. We consider the subgraph $S(Z)$ of $G(Z)$ which is the geodesic beginning with the (reverse) arrow associated to v and ending at the arrow associated to u. We orient this geodesic from v to u.

Proposition 2 *The graph $G(Z)$ admits an unique maximal arc which is equal to $S(Z)$.*

Proof. As $G(Z)$ is a tree, $S(Z)$ is homeomorphic to a segment. Notice that $G(Z)$ has only two arrows (one associated to the strict transform of $\{u = 0\}$ and the other to the strict transform of $\{v = 0\}$), both of them contained in $S(Z)$. So $\overline{G(Z) \setminus S(Z)}$ doesn't contain any arrow.

We number the irreducible components of E_Z corresponding to the vertices of $S(Z)$ from v to u. Let (i) and $(i + 1)$ be two consecutive vertices on $S(Z)$ which represent respectively the irreducible components D_i and D_{i+1}. We have to prove that $q_{D_i} < q_{D_{i+1}}$.

Let c_i (resp. c_{i+1}) be a curvetta of D_i (resp. D_{i+1}). The curve $r(c_i)$ (resp. $r(c_{i+1})$) admits a Puiseux expansion beginning by:

$u = a_{i,0}v^{m_i/n_i} + a_{i,1}v^{(m_i+1)/n_i} + \dots$ (resp. $u = a_{i+1,0}v^{m_{i+1}/n_{i+1}} + a_{i+1,1}v^{(m_{i+1}+1)/n_{i+1}} + \dots$)

The resolution of plane curve singularities computed by continued fraction expansion (for example see [9], ch. 6) implies that $m_i/n_i < m_{i+1}/n_{i+1}$. But $m_i/n_i = q_{D_i}$ and $m_{i+1}/n_{i+1} = q_{D_{i+1}}$. So, $S(Z)$ is a maximal arc as defined in the introduction.

It leaves to show that on a connected part T of $\overline{G(Z)\backslash S(Z)}$ the Hironaka quotients are constant.

The intersection of T with $S(Z)$ is composed of an unique vertex. Let us call it (i). An irreducible component D_j associated to a vertex of T is obtained by a sequence of blowing-up of points which begins with the blow-up of a smooth point (in the total transform of $\{uv = 0\}$) of D_i. From proposition 2.1 of [12], $q_{D_j} = q_{D_i}$.

Before describing the behaviour of the Hironaka quotients associated to $(X', E_{X'})$, we need to study the quotients associated to a resolution of a quasi-ordinary normal surface germ. We will use it to compute the Hironaka quotients on the irreducible components of $E_{X'}$ created by $\bar{\rho}$.

5 Quotients associated to the minimal resolution of a quasi-ordinary normal surface germ

Definition 6 *A germ (W, z) of normal surface is quasi-ordinary if there exists a finite morphism $\Phi : (W, z) \to (\mathbb{C}^2, 0)$ such that the discriminant locus is the union of the two coordinate axes of $\mathbb{C}^2$.*

Let $\Phi : (W, z) \to (\mathbb{C}^2, 0)$ be a finite morphism defined on a quasi-ordinary normal surface germ such that the discriminant locus is the union of the two coordinate axes of $\mathbb{C}^2$.

Let us denote (u, v) the coordinate of $\mathbb{C}^2$.

Remark 7 *The link of W is connected because (W, z) is an irreducible germ of complex surface. As $\{uv = 0\}$ is the discriminant locus of $\Phi : (W, z) \to (\mathbb{C}^2, 0)$, the topology of the situation implies that $\Phi^{-1}(\{u = 0\})$ (resp. $\Phi^{-1}(\{v = 0\})$) is an irreducible germ of curve in (W, z).*

Proposition 3 *(see Theorem 1.4.2 of [8]) (W, z) has a minimal good resolution $\rho_W : (\tilde{W}, E_{\tilde{W}}) \to (W, z)$ such that :*

I) the dual graph of $E_{\tilde{W}}$ is a bamboo and all the vertices represent a rational smooth curve.

Let k be the number of irreducible components of $E_{\tilde{W}}$. We orient the bamboo from the vertex (1) to the vertex (k). The vertices are indexed by this orientation.

II) the strict transform of $\Phi^{-1}(\{v = 0\})$ (resp. $\Phi^{-1}(\{u = 0\})$) is a curvetta of the irreducible component $E_1^{\tilde{W}}$ (resp. $E_k^{\tilde{W}}$) of $E_{\tilde{W}}$.

To obtain the dual graph $G(\tilde{W})$ we add to the vertex (1) (resp. (k)) of the dual graph of $E_{\tilde{W}}$ a reverse arrow indices by (v) which represents the strict transform (by $\Phi \circ \rho_{\tilde{W}}$) of $\{v = 0\}$ (resp. an arrow indices by (u) which represents the strict transform of $\{u = 0\}$). We get a graph which has the following shape:

The total transform of $\{uv = 0\}$ is $E_{\tilde{W}}^{+} = (\Phi \circ \rho_W)^{-1}(\{uv = 0\})$. For all i, $1 \le i \le k$, let x_i be a point of $E_i^{\tilde{W}}$ which is smooth in $E_{\tilde{W}}^{+}$ and (c_i, x_i) be a curvetta of $E_i^{\tilde{W}}$. Let Γ be the union of the plane curve germs $\gamma_i = (\Phi \circ \rho_W)(c_i)$ and $\Gamma^{+} = \Gamma \cup \{uv = 0\}$.

Lemma 4 *Let $\frac{m_i}{n_i}$ be the first Puiseux exponent of γ_i. For all i, $1 \le i < k$, we have $\frac{m_i}{n_i} < \frac{m_{i+1}}{n_{i+1}}$.*

Remark 8 *From section 3, the rational number $\frac{m_i}{n_i} \in \mathbb{Q}_+$ is the Hironaka quotient $q_{E_i^{\tilde{W}}}$ of Φ on $E_i^{\tilde{W}}$.*

Proof of lemma 4. We order the set $Q = \{\frac{m_i}{n_i}\}$ of the first Puiseux exponent of the irreducible components of Γ. We obtain $Q = \{s_1 < ... < s_j < ... < s_{k'}\}$ where $k' \le k$, $s_j \in \mathbb{Q}_+$. So, for all j, $1 \le j \le k'$, there exists at least one index $i(j)$ such that $\frac{m_{i(j)}}{n_{i(j)}} = s_j$.

The curve $\gamma_{i(j)}$ admits a Puiseux expansion which begins by:

$$u = a_{i(j)0}\, v^{m_{i(j)}/n_{i(j)}} + a_{i(j)1}\, v^{(m_{i(j)}+1)/n_{i(j)}} + \ldots,\ a_{i(j)0} \neq 0$$

We will say that the plane curve germ $\gamma'_{i(j)}$, having

$$u = a_{i(j)0}\, v^{m_{i(j)}/n_{i(j)}}$$

as Puiseux expansion, is the shadow of $\gamma_{i(j)}$. Let Γ' be the union of the curves $\gamma'_{i(j)}$, $1 \le j \le k'$, and let r' be the minimal resolution of the plane curve germ $\Gamma'^{+} = \Gamma' \cup \{uv = 0\}$:

$$r' : (M, E_M) \to (\mathbb{C}^2, 0).$$

The discriminant locus of Φ is $\{uv = 0\}$. Here, we begin with the resolution r' followed by a Hirzebruch-Jung construction to obtain a good resolution of (W, z). It is described in figure 4.

$$
\begin{tikzcd}
(W'', E_{W''}) \arrow[rrd, "\beta''"] \arrow[dd, "\rho''"'] \arrow[rrddd, "R''"] & & \\
& & (\tilde{W}, E_{\tilde{W}}) \arrow[dd, "\rho_W"] \\
(\bar{W}, E_{\bar{W}}) \arrow[dr, "\nu"] & & \\
& (W', E_{W'}) \arrow[r, "r'_{\Phi}"] \arrow[d, "\Phi_{r'}"'] & (W, z) \arrow[d, "\Phi"] \\
& (M, E_M) \arrow[r, "r'"] & (\mathbb{C}^2, 0)
\end{tikzcd}
$$

Figure 4: Diagram of the Hirzebruch-Jung's resolution of (W, z) constructed with the curve $\Gamma'^{+} = \Gamma' \cup \{uv = 0\}$

In figure 4, (W', E'_W) is the pull-back of Φ by r'. As explained in Section 2, the normalization $\nu : (\bar{W}, E_{\bar{W}}) \to (W', E'_W)$ followed by the minimal resolution

$$\rho'' : (W'', E_{W''}) \to (\bar{W}, E_{\bar{W}})$$

of the isolated singular points of $\bar{W}$ is a good resolution $R'' : (W'', E''_W) \to (W, z)$ of (W, z). There exist $\beta'' : (W'', E''_W) \to (\tilde{W}, E_{\tilde{W}})$ a composition of contraction of some irreducible components of E''_W such that $R'' = \rho_W \circ \beta''$.

Step I) By the minimal resolution of $\Gamma'^{+} = \Gamma' \cup \{uv = 0\}$, the dual graph $G(M)$ (endowed with an arrow (resp. a reverse arrow) representing the strict transform c_u, of $\{u = 0\}$ (resp. c_v of $\{v = 0\}$) is a bamboo with k'' ($k' \leq k''$) vertices. We orient this bamboo from the reverse arrow associated to c_v to the arrow associated to c_u. The indices (l), $1 \leq l \leq k''$, of the vertices increase with this orientation.

The dual graph of E_M with the strict transforms c_u of $\{u = 0\}$ and c_v of $\{v = 0\}$ has the following shape:

The theory of the resolution of plane curve germ obtained by computation of the continued fraction expansion of the first Puiseux exponents $s_j, 1 \leq j \leq k'$, implies that the strict transform of a germ having s_j as first Puiseux exponent meets E_M at a point of the irreducible component $D_{l(j)}$ which is a smooth point of $r'^{-1}(\Gamma'^{+})$. Moreover, we have $(l(1)) < (l(2)) < .. < (l(j)) < ... < (l(k'))$.

Step II) As $G(\tilde{W})$ is a bamboo and as $\Phi^{-1}(\{u = 0\})$ (resp. $\Phi^{-1}(\{v = 0\})$) is an irreducible germ of curve in (W, z), $\Phi_{r'} \circ \nu$ induces an isomorphism of graph $\nu_G : G(\bar{W}) \to G(M)$. Indeed:

The strict transform of $\{v = 0\}$(resp. $\{u = 0\}$) being irreducible, $(\Phi_{r'} \circ \nu)^{-1}(D_1)$ (resp. $(\Phi_{r'} \circ \nu)^{-1}(D_{k''})$) is only one irreducible component of $E_{\bar{W}}$. But there is no cycle in the graph $G(W'')$ because $G(\tilde{W})$ has no cycle. Moreover, ρ'' is only a resolution of quasi-ordinary singular points. The graph $G(W'')$ is obtained from $G(\bar{W})$ by replacing some edges by bamboos and $G(\bar{W})$ has no cycle. So, all the $(\Phi_{r'} \circ \nu)^{-1}(D_l)$ are irreducible in $E_{\bar{W}}$ and two irreducible components of $E_{\bar{W}}$ has at most one common point. We can identify $G(\bar{W})$ with $G(M)$ after putting, via ν_G, the orientation and indices of $G(M)$ on $G(\bar{W})$.

Step III) By step II, $G(\bar{W})$ is, in particular, a bamboo. The graph $G(W'')$ is obtained from $G(\bar{W})$ by replacing some edges by bamboos, it produces a new bamboo which is just an extension of $G(\bar{W})$. We lift the orientation of $G(\bar{W})$ on $G(W'')$ and we order the indices of the vertices of $G(W'')$ with the help of this orientation. So, $\beta'' : (W'', E''_W) \to (\tilde{W}, E_{\tilde{W}})$ induces a morphism between two oriented bamboos $\beta''_G : G(W'') \to G(\tilde{W})$.

For all j, $1 \leq j \leq k'$, let $D^0_{l(j)}$ be the set of the smooth points (in $r'^{-1}(\Gamma'^{+})$) of the irreducible component $D_{l(j)}$ of E_M which meets the strict transform of $\gamma_{i(j)}$.

It exists only one index $l''(j)$ such that $(\Phi_{r'} \circ \nu \circ \rho'')^{-1}(D^0_{l(j)}) = E''^0_{l''(j)}$ and the strict transform of $\gamma_{i(j)}$ via $(r' \circ \Phi_{r'} \circ \nu \circ \rho'')$ meets $E''^0_{l''(j)}$ at a point $z_{i(j)}$ which is smooth in the total transform of Γ^+. But $\gamma_{i(j)}$ is the direct image (by $\Phi \circ \rho_W$) of the chosen curvetta $(c_{i(j)}, x_{i(j)})$ of $\tilde{E}_{i(j)}$. The commutation of the diagram implies that $\beta''(E''_{l''(i(j))}) = \tilde{E}_{i(j)}$. So, $\gamma_{i(j)}$ is the only irreducible component of Γ such that $\frac{m_{i(j)}}{n_{i(j)}} = s_j$. So, $k = k'$, and $j = i(j) = i$ for all $j, 1 \leq j \leq k$. This implies:

$$s_i = \frac{m_i}{n_i} < s_{i+1} = \frac{m_{i+1}}{n_{i+1}}$$

This ends the proof of Lemma 4.

6 Behaviour of the Hironaka quotients in each step of the Hirzebruch-Jung resolution

Remark 9 *The computation of the Hironaka quotients of (f,g) in each step of the Hirzebruch-Jung resolution is based on the following principle: Lemma 2 and Remark 5 (of section 3), imply that the Hironaka quotient on an irreducible component E of the exceptional divisor $E_{\bar{Z}}$ (resp. $E_{X'}$) is equal to the contact quotient of the direct image, in (X,p), of a curvetta of E.*

We will show how the Hironaka quotients associated to the irreducible components of E_Z enable us to describe the behaviour of the Hironaka quotients on the irreducible components of $E_{\bar{Z}}$ (resp. $E_{X'}$).

6.1 Hironaka quotients associated to $\bar{Z}$

Let $\bar{E}_i$ be an irreducible component of $E_{\bar{Z}}$. Let $\bar{E}_i^0$ be the open set of the smooth points of $\bar{E}_i$ in the total transform $E_{\bar{Z}}^+ = (r \circ \phi_r \circ n)^{-1}(\Delta^+)$. By construction $(\phi_r \circ n)(\bar{E}_i^0)$ is the set D_i^0 of the smooth points, in the total transform $E_Z^+ = r^{-1}(\Delta^+)$, of an irreducible component D_i of E_Z.

Proposition 4 *The Hironaka quotient on $\bar{E}_i$ is equal to the Hironaka quotient on D_i.*

Proof. Let $\bar{z} \in \bar{E}_i^0$ and let $(\bar{c}_i, \bar{z})$ be a curvetta of $\bar{E}_i$. In section 3, we have seen that the Hironaka quotient $q_{\bar{E}_i}$ is equal to the first Puiseux exponent of $(\phi \circ r_\phi \circ n)(\bar{c}_i, \bar{z}) = (r \circ \phi_r \circ n)(\bar{c}_i, \bar{z})$. Let $z = (\phi_r \circ n)(\bar{z}) \in D_i^0$. But, as $(\gamma_i, z) = (\phi_r \circ n)(\bar{c}_i, \bar{z})$ is a germ of curve at a point of D_i^0, this implies that $(r(\gamma_i), 0)$ has the same first Puiseux exponent than $(r(c_i), 0)$ where (c_i, z) is a curvetta (at z) of D_i and which is equal to the Hironaka quotient on q_{D_i} (see section 4). This proves that $q_{D_i} = q_{\bar{E}_i}$.

As described in the introduction, we construct the partially oriented dual graph $G(\bar{Z})$ of $\bar{Z}$. Proposition 4 can also be stated in terms of dual graphs as follow:

Remark 10 *Let $n_G : G(\bar{Z}) \to G(Z)$ be the morphism of graphs induced on the dual graphs by $\phi_r \circ n : (\bar{Z}, E_{\bar{Z}}) \to (Z, E_Z)$. The Hironaka quotient of the vertex $(i) \in G(\bar{Z})$ is equal to the Hironaka quotient of the vertex $n_G(i) \in G(Z)$. Hence, n_G is an orientations preserving morphism of graphs.*

6.2 Hironaka quotients associated to X'

As in section 2, we consider the finite set $\bar{z}_i, 1 \leq i \leq n$, of the singular points of $\bar{Z}$. For each index i, we choose a sufficiently small neighbourhood $(\bar{Z}_i, \bar{z}_i)$ of $\bar{z}_i$ in $\bar{Z}$ and we denote by $\bar{\rho}_i : (Z_i'', E_{Z_i''}) \to (\bar{Z}_i, \bar{z}_i)$ the minimal resolution of the singularity $(\bar{Z}_i, \bar{z}_i)$. We have seen (Section 2) that $\bar{z}_i = \bar{E}_{i_1} \cap \bar{E}_{i_2}$ is a double point of the total transform $E_{\bar{Z}}^+ = (\phi_r \circ n)^{-1}(E_Z^+)$

($\bar{E}_{i_1}$ or $\bar{E}_{i_2}$ could be an irreducible component of the strict transform of the discriminant Δ). As $(\bar{Z}_i, \bar{z}_i)$ is a quasi-ordinary singular point, the dual graph of $E_{Z''_i} = (\bar{\rho})^{-1}(\bar{z}_i)$ is a bamboo, let us denote by k the number of its irreducible components.

Let us denote by E'_{i_1} (resp. E'_{i_2}) the irreducible component of $E_{X'}$ such that $(\bar{\rho})(E'_{i_1}) = \bar{E}_{i_1}$ (resp. $(\bar{\rho})(E'_{i_2}) = \bar{E}_{i_2}$). One extremity of this bamboo represents the irreducible component E'_1 of $E_{X'}$ which meets E'_{i_1}. We index it by (1). To obtain the dual graph $G(\bar{Z}_i)$, we add to (1) a reverse arrow indexed by (i_1) which represents E'_{i_1}.

The other extremity of this bamboo represents the irreducible component E'_k of $E_{X'}$ which meets E'_{i_2}. We index it by (k). To obtain the dual graph $G(\bar{Z}_i)$, we add to (k) an arrow indexed by (i_2) which represents E'_{i_2}. We orient $G(\bar{Z}_i)$ from (1) to (k) and we order the indices of the vertices with the help of this orientation.

The graph $G(\bar{Z}_i)$ has the following shape:

Theorem 2 *Let E'_j be an irreducible component of $E_{X'}$ and let z' be a point of ${E'}^0_j$ which is the set of the smooth points of E'_j in the total transform $(\bar{\rho})^{-1}(E^+_{\bar{Z}})$. Let (c'_j, z') be a curvetta (at z') of E'_j.*

If $\bar{\rho}(z') \neq \bar{z}_i, \bar{z}_i \in \{\bar{z}_i, 1 \leq i \leq n\}$, we have the following equality of the Hironaka quotients: $q_{E'_j} = q_{\bar{E}_j} = q_{D_j}$.

If $\bar{\rho}(z') = \bar{z}_i$, we have seen that $\bar{z}_i = \bar{E}_{i_1} \cap \bar{E}_{i_2}$ is a double point of the total transform $E^+_{\bar{Z}} = (\phi_r \circ n)^{-1}(E^+_Z)$. We have two cases

I) either $q_{\bar{E}_{i_1}} = q_{\bar{E}_{i_2}}$, then $q_{E'_j} = q_{\bar{E}_{i_1}} = q_{\bar{E}_{i_2}}$,

II) or $q_{\bar{E}_{i_1}} < q_{\bar{E}_{i_2}}$, then $q_{\bar{E}_{i_1}} < q_{E'_j} < q_{\bar{E}_{i_2}}$. More precisely, the dual graph of $(\bar{\rho})^{-1}(\bar{z}_i)$ is a bamboo. We orient this bamboo from the vertex (i_1) to the vertex (i_2). With this orientation, we order the indices (j) of the dual graph of $(\bar{\rho})^{-1}(\bar{z}_i)$ from (1) to (k). We have:

$$q_{E'_{i_1}} = q_{\bar{E}_{i_1}} < q_{E'_1} < \ldots < q_{E'_j} < \ldots < q_{E'_k} < q_{\bar{E}_{i_2}} = q_{E'_{i_2}}.$$

Proof of Theorem 2.

Let us recall that $q_{E'_j}$ is equal to the first Puiseux exponent of the plane curve germ γ_j where

$$\gamma_j = (r \circ \phi_r \circ n \circ \bar{\rho})(c'_j) = (\phi \circ r_\phi \circ n \circ \bar{\rho})(c'_j).$$

If $\bar{\rho}(z') \neq \bar{z}_i, \bar{z}_i \in \{\bar{z}_i, 1 \leq i \leq n\}$, $\bar{\rho}(E'_j)$ is an irreducible component $\bar{E}_j$ of $E_{\bar{Z}}$ and $(\phi_r \circ n \circ \bar{\rho})(E'_j) = D_j$ is an irreducible component of E_Z. Then, the first Puiseux exponent of $(r \circ \phi_r \circ n \circ \bar{\rho})(c'_j, z')$ is equal to $q_{\bar{E}_j} = q_{D_j}$.

If $\bar{\rho}(z') = \bar{z}_i$, we have seen that $\bar{z}_i = \bar{E}_{i_1} \cap \bar{E}_{i_2}$ is a double point of the total transform $E^+_{\bar{Z}} = (\phi_r \circ n)^{-1}(E^+_Z)$. Let $D_{i_1} = (\phi_r \circ n)(\bar{E}_{i_1})$, $D_{i_2} = (\phi_r \circ n)(\bar{E}_{i_2})$ and $z = (\phi_r \circ n \circ \bar{\rho})(z') \in D_{i_1} \cap D_{i_2}$. In Section 6.1 we proved that $q_{D_{i_1}} = q_{\bar{E}_{i_1}}$ and $q_{D_{i_2}} = q_{\bar{E}_{i_2}}$.

I) If $q_{\bar{E}_{i_1}} = q_{\bar{E}_{i_2}}$, we have $q_{D_{i_1}} = q_{D_{i_2}}$.

As $z = (\phi_r \circ n \circ \bar{\rho})(z') \in D_{i_1} \cap D_{i_2}$, we deduce from lemma 3 that $q_{E'_j} = q_{D_{i_1}} = q_{D_{i_2}}$.

II) If $q_{D_{i_1}} = q_{\bar{E}_{i_1}} < q_{\bar{E}_{i_2}} = q_{D_{i_2}}$, we have to study the minimal resolution $\bar{\rho}_i$: $(Z''_i, E_{Z''_i}) \to (\bar{Z}_i, \bar{z}_i)$ of the singularity $(\bar{Z}_i, \bar{z}_i)$.

For each irreducible component E'_j of $(\bar{\rho})^{-1}(\bar{z}_i)$ we choose a curvetta (c'_j, z'_j) of E'_j. We take the following notation: $D_{i_1} = (\phi_r \circ n \circ \bar{\rho})(E'_{i_1})$ and $D_{i_2} = (\phi_r \circ n \circ \bar{\rho})(E'_{i_2})$. So, we have $z_i = (\phi_r \circ n \circ \bar{\rho})(\bar{z}'_j) = D_{i_1} \cap D_{i_2}$.

But, $V_i = (\phi_r \circ n \circ \bar{\rho})(Z''_i)$ is a neighborhood of z_i in Z. Let us denote by Φ_i the restriction of $(\phi_r \circ n \circ \bar{\rho})$ on Z''_i. As Z is smooth and E_Z is a normal crossing divisor in Z, the morphism Φ_i satisfies the hypothesis of Lemma 4 where the smooth plane curve germs (D_{i_1}, z_i) and (D_{i_2}, z_i) play the role of the two axes $u = 0$ and $v = 0$. With this choice of axes at z_i in V_i, the first Puiseux exponents s_j of $c_j = \Phi_i(c'_j)$ are strictly ordered $s_1 < ... < s_j < ... < s_k$.

The curve c_j admits a Puiseux expansion which begins by:

$$x = a_{j0} y^{m_j/n_j} + a_{j1}\, y^{(m_j+1)/n_j} + \dots,\ a_{j0} \neq 0,\ m_j/n_j = s_j.$$

We will say that the plane curve germ c^*_j, having

$$x = a_{j0} y^{m_j/n_j}$$

as Puiseux expansion, is the shadow of c_j. Let $\gamma^*_j = r(c^*_j)$ and let t_j be the first Puiseux exponent of γ^*_j.

As $q_{D_{i_1}} < q_{D_{i_2}}$, the edge which represents $z_i = D_{i_1} \cap D_{i_2}$ in $G(Z)$ is an edge of the maximal arc $S(Z)$ of Proposition 2, Section 4. Resolution of plane curve germs implies that $q_{D_{i_1}} < t_1 < ... < t_j < ... < t_k < q_{D_{i_2}}$ and that t_j is also the first Puiseux exponent of $r(c_j) = \gamma_j$. But $q_{E'_j}$ is equal to the first Puiseux exponent of γ_j. This ends the proof of Theorem 2.

7 Behaviour of the dual graphs in each step of the Hirzebruch-Jung resolution

7.1 Maximal arcs in the dual graph $G(\bar{Z})$ of the normalization

Let $A(\bar{Z}) := n_G^{-1}(S(Z))$ be the inverse image of the maximal arc $S(Z)$ of $G(Z)$.

Theorem 3 *$A(\bar{Z})$ is the union of all the maximal arcs of $G(\bar{Z})$. The Hironaka quotients of the vertices of a connected component of the closure of $G(\bar{Z}) \setminus A(\bar{Z})$ are constant.*

Moreover $G(\bar{Z}) \setminus A(\bar{Z})$ doesn't contain any arrow.

Proof. By remark 10, n_G preserves the Hironaka quotients. The definition of $A(\bar{Z})$ implies that the Hironaka quotients of the vertices of a connected component of the closure of $G(\bar{Z})\backslash A(\bar{Z})$ are constant.

Again by remark 10, an edge of $G(\bar{Z})$ is oriented if and only if it is an edge of $A(\bar{Z})$. All the going-in arrows (resp. going-out) arrows are in $A(\bar{Z})$ because there are above (by n_G) the unique going-in (resp. going-out) arrow of $S(Z)$. The image, by n_G, of a vertex of $A(\bar{Z})$ is a vertex of $S(Z)$. In $S(Z)$ a vertex has exactly one going-in edge and one going-out edge. As n_G preserves the orientation of the edges, a vertex of $A(\bar{Z})$ has at least a going-in edge and a going-out edge. It allows us to show that each edge and each vertex of $A(\bar{Z})$ belong to a maximal arc of $A(\bar{Z})$.

7.2 Maximal arcs in the dual graph $G(X')$ of the good resolution of the Hirzebruch-Jung singularities of $\bar{Z}$

Let $A(X') := (\bar{\rho}_G)^{-1}(A(\bar{Z}))$ be the inverse image of $A(\bar{Z})$ by the morphism of graphs $\bar{\rho}_G : G(X') \to G(\bar{Z})$ induced by $\bar{\rho}$.

Theorem 4 *$A(X')$ is the union of all the maximal arcs of $G(X')$. The Hironaka quotients of the vertices of a connected component of the closure of $G(X')\backslash A(X')$ are constant.*

Moreover $G(X')\backslash A(X')$ doesn't contain any arrow.

Proof. The graph $G(X')$ is obtained from $G(\bar{Z})$ as follows.

If an edge of $G(\bar{Z})$ represents a point which is a smooth point of $\bar{Z}$, then we keep this edge in $G(X')$ and its extremities has the same Hironaka quotients.

If an edge (e_{ij}) of $G(\bar{Z})$ represents a Hirzebruch-Jung singular point of $\bar{Z}$, then in $G(X')$ this edge is replaced by a bamboo. If (e_{ij}) is not oriented, from point I of theorem 2, the Hironaka quotients are constant on the closure of the bamboo. So the closure of the bamboo is in the closure of $G(X')\backslash A(X')$. If (e_{ij}) is oriented, from II of theorem 2, the bamboo has the same orientation and is included in $A(X')$ by construction. In particular, the inverse image by $\bar{\rho}_G$ of a maximal arc of $A(\bar{Z})$ is a maximal arc of $A(X')$.

7.3 Maximal arcs in the dual graph $G(\tilde{X})$ of the minimal resolution of ϕ

Let $\beta_1 : (X', E_{X'}) \to (X_1, E_{X_1})$ be the contraction of an irreducible component of $E_{X'}$ of self-intersection -1 which is not a rupture component. A maximal sequence of such blowing-downs gives a morphism $\beta : (X', E_{X'}) \to (\tilde{X}, E_{\tilde{X}})$. Then the contraction of $E_{\tilde{X}}$ denoted $\rho : (\tilde{X}, E_{\tilde{X}}) \to (X, p)$ is the minimal resolution of ϕ (see [4] (Theorem 5.9 p. 87) or [2] (Theorem 6.2 p. 86)).

Lemma 5 *Let $\beta_{1G} : G(X') \to G(X_1)$ be the morphism of graphs induced by β_1. The subgraph $A(X_1) := \beta_{1G}(A(X'))$ is the union of the maximal arcs of $G(X_1)$. The Hironaka quotients of the vertices of a connected component of the closure of $G(X_1)\backslash A(X_1)$ are constant.*

Moreover $G(X_1)\backslash A(X_1)$ doesn't contain any arrow.

By finite iterations, the above lemma gives a proof of Theorem 1 for the minimal resolution of ϕ.

Proof of lemma 5. By hypothesis, β_1 is the contraction of an irreducible E'_i which meets one or two irreducible components of the total transform $E^+_{X'}$.

If E'_i meets only one irreducible component of $E^+_{X'}$, this neighbour is not an arrow because $G(X')$ is a connected graph. Remark 1 implies that the only edge which arrives to (i) is not in a maximal arc. $G(X_1)$ is obtained from $G(X')$ by deleting the vertex (i) and the only non-oriented edge which meets (i). In this case the restriction on $A(X')$ of the morphism β_{1G} induces an isomorphism to $A(X_1)$.

If E'_i meets two irreducible components of $E^+_{X'}$, then two cases occur.

a) If (i) is not a vertex of $A(X')$, then from remark 1, the two edges which meet (i) are not oriented and the contraction of (i) deletes (i) and the two edges are replaced by a unique non oriented edge. In this case, the restriction on $A(X')$ of the morphism β_{1G} induces an isomorphism to $A(X_1)$.

b) If (i) is a vertex of $A(X')$, notice first that (i) meets at most one arrow because $G(X')$ is connected and the existence of β_1 implies that R is not minimal. Let (e_{ij}) be an edge which meets (i). $A(X_1)$ is obtained from $A(X')$ by contracting (e_{ij}).

In this case the restriction of β_{1G} on $G(X')\backslash A(X')$ induces an isomorphism to $G(X_1)\backslash A(X_1)$.

The contraction of $A(X')$ in $A(X_1)$ has one of the three following shapes:

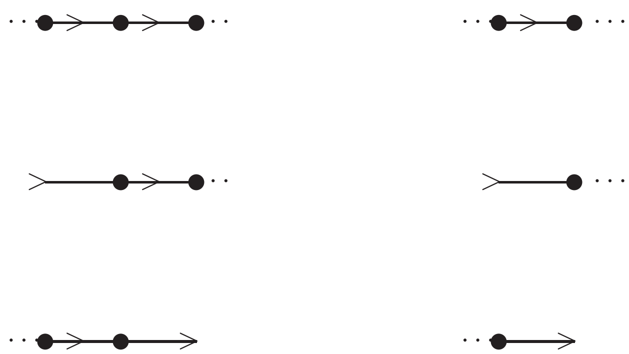

Figure 5: The shapes of the possible contractions of $A(X')$ in $A(X_1)$

7.4 Maximal arcs in the dual graph $G(Y)$ of any good resolution of ϕ

Let $\pi : (Y, E_Y) \to (X, p)$ be a good resolution of ϕ. As proved in [4] (Theorem 5.9 p. 87) and [2] (Theorem 6.2 p. 86), there exists a sequence of contractions of irreducible components of self-intersection -1 which are not rupture components such that the following diagram commutes.

$$\begin{tikzcd} (Y, E_Y) \arrow[rr, "\gamma"] \arrow[dr, "\pi"'] & & (\tilde{X}, E_{\tilde{X}}) \arrow[dl, "\rho"] \\ & (X, p) & \end{tikzcd}$$

In section 7.3 we have proved theorem 1 for $G(\tilde{X})$. By iteration it is enough to prove theorem 1 when γ is a blowing-up of a point of $E_{\tilde{X}}$. As $\tilde{X}$ is smooth, this blowing-up creates an irreducible component of the exceptional divisor of self-intersection -1. As in the proof of lemma 5, the subgraph $A(Y) := \gamma_G^{-1}(A(\tilde{X}))$ is a union of maximal arcs and satisfies theorem 1.

8 Examples

In the minimal resolution of Δ^+ the strict transforms of the discriminant curve are represented by edges ended with a star.

8.1 Exemple 1

Let $\phi = (f, g) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ defined by

$$f(x, y) = (x^2 - y^3)y(y + x^5)(x + y + x^3) \text{ and } g(x, y) = x(y + 2x^5)(x + y).$$

The critical locus of ϕ admits five irreducible components. Four of them are smooth and tangent to $\{y = -x\}$, $\{y = x\}$ and $\{y = 0\}$ for two of them. The fifth one is tangent to $\{x = 0\}$ and topologically equivalent to $\{x^2 - y^3 = 0\}$.

The graph $G(Z)$ is in Figure 6.

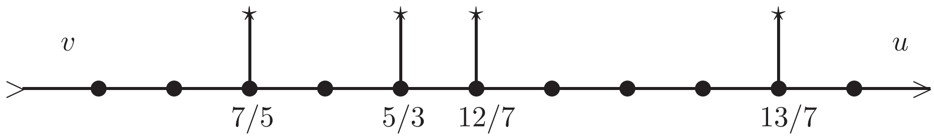

Figure 6: Graph of the minimal resolution r of Δ^+

Each vertex of $G(Z)$ belongs to the maximal arc $S(Z)$.

The graph of the minimal resolution ρ, weighted with the Hironaka quotients of (f, g), is in Figure 7.

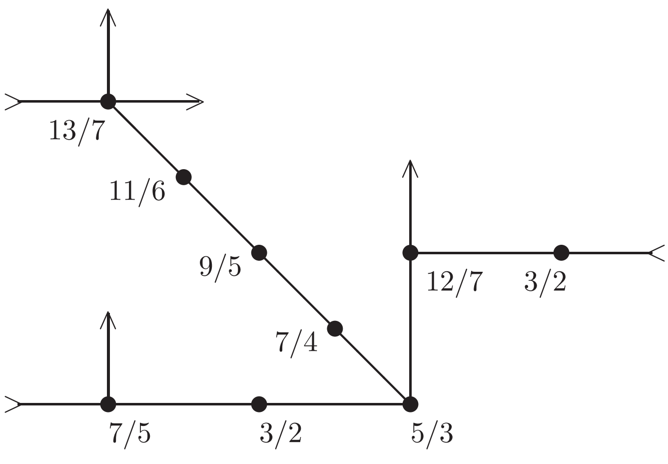

Figure 7: Graph of the minimal resolution ρ

In this example, the subgraph $A(\tilde{X})$ coincides with $G(\tilde{X})$.

Notice that the subgraphs $A(\tilde{X})$ and $G(\tilde{X})$ will have similar shapes when

$$f(x,y) = (x^2 - y^3)y(y + x^k)(x + y + x^l) \text{ and } g(x,y) = x(y + 2x^k)(x + y)$$

where k, l are integers strictly greater than 1.

8.2 Exemple 2

Let us consider the surface $(X, 0)$ of equation:

$$z^3 = (y^3 - x^2)(y^3 - (x + y)^2)$$

and let $\phi : (X, 0) \to (\mathbb{C}^2, 0)$ be the projection on the (x, y)-plane. Notice that this projection is not a generic one.

The discriminant locus of ϕ is $\Delta : (v^3 - u^2)(v^3 - (u + v)^2) = 0$. The minimal resolution tree of Δ^+ is in Figure 8.

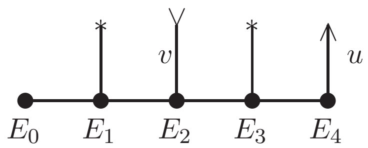

Figure 8: Graph of the minimal resolution r of Δ^+

The vertices E_2, E_3, E_4 belong to the maximal arc $S(Z)$.

The set of Hironaka quotients associated to $G(Z)$ is $\left\{1, \frac{3}{2}, 2\right\}$ (1 for E_0, E_1, E_2, 3/2 for E_3 and 3 for E_4).

The dual graph $G(X')$ of the Hirzebruch-Jung good resolution $\rho' : (X', E_{X'}) \to (X, 0))$ of ϕ is in Figure 9. We represent the subgraph $A(X')$ by a double-line joining the arrows associated to the strict transforms of $\{f = 0\}$ and $\{g = 0\}$.

The dual graph $G(\tilde{X})$ of the minimal good resolution $\rho : (\tilde{X}, E_{\tilde{X}}) \to (X, 0))$ of ϕ, is obtained from the one in Figure 9 by blowing-down E'_0, E'_4 and 4 other vertices of self-intersection -1 : $E'_{1,2}, E'_{2,3}, E'_{1,0}, E'_{3,0}$.

In $G(\tilde{X})$ each irreducible component of the exceptional divisor associated to the vertices of $G(\tilde{X})$ is of genus zero and of self-intersection equal to -2, except the one intersected by the strict transform of $\{g = 0\}$ which has self-intersection -3. We weight $G(\tilde{X})$ with the Hironaka quotients of $(f = u \circ \phi, g = v \circ \phi)$. It is represented in Figure 10.

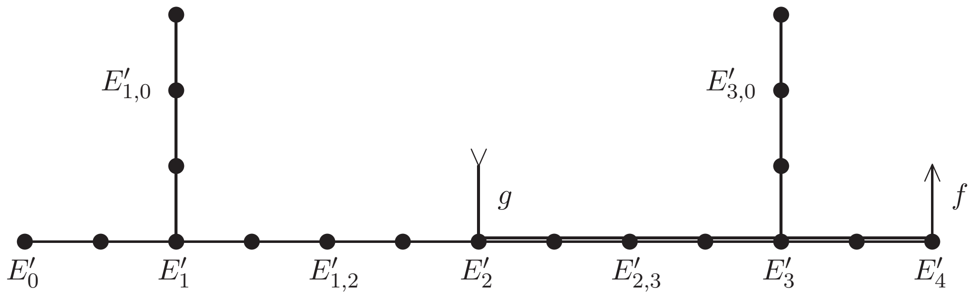

Figure 9: Graph of the Hirzebruch-Jung resolution of ϕ

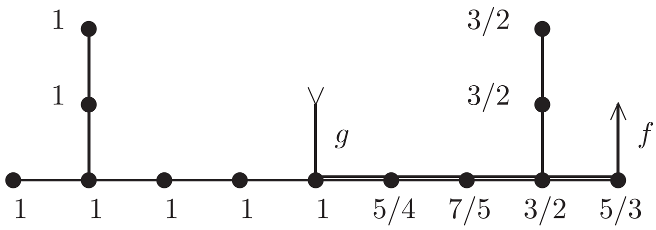

Figure 10: Graph of the minimal resolution of ϕ

In this example $A(\tilde{X})$ (respectively $A(X')$) is strictly included in $G(\tilde{X})$ (respectively $G(X')$) and $\overline{G(\tilde{X})\setminus A(\tilde{X})}$ (respectively $\overline{G(X')\setminus A(X')}$) admits two connected components on which the Hironaka quotients are respectively equal to 1 and 3/2.

8.3 Exemple 3

Let us consider the surface $(X,0)$ of the following equation:

$$z^2 = (y+x^3)(y+x^2)(x^{34}-y^{13}).$$

1. Let us first consider the case where $\phi_1 : (X,0) \to (\mathbb{C}^2,0)$ is the projection on the $(x, x+y)$-plane. It is a generic projection.

   The discriminant locus of $\phi_1 = (f_1, g_1)$ is the curve Δ_1 which admits three components with Puiseux expansions given by :

   $$v = u - u^2$$

   $$v = u - u^3$$

   $$v = u + u^{34/13}$$

   Notice that the three components of Δ_1 admit 1 as first Puiseux exponent and respectively $2, 3, 34/13$ as second Puiseux exponent.

The coordinate axes are transverse to the discriminant locus of ϕ_1. Hence the maximal arc of the tree of the minimal embedded resolution of Δ_1^+ has a unique vertex of Hironaka quotient equal to one. Moreover the Hironaka quotients are constant in the tree $G(Z)$ of the minimal embedded resolution of Δ_1^+. The dual graph $G(Z)$ is in Figure 11.

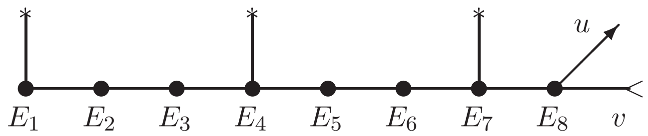

Figure 11: Graph of the minimal resolution of Δ_1^+

The Hironaka quotient associated to each irreducible component of E_Z is equal to one. Only the vertex E_8 of $G(Z)$ belongs to $S(Z)$.

The dual graph $G(X')$ of R admits a cycle created by the normalization. The irreducible component E'_0 is obtained by the resolution $\bar{\rho}$. The irreducible components of the exceptional divisor associated to the vertices of $G(X')$ have a genus equal to zero. The subgraph $\overline{G(X')\backslash A(X')}$ is connected of Hironaka quotient equal to one.

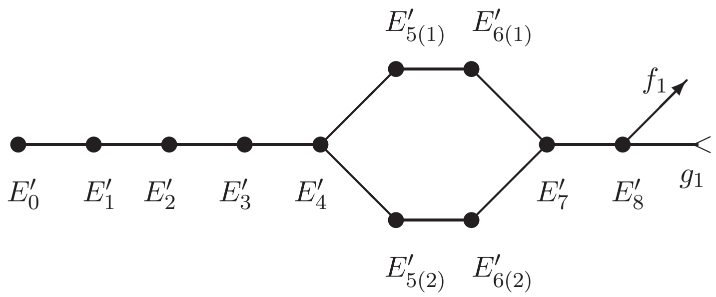

Figure 12: The graph of the Hirzebruch-Jung resolution of ϕ_1

The minimal good resolution ρ is obtained by blowing down E'_3. Its dual graph is in Figure 13.

2. Now let us consider the projection $\phi_2 = (f_2, g_2)$ on the (x, y)-plane. Notice that ϕ_1 and ϕ_2 have the same critical locus.

   The discriminant locus of ϕ_2 is the curve $\Delta_2 : (v + u^3)(v + u^2)(u^{34} - v^{13}) = 0$.

   The first Puiseux exponents of the components of Δ_2 are $2, 3, 34/13$.

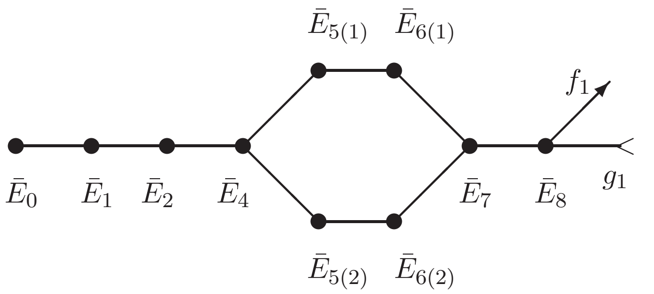

Figure 13: The graph of the minimal resolution of ϕ_1

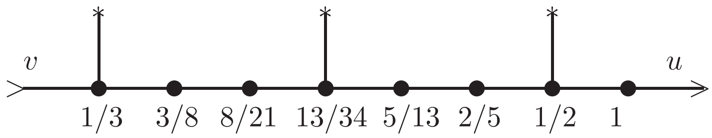

Figure 14: Graph of the minimal resolution of Δ_2^+

The tree $G(Z)$ of the minimal embedded resolution of Δ_2^+ is in Figure 14.

Each vertex of $G(Z)$ belongs to $S(Z)$.

The subgraph $\overline{G(X')\backslash A(X')}$ admits a unique connected component corresponding to a full-torus. Its Hironaka quotient is equal to 1/3.

The graph of the minimal resolution of ϕ_2 is obtained by blowing-down E'_3 and E'_8.

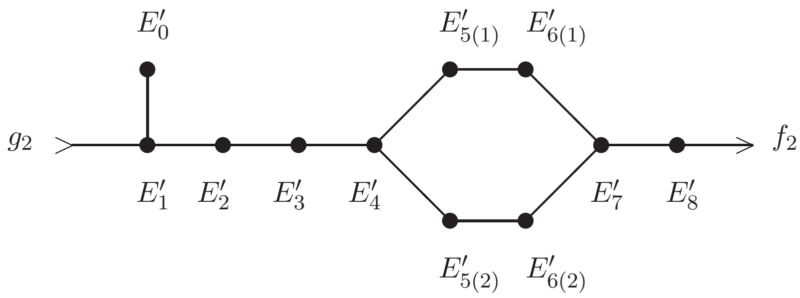

Figure 15: The graph of the Hirzebruch-Jung resolution of ϕ_2

The second Puiseux exponents of the components of Δ_1 are the Hironaka quotients of the rupture vertices of the minimal resolution of ϕ_2. This comes from the fact that,

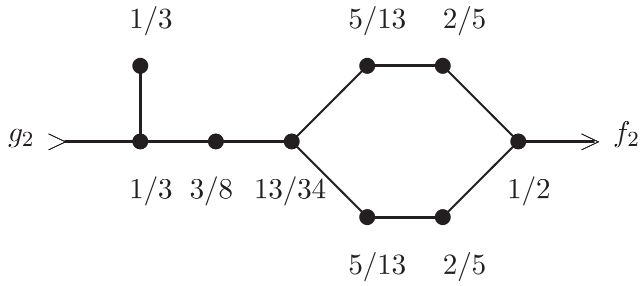

Figure 16: Graph of the minimal resolution of ϕ_2

in case 1 and 2, the functions f_1, g_1, f_2, g_2 belong to the same pencils Λ generated by x and y. In case 1, f_1 and g_1 are generic elements of the pencil, and in case 2, g_2 is not generic anymore.

As proved in [3] there exists an iterative process to compute the Puiseux exponents of the discriminant curve of a finite morphism $(X, p) \to (\mathbb{C}^2, 0)$.

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